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ON CO-GORENSTEIN MODULES, MINIMAL FLAT RESOLUTIONS AND DUAL BASS NUMBERS

BY

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Abstract. The dual of a Gorenstein module is called a co-Gorenstein module, defined by Lingguang Li. In this paper, we prove that if R is a local U-ring and M is an Artinian R-module, then M is a co-Gorenstein R-module if and only if the complex $\operatorname{Hom}_{\widehat{R}}(\mathcal{C}(\mathcal{U},\widehat{R}),M)$ is a minimal flat resolution for M when we choose a suitable triangular subset $\mathcal{U}$ on $\widehat{R}$. Moreover we characterize the co-Gorenstein modules over a local U-ring and Cohen–Macaulay local U-ring.

1. Introduction. Throughout this paper, R is a commutative Noetherian ring and M is an R-module. If R is a local ring with maximal ideal $\mathfrak{m}$ and residue field $k=R/\mathfrak{m}$, then $\widehat{R}$ denotes the $\mathfrak{m}$-adic completion of R and $E=E_R(k)$ denotes the injective hull of k.

In [1], using the minimal injective resolution of M, the Bass numbers $\mu^i(\mathfrak{p},M)$ are defined, and it is proved that $\mu^i(\mathfrak{p},M)=\dim_{k(\mathfrak{p})}\operatorname{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}),M_\mathfrak{p})$ for all $i\geq 0$ and all prime ideals $\mathfrak{p}$ of R.

In [9], Enochs and Xu define the dual Bass numbers $\pi_i(\mathfrak{p},M)$ by using the minimal flat resolution of M and they proved that, for all $i\geq 0$ and all prime ideals $\mathfrak{p}$ of R,

$$\pi_i(\mathfrak{p},M)=\dim_{k(\mathfrak{p})}\operatorname{Tor}_i^{R_\mathfrak{p}}(k(\mathfrak{p}),\operatorname{Hom}_R(R_\mathfrak{p},M)) \tag{1}$$

whenever M is a cotorsion R-module.

In [29], Tang and the second named author introduce the concept of co-Cohen–Macaulay modules over local rings and study their properties. In [11]–[12], Li studies the vanishing properties of dual Bass numbers and proves that the colocalization of co-Cohen–Macaulay modules preserves the co-Cohen–Macaulayness under certain conditions. Moreover, Li [11] defines co-Gorenstein modules by vanishing properties of the dual Bass numbers of modules.

Let us mention here that the Bass numbers, dual Bass numbers and the colocalization technique are successfully used in the study of coalge-

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bras over a field in [4], [13], [24] and [25]. In the non-commutative context, Cohen–Macaulay modules are usually studied in relation to tame representation type in the integral representation theory, vector bundles, and classical orders (see [7], [26] and [27]).

The Cousin complex $\mathcal{C}_R(M)$ was introduced by Sharp [19]. If M is a finitely generated R-module, then M is a Cohen–Macaulay R-module if and only if the Cousin complex $\mathcal{C}_R(M)$ is exact (see [20, Theorem 2.4]). Moreover, by [20, Section 3], M is a Gorenstein R-module if and only if $\mathcal{C}_R(M)$ provides a minimal injective resolution for M. On the other hand, Sharp and the second author [23] introduced the modules of generated fractions as a generalization of modules of fractions. Let $\mathcal{U} = \{U_i : i \in \mathbb{N}\}$ be any chain of triangular subsets on R. Then one can construct an associated complex $\mathcal{C}(\mathcal{U}, R)$ which consists of generalized fractions modules. Furthermore, in [29, Theorem 3.10], Tang and the second author have proved that if we choose a suitable triangular subset $\mathcal{U}$ on R, then M is a co-Cohen–Macaulay R-module with respect to $\mathcal{U}$ if and only if the induced complex $\operatorname{Hom}_R(\mathcal{C}(\mathcal{U}, R), M)$ is exact.

The following conjecture is discussed by Li [11]:

Conjecture 1.1. *Let R be a U-ring and M be an Artinian R-module. One can choose a suitable chain $\mathcal{U}$ of triangular subsets on R such that M is a co-Gorenstein R-module with respect to $\mathcal{U}$ if and only if the induced complex* $\operatorname{Hom}_R(\mathcal{C}(\mathcal{U}, R), M)$ *is a minimal flat resolution of M.*

In Section 3, we prove the conjecture under certain conditions (see Theorem 3.9). We also characterize co-Gorenstein modules by means of the local cohomology module $H_{\mathfrak{m}}^{\dim R}(R)$ over a Cohen–Macaulay U-ring R.

2. Preliminaries and notation. In [14], Melkersson and Schenzel introduced the colocalization of modules as follows. Let R be a ring, S be a multiplicatively closed subset of R and M be an R-module. The R_S-module $\operatorname{Hom}_R(R_S, M)$ is called the *colocalization* of M with respect to S. As a dual notion of $\operatorname{Supp}_R(M) = \{\mathfrak{p} \in \operatorname{Spec}(R) : M_{\mathfrak{p}} \neq 0\}$, the *co-support* of M is defined to be $\operatorname{cosupp}_R(M) = \{\mathfrak{p} \in \operatorname{Spec}(R) : \operatorname{Hom}_R(R_{\mathfrak{p}}, M) \neq 0\}$. Also in [14], the *codimension* of M is defined to be $\operatorname{codim}_R M = \sup\{\dim(R/\mathfrak{p}) : \mathfrak{p} \in \operatorname{cosupp}_R(M)\}$.

The *Krull dimension* $\operatorname{Kdim}_R M$ of an Artinian module M, introduced by Roberts [17] is defined inductively as follows. When $M = 0$, we set $\operatorname{Kdim}_R M = -1$. Then by induction, for any integer $\alpha \geq 0$, we set $\operatorname{Kdim}_R M = \alpha$ if (i) $\operatorname{Kdim}_R M < \alpha$ is false, and (ii) for any ascending chain $M_0 \subseteq M_1 \subseteq \cdots$ of submodules of M there exists an integer n such that $\operatorname{Kdim}_R(M_{i+1}/M_i) < \alpha$ for all $i > n$.

Note that if $(R, \mathfrak{m})$ is local and $M \neq 0$ is an Artinian R-module, then by [17, Theorem 6] we have

$$\operatorname{Kdim}_R M = \inf\big\{i \in \mathbb{Z} : \text{there exist } x_1, \dots, x_i \in \mathfrak{m} \text{ with } l_R(0 :_M (x_1, \dots, x_i)) < \infty\big\}.$$

We also recall that $\operatorname{Ann}_R(M) = \{x \in R : xM = 0\}$ and $\mathcal{V}(\operatorname{Ann}_R(M)) = \{\mathfrak{p} \in \operatorname{Spec}(R) : \operatorname{Ann}_R(M) \subset \mathfrak{p}\}$.

Let $x_1, \dots, x_n$ be a sequence of elements of R. We say that $x_1, \dots, x_n$ is a *poor M-cosequence* if

$$0 :_M (x_1, \dots, x_{i-1}) \xrightarrow{x_i} 0 :_M (x_1, \dots, x_{i-1})$$

is surjective for $i = 1, \dots, n$; it is an *M-cosequence* if, in addition, we have $0 :_M (x_1, \dots, x_n) \neq 0$. For an ideal I of R with $0 :_M I \neq 0$, the *cograde* of M with respect to I, denoted by $\operatorname{cograde}_I M$, is the length of a maximal M-cosequence in I. Notice that the lengths of any two maximal M-cosequences in I are equal.

