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REPRESENTATION NUMBERS OF FIVE SEXTENARY QUADRATIC FORMS

BY

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Abstract. For nonnegative integers a, b, c and positive integer n, let $N(a,b,c;n)$ denote the number of representations of n by the form

$$\sum_{i=1}^{a}(x_i^2+x_iy_i+y_i^2)+2\sum_{j=1}^{b}(u_j^2+u_jv_j+v_j^2)+4\sum_{k=1}^{c}(r_k^2+r_ks_k+s_k^2).$$

Explicit formulas for $N(a,b,c;n)$ for some small values were determined by Alaca, Alaca and Williams, by Chan and Cooper, by Köklüce, and by Lomadze. We establish formulas for $N(2,1,0;n)$, $N(2,0,1;n)$, $N(1,2,0;n)$, $N(1,0,2;n)$ and $N(1,1,1;n)$ by employing the (p,k)-parametrization of three 2-dimensional theta functions due to Alaca, Alaca and Williams.

1. Introduction. Let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$ and $\mathbb{C}$ denote the sets of positive integers, nonnegative integers, integers and complex numbers, respectively. For $a,b,c\in\mathbb{N}_0$ and $n\in\mathbb{N}$, let $N(a,b,c;n)$ denote the number of representations of n by the form

$$\sum_{i=