For an element $\mathfrak{p} \in \operatorname{cosupp}_R(M)$ we define the *co-height* of $\mathfrak{p}$ with respect to M, written $\operatorname{coht}_M \mathfrak{p}$, by

$$\operatorname{coht}_M \mathfrak{p} = \sup\big\{n \in \mathbb{Z} : \text{there exists a chain } \mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n \text{ of length } n \text{ such that } \mathfrak{p}_i \in \operatorname{cosupp}_R(M) \text{ for all } 0 \leq i \leq n\big\}.$$

It is obvious that $\operatorname{coht}_M \mathfrak{p} = \operatorname{codim}_{R_\mathfrak{p}} \operatorname{Hom}_R(R_\mathfrak{p}, M)$.

A property of finitely generated modules is that $\operatorname{Ann}_R(M/\mathfrak{p}M) = \mathfrak{p}$ for any $\mathfrak{p} \in \mathcal{V}(\operatorname{Ann}_R(M))$. However the dual of this property, that is, $\operatorname{Ann}_R(0 :_M \mathfrak{p}) = \mathfrak{p}$ for any $\mathfrak{p} \in \mathcal{V}(\operatorname{Ann}_R(M))$, is not true for any Artinian module. For example, let $(R, \mathfrak{m})$ be a Noetherian local domain of dimension 2 such that $\widehat{R}$, the $\mathfrak{m}$–adic completion of R, has an associated prime $\mathfrak{q}$ of dimension 1. Then, by [6], the Artinian R-module $H^1_\mathfrak{m}(R)$ satisfies $\operatorname{Ann}_R(0 :_{H^1_\mathfrak{m}(R)} \mathfrak{p}) = \mathfrak{p}$ for no $\mathfrak{p} \in \mathcal{V}(\operatorname{Ann}_R(H^1_\mathfrak{m}(R)))$.

Proposition 2.1 ([5, Proposition 2.1]). *Let $(R, \mathfrak{m})$ be a Noetherian local ring and M be an Artinian R-module. If one of the following two conditions is satisfied:*

(i) *R is complete with respect to the $\mathfrak{m}$-adic topology,*
(ii) *M contains a submodule isomorphic to the injective hull of $R/\mathfrak{m}$,*

then $\operatorname{Ann}_R(0 :_M \mathfrak{p}) = \mathfrak{p}$ *for any* $\mathfrak{p} \in \mathcal{V}(\operatorname{Ann}_R(M))$.

Definition 2.2. A commutative Noetherian ring R is called a *U-ring* if $\operatorname{Ann}_R(0 :_M \mathfrak{p}) = \mathfrak{p}$ for any Artinian R-module M and any $\mathfrak{p} \in \mathcal{V}(\operatorname{Ann}_R(M))$.

Notice that, in view of the above proposition, any complete local ring is a U-ring.

REMARK 2.3. Let R be a commutative Noetherian ring.

(i) Recall that an R-module I is *faithfully injective* if it is injective and $\operatorname{Hom}_R(M, I) = 0$ only if $M = 0$. It is well known that if $(R, \mathfrak{m})$ is local, then $E = E(R/\mathfrak{m})$ is faithfully injective.

(ii) If $(R, \mathfrak{m})$ is local, then any Artinian R-module M has a natural $\widehat{R}$-module structure and, by [10, Lemma 1.14], $M \otimes_R \widehat{R} \cong M$. Here $\widehat{R}$ is the completion of R in the $\mathfrak{m}$-adic topology.

(iii) By [30, Section 3.3], all Artinian R-modules are cotorsion and if M is an Artinian R-module then, by [30, Proposition 5.2.8],

$$\pi_i(\mathfrak{p}, M) = \dim_{k(\mathfrak{p})} \operatorname{Tor}_i^{R_\mathfrak{p}}(k(\mathfrak{p}), \operatorname{Hom}_R(R_\mathfrak{p}, M))$$

for all $\mathfrak{p} \in \operatorname{cosupp}_R(M)$ and $i \geq 0$.

One can easily prove the following two lemmas.

LEMMA 2.4. *Let $(R, \mathfrak{m})$ be a local ring and M be an Artinian R-module. Let $x_1, \dots, x_n$ be a sequence of elements of R or $\widehat{R}$. Then $x_1, \dots, x_n$ is an M-cosequence if and only if $x_1, \dots, x_n$ is a $\operatorname{Hom}_{\widehat{R}}(M, E)$-sequence.*

LEMMA 2.5 ([12, Lemma 2.6]). *Let R be a Noetherian ring and M be an Artinian R-module such that $\operatorname{Ann}_R(0 :_M \mathfrak{p}) = \mathfrak{p}$ for any $\mathfrak{p} \in \mathcal{V}(\operatorname{Ann}_R M)$. Then*

$$\operatorname{Kdim}_R M = \operatorname{codim}_R M.$$

REMARK 2.6. Let S be a ring. Then with the notation of [18] the following statements hold:

(i) Consider the triple $({}_RA,\ {}_RB_S, C_S)$, where A is finitely generated and C is injective. Then, by [18, Theorem 9.51], there is an isomorphism,

$$\operatorname{Tor}_i^R(A, \operatorname{Hom}_S(B, C)) \cong \operatorname{Hom}_S(\operatorname{Ext}_R^i(A, B), C)$$

for any integer $i \geq 0$.

(ii) If the R-module B in the triple $({}_RA,\ {}_SB_R,\ {}_SC)$ is projective, then, by [18, Exercise 9.21], there exists an isomorphism

$$\operatorname{Ext}_S^i(B \otimes_R A, C) \cong \operatorname{Ext}_R^i(A, \operatorname{Hom}_S(B, C))$$

for any $i \geq 0$.

LEMMA 2.7. *Let $(R, \mathfrak{m})$ be a local U-ring and M be an Artinian R-module. Then*

$$\operatorname{codim}_R M \geq \dim_{\widehat{R}} \operatorname{Hom}_{\widehat{R}}(M, E).$$

Proof. Let $x_1, \dots, x_i$ be a sequence of elements of R. Then by Remark 2.3 and Matlis duality,

$$\begin{aligned} 0 :_M (x_1,\dots,x_i) &\cong 0 :_M (x_1,\dots,x_i) \otimes_R \widehat{R} \cong \operatorname{Hom}_{\widehat{R}}(\widehat{R}/(x_1,\dots,x_i)\widehat{R}, M) \\ &\cong \operatorname{Hom}_{\widehat{R}}\big(\widehat{R}/(x_1,\dots,x_i)\widehat{R}, \operatorname{Hom}_{\widehat{R}}(\operatorname{Hom}_{\widehat{R}}(M,E),E)\big) \\ &\cong \operatorname{Hom}_{\widehat{R}}\big(\widehat{R}/(x_1,\dots,x_i)\widehat{R} \otimes \operatorname{Hom}_{\widehat{R}}(M,E),E\big) \\ &\cong \operatorname{Hom}_{\widehat{R}}\big(\operatorname{Hom}_{\widehat{R}}(M,E)/(x_1,\dots,x_i)\operatorname{Hom}_{\widehat{R}}(M,E),E\big). \end{aligned}$$

Hence, in view of Lemma 2.5 and [3, p. 413], we have

$$\operatorname{codim}_R M \geq \dim_{\widehat{R}} \operatorname{Hom}_{\widehat{R}}(M,E). \ ∎$$

PROPOSITION 2.8. *Let $(R,\mathfrak{m})$ be a local ring and M be an Artinian R-module. Then*

(i) $\operatorname{Supp}_{\widehat{R}}(\operatorname{Hom}_{\widehat{R}}(M,E)) = \operatorname{cosupp}_{\widehat{R}}(M)$.

(ii) *Let $i \geq 0$ and $\mathfrak{p} \in \operatorname{Spec}(\widehat{R})$. The dual Bass number $\pi_i(\mathfrak{p},M)$ is zero if and only if the Bass number $\mu^i(\mathfrak{p},\operatorname{Hom}_{\widehat{R}}(M,E))$ is zero.*

Proof. (i) Let $\mathfrak{p} \in \operatorname{Spec}(\widehat{R})$. We have the following isomorphisms:

$$\begin{aligned} \operatorname{Hom}_{\widehat{R}}(\operatorname{Hom}_{\widehat{R}}(M,E)_{\mathfrak{p}},E) &\cong \operatorname{Hom}_{\widehat{R}}(\operatorname{Hom}_{\widehat{R}}(M,E) \otimes_{\widehat{R}} \widehat{R}_{\mathfrak{p}},E) \\ &\cong \operatorname{Hom}_{\widehat{R}}\big(\widehat{R}_{\mathfrak{p}}, \operatorname{Hom}_{\widehat{R}}(\operatorname{Hom}_{\widehat{R}}(M,E),E)\big) \cong \operatorname{Hom}_{\widehat{R}}(\widehat{R}_{\mathfrak{p}},M). \end{aligned}$$

Therefore the result is a consequence of Remark 2.3.

(ii) Let $\mathfrak{p} \in \operatorname{Spec}(\widehat{R})$ and $i \geq 0$. Then (i) and Remark 2.6(i) yield

$$\begin{aligned} \operatorname{Tor}_i^{\widehat{R}_{\mathfrak{p}}}(k(\mathfrak{p}),\operatorname{Hom}_{\widehat{R}}(\widehat{R}_{\mathfrak{p}},E)) &\cong \operatorname{Tor}_i^{\widehat{R}_{\mathfrak{p}}}\big(k(\mathfrak{p}),\operatorname{Hom}_{\widehat{R}}(\operatorname{Hom}_{\widehat{R}}(M,E)_{\mathfrak{p}},E)\big) \\ &\cong \operatorname{Hom}_{\widehat{R}}\big(\operatorname{Ext}^i_{\widehat{R}_{\mathfrak{p}}}(k(\mathfrak{p}),(\operatorname{Hom}_{\widehat{R}}(M,E))_{\mathfrak{p}}),E\big). \end{aligned}$$

Hence the assertion follows by definitions. ∎

REMARK 2.9. In the situation of Proposition 2.8, the arguments used in the proof also show that

$$\pi_i(\mathfrak{m},M) = 0 \quad \text{if and only if} \quad \mu^i(\mathfrak{m}\widehat{R},\operatorname{Hom}_{\widehat{R}}(M,E)) = 0.$$

We recall from [29, Definition 2.12] the following:

DEFINITION 2.10. Assume that $(R,\mathfrak{m})$ is a local ring and that M is a non-zero Artinian R-module. Then M is called a *co-Cohen–Macaulay module* if

$$\operatorname{cograde}_{\mathfrak{m}} M = \operatorname{Kdim}_R M.$$

Following [15, p. 420], we denote by $\mathcal{U} = \{U_i : i \in \mathbb{N}\}$ a chain of triangular subsets on R. Then each U_i leads to a module of generalized fractions $U_i^{-i}R$ (see [23]) and we can, in fact, arrange these modules into the complex

$$\mathcal{C}(\mathcal{U},R) : 0 \to R \to U_1^{-1}R \to U_2^{-2}R \to \cdots \to U_i^{-i}R \xrightarrow{d_i} U_{i+1}^{-i-1}R \to \cdots .$$

Note that the following theorem, proved in [29], is concerned with the exactness of the induced complex $\operatorname{Hom}_R(\mathcal{C}(\mathcal{U}, R), M)$.

THEOREM 2.11. *Let M be an Artinian R-module. Then in the above notation,* $\operatorname{Hom}_R(\mathcal{C}(\mathcal{U}, R), M)$ *is exact if and only if, for all $n \in \mathbb{N}$, each element of U_n is a poor M-cosequence in the sense of* [29, p. 2175].

Proof. See [29, Theorem 3.3]. ∎

3. Co-Cohen–Macaulay and co-Gorenstein modules. Let M be a non-zero finitely generated R-module. Following [20, Theorem 3.6], M is defined to be *Gorenstein* if, for all $\mathfrak{p} \in \operatorname{Supp}_R(M)$, $\mu^i(\mathfrak{p}, M)$ is zero if and only if $\dim_{R_\mathfrak{p}} M_\mathfrak{p} \neq i$.

Next we recall the definitions of dual of a Gorenstein module introduced in [11].

DEFINITION 3.1. Let R be a Noetherian ring and M be an R-module. M is called a *co-Gorenstein* R-module if has the following property: the dual Bass number $\pi_i(\mathfrak{p}, M)$ is non-zero if and only if $i = \operatorname{coht}_M \mathfrak{p}$ for any $\mathfrak{p} \in \operatorname{cosupp}_R(M)$.

PROPOSITION 3.2. *Let $(R, \mathfrak{m})$ be a Noetherian local ring. A Noetherian R-module M is a Gorenstein R-module if and only if* $\operatorname{Hom}_R(M, E)$ *is a co-Gorenstein R-module.*

Proof. By [10, Lemma 1.15], M is a Noetherian R-module if and only if $\operatorname{Hom}_R(M, E)$ is an Artinian R-module. By Remark 2.6, there exists an isomorphism

$$\operatorname{Tor}_i^{R_\mathfrak{p}}\big(k(\mathfrak{p}), \operatorname{Hom}_R(R_\mathfrak{p}, \operatorname{Hom}_R(M, E))\big) \cong \operatorname{Hom}_R\big(\operatorname{Ext}^i_{R_\mathfrak{p}}(k(\mathfrak{p}), M_\mathfrak{p}), E\big).$$

Hence the result follows from Remark 2.3(iii) and Definition 3.1. ∎

The following two propositions, which establish characterizations of co-Gorenstein modules, are proved in [11].

PROPOSITION 3.3. *Let R be a U-ring and M be an Artinian R-module. The following conditions are equivalent:*

(i) *M is a co-Gorenstein R-module.*
(ii) $\operatorname{Hom}_R(R_\mathfrak{p}, M)$ *is a co-Gorenstein $R_\mathfrak{p}$-module for any* $\mathfrak{p} \in \operatorname{cosupp}_R(M)$.
(iii) $\operatorname{Hom}_R(R_\mathfrak{m}, M)$ *is a co-Gorenstein $R_\mathfrak{m}$-module for any maximal ideal $\mathfrak{m}$ in* $\operatorname{cosupp}_R(M)$.
(iv) *$\pi_i(\mathfrak{m}, M) \neq 0$ if and only if $i = \operatorname{coht}_M \mathfrak{m}$ for any maximal ideal $\mathfrak{m}$ in* $\operatorname{cosupp}_R(M)$.

PROPOSITION 3.4. *Let R be a local U-ring and M be an Artinian R-module. Then the following conditions are equivalent:*

(i) *M is a co-Gorenstein R-module.*
(ii) *M is a co-Cohen–Macaulay R-module and* $\operatorname{codim}_R M = \operatorname{flatdim}_R M$.

Proposition 3.5. *Let $(R,\mathfrak{m})$ be a local U-ring and M be an Artinian R-module. Then M is a co-Gorenstein R-module if and only if* $\operatorname{Hom}_{\widehat{R}}(M,E)$ *is a Gorenstein $\widehat{R}$-module.*

Proof. ($\Rightarrow$) Let M be co-Gorenstein. By Lemmas 2.4 and 2.7, we have

$$\operatorname{cograde}_{\mathfrak{m}} M \le \operatorname{depth}_{\widehat{R}} \operatorname{Hom}_{\widehat{R}}(M,E) \le \dim_{\widehat{R}} \operatorname{Hom}_{\widehat{R}}(M,E) \le \operatorname{codim}_R M.$$

On the other hand, by Proposition 3.4, M is a co-Cohen–Macaulay R-module and $\operatorname{codim}_R M = \operatorname{flatdim}_R M$. Let $\operatorname{codim}_R M = n$. In view of Remark 2.3(ii), for M regarded as an $\widehat{R}$-module, there exists a flat resolution of length n,

$$0 \to F_n \to \cdots \to F_1 \to F_0 \to M \to 0.$$

By [8, Theorem 3.2.9], applying the exact functor $\operatorname{Hom}_{\widehat{R}}(-,E)$, we obtain the injective resolution

$$0 \to \operatorname{Hom}_{\widehat{R}}(M,E) \to \operatorname{Hom}_{\widehat{R}}(F_0,E) \to \cdots \to \operatorname{Hom}_{\widehat{R}}(F_n,E) \to 0.$$

Therefore $\operatorname{inj.dim}_{\widehat{R}} \operatorname{Hom}_{\widehat{R}}(M,E) \le n$. On the other hand, $\operatorname{Hom}_{\widehat{R}}(M,E)$ is a finitely generated $\widehat{R}$-module. So, [3, Theorem 3.1.17] yields

$$n = \operatorname{codim}_R M = \dim_{\widehat{R}} \operatorname{Hom}_{\widehat{R}}(M,E) \le \operatorname{inj.dim}_{\widehat{R}} \operatorname{Hom}_{\widehat{R}}(M,E) \le n.$$

Hence

$$n = \operatorname{cograde}_{\mathfrak{m}} M = \operatorname{codim}_R M,$$

and finally we obtain

$$\begin{aligned}\dim_{\widehat{R}} \operatorname{Hom}_{\widehat{R}}(M,E) =&\ \operatorname{depth}_{\widehat{R}} \operatorname{Hom}_{\widehat{R}}(M,E)\\ =&\ \operatorname{inj.dim}_{\widehat{R}} \operatorname{Hom}_{\widehat{R}}(M,E) = \operatorname{depth} \widehat{R}.\end{aligned}$$

Consequently, by [20, Theorem 3.11], $\operatorname{Hom}_{\widehat{R}}(M,E)$ is a Gorenstein $\widehat{R}$-module.

($\Leftarrow$) By Remark 2.9, $\mu^i(\mathfrak{m}\widehat{R}, \operatorname{Hom}_{\widehat{R}}(M,E)) = 0$ if and only if $\pi_i(\mathfrak{m},M) = 0$ for all $i \ge 0$. Since $\operatorname{Hom}_{\widehat{R}}(M,E)$ is a Gorenstein $\widehat{R}$-module, we have

$$\operatorname{inj.dim}_{\widehat{R}} \operatorname{Hom}_{\widehat{R}}(M,E) = \operatorname{ht}_{\operatorname{Hom}_{\widehat{R}}(M,E)} \mathfrak{m}\widehat{R} = n$$

for some integer n. Let

$$0 \to \operatorname{Hom}_{\widehat{R}}(M,E) \to E^0 \to E^1 \to \cdots \to E^n \to 0$$

be an injective resolution for $\operatorname{Hom}_{\widehat{R}}(M,E)$. By [8, Theorem 3.2.16], $\operatorname{Hom}_{\widehat{R}}(E^i,E)$ is a flat $\widehat{R}$-module for all $0 \le i \le n$. Therefore we have the following flat resolution for M:

$$0 \to \operatorname{Hom}_{\widehat{R}}(E^n,E) \to \cdots \to \operatorname{Hom}_{\widehat{R}}(E^1,E) \to \operatorname{Hom}_{\widehat{R}}(E^0,E) \to M \to 0.$$

Hence $\operatorname{flatdim}_R M \le n$.

On the other hand, by [12, Corollary 5.11], $\operatorname{codim}_R M \le \operatorname{flatdim}_R M \le n$. Thus by Lemma 2.7, $\operatorname{coht}_M \mathfrak{m} = \operatorname{ht}_{\operatorname{Hom}_{\widehat{R}}(M,E)} \mathfrak{m}\widehat{R}$. Hence the result follows from Remark 2.9 and Proposition 3.3. ∎

COROLLARY 3.6. *Let R be a local U-ring and M be a co-Gorenstein R-module. Then* $\operatorname{Ann}_R(M) = 0$, *$R$ is a Cohen–Macaulay ring and* $\dim R = \operatorname{codim} M = \operatorname{flatdim}_R M$.

Proof. Apply [20, Theorem 4.12], [2, Remark 10.2.2(ii)] and Proposition 3.5. ∎

PROPOSITION 3.7. *Let $(R, \mathfrak{m})$ be a local U-ring and M be an Artinian R-module. Then M is a co-Gorenstein R-module if and only if M is a co-Gorenstein $\widehat{R}$-module.*

Proof. (⇒) Let M be a co-Gorenstein R-module. Then by Propositions 2.8(i) and 3.5, $\operatorname{Hom}_{\widehat{R}}(M, E)$ is a Gorenstein $\widehat{R}$-module and

$$\operatorname{Spec}(\widehat{R}) = \operatorname{Supp}_{\widehat{R}}(\operatorname{Hom}_{\widehat{R}}(M, E)) = \operatorname{cosupp}_{\widehat{R}}(M).$$

Hence the result follows from Proposition 2.8(ii) and Definition 3.1.

(⇐) Let M be a co-Gorenstein $\widehat{R}$-module. Then by definition we have $\pi_i(\mathfrak{p}, M) \neq 0$ if and only if $\operatorname{ht} \mathfrak{p} = i$ for any $i \geq 0$ and any $\mathfrak{p} \in \operatorname{cosupp}_{\widehat{R}}(M)$. Therefore by Proposition 2.8(ii), $\mu^i(\mathfrak{p}, \operatorname{Hom}_{\widehat{R}}(M, E)) \neq 0$ if and only if $\operatorname{ht} \mathfrak{p} = i$, for any $i \geq 0$ and any $\mathfrak{p} \in \operatorname{cosupp}_{\widehat{R}}(M) = \operatorname{Supp}_{\widehat{R}}(\operatorname{Hom}_{\widehat{R}}(M, E))$. Consequently, by [20, Theorem 3.11], $\operatorname{Hom}_{\widehat{R}}(M, E)$ is a Gorenstein $\widehat{R}$-module, and by Proposition 3.5, M is a co-Gorenstein R-module. ∎

Now we recall the notion of the Cousin complex, due to Sharp [19].

DEFINITION 3.8. Let $(R, \mathfrak{m})$ be a Noetherian local ring and M be an R-module.

(a) A *filtration* of $\operatorname{Spec}(R)$ is a descending sequence $\mathcal{F} = (F_i)_{i\geq 0}$ of subsets of $\operatorname{Spec}(R)$ with the property that for every $i \geq 0$, each member of $\partial F_i = F_i \setminus F_{i+1}$ is a minimal member of F_i with respect to inclusion. We say that the filtration $\mathcal{F}$ *admits an R-module M* if $\operatorname{Supp}_R(M) \subseteq F_0$.

(b) Let $\mathcal{F} = (F_i)_{i\geq 0}$ be a filtration of $\operatorname{Spec}(R)$ which admits an R-module M. An obvious modification of the construction given in [19, §2] defines the complex

$$\mathcal{C}(\mathcal{F}, M) : 0 \to M \xrightarrow{d^{-1}} M^0 \xrightarrow{d^0} M^1 \to \cdots \to M^i \xrightarrow{d^i} M_{i+1} \to \cdots,$$

called the *Cousin complex* for M with respect to $\mathcal{F}$, where $M^0 = \bigoplus_{\mathfrak{p}\in\partial F_0} M_{\mathfrak{p}}$ and $M^i = \bigoplus_{\mathfrak{p}\in\partial F_i} (\operatorname{Coker} d^{i-2})_{\mathfrak{p}}$ for all $i > 0$; for $m \in M$ and $\mathfrak{p} \in \partial F_0$, the component of $d^1(m)$ in $M_{\mathfrak{p}}$ is $m/1$; and for $i > 0$, $x \in M^{i-1}$ and $\mathfrak{q} \in \partial F_i$, the component of $d^{i-1}(x)$ in $(\operatorname{Coker} d^{i-2})_{\mathfrak{q}}$ is $\pi(x)/1$, where $\pi : M^{i-1} \to \operatorname{Coker} d^{i-2}$ is the canonical epimorphism.

(c) If M is an R-module then $\mathcal{H}(M) = (H_i)_{i\geq 0}$ denotes the *M-height filtration* of $\operatorname{Spec}(R)$, defined by $H_i = \{\mathfrak{p} \in \operatorname{Supp}_R(M) : \operatorname{ht}_M \mathfrak{p} \geq i\}$ for $i \geq 0$. We denote by $\mathcal{C}(M)$ the Cousin complex for M with respect to $\mathcal{H}(M)$.

(d) Denote by $\mathcal{C}(\mathcal{D}(R), M)$ the Cousin complex of M with respect to the *dimension filtration* $\mathcal{D}(R) = (D_i)_{i\geq 0}$ of the spectrum of a local ring R, where $D_i = \{\mathfrak{p} \in \operatorname{Spec}(R) : \dim(R/\mathfrak{p}) \leq \dim R - i\}$ for all $i \geq 0$.

Now we state the main result of this section:

THEOREM 3.9. *Let R be a local U-ring with $\dim R = d$ and M be an Artinian R-module. Then M is a co-Gorenstein R-module if and only if $\operatorname{Hom}_{\widehat{R}}(\mathcal{C}(\mathcal{U}, \widehat{R}), M)$ is a minimal flat resolution of M over the ring $\widehat{R}$, where $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ and $U_n = \{(x_1, \dots, x_n) :$ there exists $j \geq 0$ such that $\{x_1, \dots, x_j\}$ is a subset of a system of parameters for $\widehat{R}$, and $x_{j+1} = x_{j+2} = \cdots = x_n = 1\}$.*

Proof. ($\Rightarrow$) Let M be a co-Gorenstein R-module. Then, by Proposition 3.5, $\operatorname{Hom}_{\widehat{R}}(M, E)$ is a Gorenstein $\widehat{R}$-module. Hence, by [20, Theorem 4.12],

$$\operatorname{Ann}_{\widehat{R}}(\operatorname{Hom}_{\widehat{R}}(M, E)) = 0,$$

so that $\operatorname{Supp}_{\widehat{R}}(\operatorname{Hom}_{\widehat{R}}(M, E)) = \operatorname{Spec}(\widehat{R})$. By Corollary 3.6, $\widehat{R}$ is Cohen–Macaulay and, by [22, Theorem 1.7], we have

$$\mathcal{C}_{\widehat{R}}(\widehat{R}) \otimes \operatorname{Hom}_{\widehat{R}}(M, E) \cong \mathcal{C}_{\widehat{R}}(\operatorname{Hom}_{\widehat{R}}(M, E)),$$

where $\mathcal{C}_{\widehat{R}}(\widehat{R})$ and $\mathcal{C}_{\widehat{R}}(\operatorname{Hom}_{\widehat{R}}(M, E))$ are Cousin complexes of, respectively, $\widehat{R}$ and $\operatorname{Hom}_{\widehat{R}}(M, E)$ with respect to the height filtration. Notice that in this case $\operatorname{ht} \mathfrak{p} = \operatorname{ht}_{\operatorname{Hom}_{\widehat{R}}(M,E)} \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec}(\widehat{R})$.

Since $\operatorname{Hom}_{\widehat{R}}(M, E)$ is Gorenstein, $\mathcal{C}_{\widehat{R}}(\operatorname{Hom}_{\widehat{R}}(M, E))$ is a minimal injective resolution for $\operatorname{Hom}_{\widehat{R}}(M, E)$ (see [20, §3]). Now, applying the exact functor $\operatorname{Hom}_{\widehat{R}}(-, E)$ on $\mathcal{C}_{\widehat{R}}(\operatorname{Hom}_{\widehat{R}}(M, E))$, we get the following isomorphisms of complexes:

$$\begin{aligned}\operatorname{Hom}_{\widehat{R}}\big(\mathcal{C}_{\widehat{R}}(\operatorname{Hom}_{\widehat{R}}(M, E)), E\big) &\cong \operatorname{Hom}_{\widehat{R}}\big(\mathcal{C}_{\widehat{R}}(\widehat{R}) \otimes \operatorname{Hom}_{\widehat{R}}(M, E), E\big)\\ &\cong \operatorname{Hom}_{\widehat{R}}\big(\mathcal{C}_{\widehat{R}}(\widehat{R}), \operatorname{Hom}_{\widehat{R}}(\operatorname{Hom}_{\widehat{R}}(M, E), E)\big) \cong \operatorname{Hom}_{\widehat{R}}(\mathcal{C}_{\widehat{R}}(\widehat{R}), M).\end{aligned}$$

Therefore the complex $\operatorname{Hom}_{\widehat{R}}(\mathcal{C}_{\widehat{R}}(\widehat{R}), M)$ is exact.

On the other hand, since $\mathcal{C}_{\widehat{R}}(\operatorname{Hom}_{\widehat{R}}(M, E))$ is a minimal injective resolution of the Gorenstein $\widehat{R}$-module $\operatorname{Hom}_{\widehat{R}}(M, E)$, the ith term $\operatorname{Hom}_{\widehat{R}}(\bigoplus_{\operatorname{ht}\mathfrak{p}=i}(\operatorname{Coker} d^{i-2})_{\mathfrak{p}}, E)$ of $\operatorname{Hom}_{\widehat{R}}(\mathcal{C}_{\widehat{R}}(\operatorname{Hom}_{\widehat{R}}(M, E)), E)$ is a flat precover of $\operatorname{Im} d_i$ for all $i \geq 0$, where $d_i = \operatorname{Hom}_{\widehat{R}}(d^{i-1}, E)$ and d^{i-1} is the ith differentiation of the Cousin complex $\mathcal{C}_{\widehat{R}}(\operatorname{Hom}_{\widehat{R}}(M, E))$. Therefore $\operatorname{Hom}_{\widehat{R}}(\mathcal{C}_{\widehat{R}}(\operatorname{Hom}_{\widehat{R}}(M, E)), E)$ is a flat resolution for M as an $\widehat{R}$-module. Since

$\bigoplus_{\operatorname{ht}\mathfrak{p}=i}(\operatorname{Coker} d^{i-2})_{\mathfrak{p}}$ is an injective $\widehat{R}$-module, [20, Theorem 3.11] yields

$$\bigoplus_{\operatorname{ht}\mathfrak{p}=i}(\operatorname{Coker} d^{i-2})_{\mathfrak{p}} \cong \bigoplus_{\operatorname{ht}\mathfrak{p}=i}\mu^i\big(\mathfrak{p},\operatorname{Hom}_{\widehat{R}}(M,E)\big)E(\widehat{R}/\mathfrak{p}).$$

Hence we derive the isomorphism

$$\operatorname{Hom}_{\widehat{R}}\Big(\bigoplus_{\operatorname{ht}\mathfrak{p}=i}(\operatorname{Coker} d^{i-2})_{\mathfrak{p}},E\Big) \cong \prod_{\operatorname{ht}\mathfrak{p}=i}\operatorname{Hom}_{\widehat{R}}\big(E(\widehat{R}/\mathfrak{p}),E\big)^{\mu^i(\mathfrak{p},\operatorname{Hom}_{\widehat{R}}(M,E))}.$$

Since

$$\begin{aligned}\operatorname{Hom}_{\widehat{R}}(E(\widehat{R}/\mathfrak{p}),E) &\cong \operatorname{Hom}_{\widehat{R}}(E(\widehat{R}/\mathfrak{p})\underset{\widehat{R}}{\otimes}\widehat{R}_{\mathfrak{p}},E)\\ &\cong \operatorname{Hom}_{\widehat{R}}\big(E(\widehat{R}/\mathfrak{p}),\operatorname{Hom}_{\widehat{R}}(\widehat{R}_{\mathfrak{p}},E)\big)\end{aligned}$$

and $\operatorname{Hom}_{\widehat{R}}(\widehat{R}_{\mathfrak{p}},E)$ is an $\widehat{R}_{\mathfrak{p}}$-injective module, we see that $\operatorname{Hom}_{\widehat{R}}(\widehat{R}_{\mathfrak{p}},E) \cong \bigoplus_{\mathfrak{q}\subseteq\mathfrak{p}}\gamma(\mathfrak{q})E_{\widehat{R}}(\widehat{R}/\mathfrak{q})$, where $\gamma(\mathfrak{q})\in\mathbb{N}\cup\{0\}$. We set $\operatorname{Hom}_{\widehat{R}}(\widehat{R}_{\mathfrak{p}},E)\cong A\oplus B$, where $A=\bigoplus_{\mathfrak{q}=\mathfrak{p}}E_{\widehat{R}}(\widehat{R}/\mathfrak{q})$ and $B=\bigoplus_{\mathfrak{q}\neq\mathfrak{p},\,\mathfrak{q}\subset\mathfrak{p}}E_{\widehat{R}}(\widehat{R}/\mathfrak{q})$.

Note that if $\mathfrak{q}\neq\mathfrak{p}$ and $\mathfrak{q}\subset\mathfrak{p}$, then $\operatorname{Hom}_{\widehat{R}}(E(\widehat{R}/\mathfrak{p}),E(\widehat{R}/\mathfrak{q}))=0$ and therefore $\operatorname{Hom}_{\widehat{R}}(E(\widehat{R}/\mathfrak{p}),\prod_{\mathfrak{q}\subset\mathfrak{p},\,\mathfrak{q}\neq\mathfrak{p}}E(\widehat{R}/\mathfrak{q}))=0$. Since B is a direct summand of $\prod_{\mathfrak{q}\subset\mathfrak{p},\,\mathfrak{q}\neq\mathfrak{p}}E(\widehat{R}/\mathfrak{q})$, it follows that $\operatorname{Hom}_{\widehat{R}}(E(\widehat{R}/\mathfrak{p}),B)=0$, and hence

$$\operatorname{Hom}_{\widehat{R}}(E(\widehat{R}/\mathfrak{p}),E)\cong\operatorname{Hom}_{\widehat{R}}(E(\widehat{R}/\mathfrak{p}),A)\cong\operatorname{Hom}_{\widehat{R}}(E(\widehat{R}/\mathfrak{p}),E(\widehat{R}/\mathfrak{p})^X)$$

for some set X. Since $\mu^i(\mathfrak{p},\operatorname{Hom}_{\widehat{R}}(M,E))$ is finite, for each prime ideal $\mathfrak{p}$ of $\widehat{R}$ with $\operatorname{ht}\mathfrak{p}=i$ there exists a set $X_{\mathfrak{p}}$ such that

$$\operatorname{Hom}_{\widehat{R}}(E(\widehat{R}/\mathfrak{p}),E)^{\mu^i(\mathfrak{p},\operatorname{Hom}_{\widehat{R}}(M,E))}\cong\operatorname{Hom}_{\widehat{R}}(E(\widehat{R}/\mathfrak{p}),E(\widehat{R}/\mathfrak{p})^{X_{\mathfrak{p}}})=T_{\mathfrak{p}}.$$

Consequently, we get an isomorphism $\operatorname{Hom}_{\widehat{R}}(\bigoplus_{\operatorname{ht}\mathfrak{p}=i}(\operatorname{Coker} d^{i-2})_{\mathfrak{p}},E)\cong\prod_{\operatorname{ht}\mathfrak{p}=i}T_{\mathfrak{p}}$.

Hence, using [30, Theorem 4.1.15 and Proposition 3.1.2], we conclude that, for all $i\geq 0$, the modules $\operatorname{Hom}_{\widehat{R}}(\bigoplus_{\operatorname{ht}\mathfrak{p}=i}(\operatorname{Coker} d^{i-2})_{\mathfrak{p}},E)$ and $\operatorname{Im} d_i$ are cotorsion. Therefore, by [30, Lemma 5.2.6] and the above arguments, $\operatorname{Hom}_{\widehat{R}}(\bigoplus_{\operatorname{ht}\mathfrak{p}=i}(\operatorname{Coker} d^{i-2})_{\mathfrak{p}},E)$ is a flat cover of $\operatorname{Im} d_i$ for any $i\geq 0$.

So, by [30, Definition 5.2.1], the complex $\operatorname{Hom}_{\widehat{R}}(\mathcal{C}_{\widehat{R}}(\operatorname{Hom}_{\widehat{R}}(M,E)),E)$ is a minimal flat resolution of M, viewed as an $\widehat{R}$-module. Consequently, by the first part of the proof, $\operatorname{Hom}_{\widehat{R}}(\mathcal{C}_{\widehat{R}}(\widehat{R}),M)$ is a minimal flat resolution of M, viewed as an $\widehat{R}$-module.

On the other hand, since $\operatorname{Supp}_{\widehat{R}}(\operatorname{Hom}_{\widehat{R}}(M,E))=\operatorname{Spec}(\widehat{R})$ and $\widehat{R}$ is Cohen–Macaulay, we have $\operatorname{ht}\mathfrak{p}+\dim(\widehat{R}/\mathfrak{p})=\dim\widehat{R}=\dim\operatorname{Hom}_{\widehat{R}}(M,E)$ for all prime ideals $\mathfrak{p}$ of R. Hence $\operatorname{ht}\mathfrak{p}\geq i$ if and only if $\dim(\widehat{R}/\mathfrak{p})\leq\dim\widehat{R}-i$;

so $\mathcal{C}_{\widehat{R}}(\widehat{R}) \cong \mathcal{C}_{\widehat{R}}(\mathcal{D}(\widehat{R}), \widehat{R})$. Note that, by [16, Example 3.8], $\mathcal{C}_{\widehat{R}}(\mathcal{D}(\widehat{R}), \widehat{R}) \cong \mathcal{C}_{\widehat{R}}(\mathcal{V}, \widehat{R})$, where $\mathcal{V} = \{V_i : i \in \mathbb{N}\}$ and

$$V_i = \Big\{(v_1, \ldots, v_i) : \forall j = 1, \ldots, i, \sum_{k=1}^{j} \widehat{R}v_k \not\subseteq \mathfrak{p} \text{ for all } \mathfrak{p} \in \operatorname{Spec}(\widehat{R}) \\ \text{for which } \dim(\widehat{R}/\mathfrak{p}) = \dim \widehat{R} - (i-1)\Big\}.$$

Since $\widehat{R}$ is Cohen–Macaulay, it is easy to see that the $\widehat{R}$-modules $V_i^{-i}\widehat{R}$ and $U_i^{-i}\widehat{R}$ are isomorphic for all $i \geq 1$. Therefore by [16, Examples 3.7 and 3.8] we have the isomorphisms of complexes

$$\mathcal{C}_{\widehat{R}}(\mathcal{D}(\widehat{R}), \widehat{R}) \cong \mathcal{C}_{\widehat{R}}(\widehat{R}) \cong \mathcal{C}_{\widehat{R}}(\mathcal{V}, \widehat{R}) \cong \mathcal{C}_{\widehat{R}}(\mathcal{U}, \widehat{R}).$$

It follows that $\operatorname{Hom}_{\widehat{R}}(\mathcal{C}_{\widehat{R}}(\mathcal{U}, \widehat{R}), M)$ is a minimal flat resolution of M, viewed as an $\widehat{R}$-module.

($\Leftarrow$) Let $\operatorname{Hom}_{\widehat{R}}(\mathcal{C}_{\widehat{R}}(\mathcal{U}, \widehat{R}), M)$ be a minimal flat resolution for M, viewed as an $\widehat{R}$-module. Then $\operatorname{flatdim}_{\widehat{R}} M = \dim \widehat{R} = d$ and, by [29, Theorem 3.3 and 3.10], M is co-Cohen–Macaulay and, for all $i \geq 1$, each element of U_i is a poor M-cosequence. So, by the same argument as in the proof of Lemma 2.7, we get

$$\dim \widehat{R} = d \leq \operatorname{codepth}_{\widehat{R}} M = \operatorname{Kdim}_{\widehat{R}} M = \dim_{\widehat{R}} \operatorname{Hom}_{\widehat{R}}(M, E) \leq \dim \widehat{R}.$$

Hence, by Propositions 3.4 and 3.7, M is a co-Gorenstein R-module. ∎

LEMMA 3.10. *Let R be a local U-ring and M be a non-zero Artinian R-module. If M is co-Gorenstein R-module then a sequence $\underline{x} = \{x_1, \ldots, x_n\}$ of elements of R is an R-sequence if and only if $\underline{x}$ is an M-cosequence.*

Proof. Apply Proposition 3.5, [20, Corollary 4.13] and Lemma 2.4. ∎

PROPOSITION 3.11. *Let R be a local U–ring and M be an Artinian R-module. Then the following conditions are equivalent:*

(i) *M is a co-Gorenstein R-module.*
(ii) *For every (equivalently, for some) sequence $\underline{x} = \{x_1, \ldots, x_n\}$ of elements of $\widehat{R}$ which is maximal with respect to the property of being both an M-cosequence and an $\widehat{R}$-sequence, the $\widehat{R}/\underline{x}\widehat{R}$-module $(0 :_M \underline{x})$ is flat.*

Proof. (i)$\Rightarrow$(ii). In view of Lemma 3.10, there is a sequence $\underline{x} = \{x_1, \ldots, x_n\}$ of elements of $\widehat{R}$ which is maximal with respect to the property of being both an M-cosequence and an $\widehat{R}$-sequence. Hence, by Proposition 3.5 and [20, Theorem 3.11(ix)], $\operatorname{Hom}_{\widehat{R}}(M, E)/\underline{x}\operatorname{Hom}_{\widehat{R}}(M, E)$ is an injective $\widehat{R}/\underline{x}\widehat{R}$-module. Therefore

$$\begin{aligned}\operatorname{Hom}_{\widehat{R}}((0:{}_M\underline{x}),E) &\cong \operatorname{Hom}_{\widehat{R}}(\operatorname{Hom}_{\widehat{R}}(\widehat{R}/\underline{x}\widehat{R},M),E)\\ &\cong \widehat{R}/\underline{x}\widehat{R}\otimes_{\widehat{R}}\operatorname{Hom}_{\widehat{R}}(M,E)\\ &\cong \operatorname{Hom}_{\widehat{R}}(M,E)/\underline{x}\operatorname{Hom}_{\widehat{R}}(M,E)\end{aligned}$$

is an injective $\widehat{R}/\underline{x}\widehat{R}$-module. Hence

$$\begin{aligned}(0:_M\underline{x}) &\cong \operatorname{Hom}_{\widehat{R}}(\operatorname{Hom}_{\widehat{R}}((0:_M\underline{x}),E),E)\\ &\cong \operatorname{Hom}_{\widehat{R}/\underline{x}\widehat{R}}\big(\operatorname{Hom}_{\widehat{R}}((0:_M\underline{x}),E),\operatorname{Hom}_{\widehat{R}}(\widehat{R}/\underline{x}\widehat{R},E)\big)\\ &\cong \operatorname{Hom}_{\widehat{R}/\underline{x}\widehat{R}}\big(\operatorname{Hom}_{\widehat{R}}((0:_M\underline{x}),E),E_{\widehat{R}/\underline{x}\widehat{R}}(\widehat{R}/\mathfrak{m}\widehat{R})\big)\end{aligned}$$

is a flat $\widehat{R}/\underline{x}\widehat{R}$-module.

(ii)⇒(i). Suppose that $\underline{x}=\{x_1,\dots,x_n\}$ is a sequence maximal with respect to the property of being both an $\widehat{R}$-sequence and an M-cosequence, and that $(0:_M x)$ is a flat $\widehat{R}/\underline{x}\widehat{R}$-module. Therefore $\operatorname{Hom}_{\widehat{R}/\underline{x}\widehat{R}}((0:_M x),E_{\widehat{R}/\underline{x}\widehat{R}}(\widehat{R}/\mathfrak{m}\widehat{R}))$ is an injective $\widehat{R}/\underline{x}\widehat{R}$-module. But

$$\begin{aligned}\operatorname{Hom}_{\widehat{R}/\underline{x}\widehat{R}}\big((0:_M x),E_{\widehat{R}/\underline{x}\widehat{R}}(\widehat{R}/\mathfrak{m}\widehat{R})\big)&\cong\operatorname{Hom}_{\widehat{R}/\underline{x}\widehat{R}}\big((0:_M x),\operatorname{Hom}_{\widehat{R}}(\widehat{R}/\underline{x}\widehat{R},E)\big)\\ &\cong \operatorname{Hom}_{\widehat{R}}((0:_M x),E)\cong \operatorname{Hom}_{\widehat{R}}(M,E)/\underline{x}\operatorname{Hom}_{\widehat{R}}(M,E).\end{aligned}$$

Therefore $\operatorname{Hom}_{\widehat{R}}(M,E)/\underline{x}\operatorname{Hom}_{\widehat{R}}(M,E)$ is an injective $\widehat{R}/\underline{x}\widehat{R}$-module. By Lemma 2.4, $\underline{x}$ is a sequence maximal with respect to the property of being both an $\widehat{R}$-sequence and a $\operatorname{Hom}_{\widehat{R}}(M,E)$-sequence. Hence by [20, Theorem 3.11(ix)] and Proposition 3.5, M is a co-Gorenstein R-module. ∎

We give some characterizations of co-Gorenstein modules over a Cohen–Macaulay local U-ring that are dual to the characterization of a Gorenstein module over a complete Cohen–Macaulay local ring established in [21].

Proposition 3.12. *Let $(R,\mathfrak{m})$ be a Noetherian local ring. For each integer $n\geq 1$, the following two statements are equivalent:*

(i) *R is a Cohen–Macaulay ring of Krull dimension n.*
(ii) $\operatorname{flatdim}_R H^n_{\mathfrak{m}}(R)=n$.

Proof. Since $H^n_{\mathfrak{m}}(R)$ is an Artinian R-module, it has natural structure of an $\widehat{R}$-module. Thus $\operatorname{Hom}_{\widehat{R}}(H^n_{\mathfrak{m}}(R),E)$ is a non-zero finitely generated $\widehat{R}$-module.

(i)⇒(ii). By [3, 2.1.8(b), 3.3.8], $\widehat{R}$ is a Cohen–Macaulay local ring with a canonical module $\omega_{\widehat{R}}$. Hence by [3, Theorem 3.5.8], $\operatorname{flatdim}_{\widehat{R}} H^n_{\mathfrak{m}}(R)=\operatorname{inj.dim}_{\widehat{R}}\omega_{\widehat{R}}=n$.

(ii)⇒(i). Assume that $\operatorname{flatdim}_{\widehat{R}} H^n_{\mathfrak{m}}(R)=n$. Then $\operatorname{Hom}_{\widehat{R}}(H^n_{\mathfrak{m}}(R),E)$ is a non-zero finitely generated $\widehat{R}$-module of finite injective dimension; hence $\widehat{R}$ is a Cohen–Macaulay ring. Therefore R is a Cohen–Macaulay ring and so $H^i_{\mathfrak{m}}(R)=0$ for all $i\neq\dim R$. Since $H^n_{\mathfrak{m}}(R)\neq 0$, it follows that $n=\dim R$. ∎

COROLLARY 3.13. *Let $(R, \mathfrak{m})$ be a local U-ring. For each integer $n \geq 1$, the following two conditions are equivalent:*

(i) *R is a Cohen–Macaulay ring of Krull dimension n.*
(ii) *$H^n_{\mathfrak{m}}(R)$ is a co-Gorenstein R-module.*

Moreover, $\pi_n(\mathfrak{m}, H^n_{\mathfrak{m}}(R)) = 1$.

Proof. (ii)⇒(i). Apply Corollary 3.6.

(i)⇒(ii). Let R be a Cohen–Macaulay ring of dimension n. By [28, Proposition 2.6], $H^n_{\mathfrak{m}}(R)$ is a co-Cohen–Macaulay R-module of Krull dimension n. On the other hand, Proposition 3.12 and Lemma 2.5 yield

$$\operatorname{flatdim}_R H^n_{\mathfrak{m}}(R) = n = \operatorname{Kdim}_R H^n_{\mathfrak{m}}(R) = \operatorname{codim}_R H^n_{\mathfrak{m}}(R).$$

Hence, by Proposition 3.4, $H^n_{\mathfrak{m}}(R)$ is a co-Gorenstein R-module.

Now we show $\pi_n(\mathfrak{m}, H^n_{\mathfrak{m}}(R)) = 1$. For all $i \geq 0$ we have $\pi_i(\mathfrak{m}, H^n_{\mathfrak{m}}(R)) = \dim_k \operatorname{Tor}^R_i(k, H^n_{\mathfrak{m}}(R))$. By Remark 2.6, there exist the isomorphisms

$$\begin{aligned}\operatorname{Tor}^R_i(k, H^n_{\mathfrak{m}}(R)) &\cong \operatorname{Hom}_{\widehat{R}}\big(\operatorname{Ext}^i_R(k, \operatorname{Hom}_{\widehat{R}}(H^n_{\mathfrak{m}}(R), E), E)\big)\\ &\cong \operatorname{Hom}_{\widehat{R}}\big(\operatorname{Ext}^i_R(k \otimes_{\widehat{R}} \widehat{R}, \operatorname{Hom}_{\widehat{R}}(H^n_{\mathfrak{m}}(R), E)), E\big)\\ &\cong \operatorname{Hom}_{\widehat{R}}\big(\operatorname{Ext}^i_{\widehat{R}}(k, \operatorname{Hom}_R(\widehat{R}, \operatorname{Hom}_{\widehat{R}}(H^n_{\mathfrak{m}}(R), E))), E\big)\\ &\cong \operatorname{Hom}_{\widehat{R}}\big(\operatorname{Ext}^i_{\widehat{R}}(k, \operatorname{Hom}_{\widehat{R}}(H^n_{\mathfrak{m}}(R), E)), E\big).\end{aligned}$$

Since $\operatorname{Ext}^i_{\widehat{R}}(k, \operatorname{Hom}_{\widehat{R}}(H^i_{\mathfrak{m}}(R), E))$ is a finitely generated $\widehat{R}$-module and is a vector space over $\widehat{R}/\mathfrak{m}\widehat{R}$, we see that for a positive integer r, there is an isomorphism

$$\operatorname{Ext}^i_{\widehat{R}}(k, \operatorname{Hom}_{\widehat{R}}(H^i_{\mathfrak{m}}(R), E)) \cong k^r,$$

and hence

$$\begin{aligned}&\operatorname{Hom}_{\widehat{R}}\big(\operatorname{Ext}^i_{\widehat{R}}(k, \operatorname{Hom}_{\widehat{R}}(H^i_{\mathfrak{m}}(R), E)), E\big)\\ &\qquad\cong \operatorname{Hom}_{\widehat{R}}\Big(\bigoplus_{t=1}^{r} k, E\Big) \cong \bigoplus_{t=1}^{r} \operatorname{Hom}_{\widehat{R}}(k, E) \cong k^r.\end{aligned}$$

Therefore $\dim_k \operatorname{Tor}^R_i(k, H^n_{\mathfrak{m}}(R)) = \dim_k \operatorname{Ext}^i_{\widehat{R}}(k, \operatorname{Hom}_{\widehat{R}}(H^i_{\mathfrak{m}}(R), E))$.

Consequently, for all $i \geq 0$, $\pi_i(\mathfrak{m}, H^n_{\mathfrak{m}}(R)) = \mu^i(\mathfrak{m}\widehat{R}, \operatorname{Hom}_{\widehat{R}}(H^n_{\mathfrak{m}}(R), E))$, and by [21, Corollary 1.6], we get

$$\pi_i(\mathfrak{m}, H^i_{\mathfrak{m}}(R)) = \begin{cases} 1, & i = n,\\ 0, & i \neq n. \ \blacksquare \end{cases}$$

COROLLARY 3.14. *Let $(R, \mathfrak{m})$ be a Cohen–Macaulay local U-ring with $\dim R = n$. Then any co-Gorenstein R-module is M isomorphic to the direct sum of a finite number of copies of $H^i_{\mathfrak{m}}(R)$; more precisely, $M \cong$*

$\bigoplus \pi_n(\mathfrak{m}, M)H^n_{\mathfrak{m}}(R)$. Furthermore, if $h \neq t > 0$, then $\bigoplus tH^n_{\mathfrak{m}}(R)$ and $\bigoplus hH^n_{\mathfrak{m}}(R)$ are non-isomorphic co-Gorenstein R-modules.

Proof. Assume that M is a co-Gorenstein R-module. By Proposition 3.5 and [21, Corollary 2.7], we have

$$\operatorname{Hom}_{\widehat{R}}(M, E) \cong \bigoplus \mu^n(\mathfrak{m}\widehat{R}, \operatorname{Hom}_{\widehat{R}}(M, E)) \operatorname{Hom}_{\widehat{R}}(H^n_{\mathfrak{m}}(R), E).$$

Therefore

$$M \cong \bigoplus \mu^n(\mathfrak{m}\widehat{R}, \operatorname{Hom}_{\widehat{R}}(M, E))H^n_{\mathfrak{m}}(R).$$

Since, by an argument used in the proof of Corollary 3.13, we get

$$\mu^n(\mathfrak{m}\widehat{R}, \operatorname{Hom}_{\widehat{R}}(M, E)) = \pi_n(\mathfrak{m}, M),$$

it follows that $M \cong \bigoplus \pi_n(\mathfrak{m}, M)H^n_{\mathfrak{m}}(R)$.

Now suppose that h, t are two positive integers such that $\bigoplus hH^n_{\mathfrak{m}}(R) \cong \bigoplus tH^n_{\mathfrak{m}}(R)$. Then $h = t$ by [21, 1.6, 2.1, 2.7] and the above arguments. ■

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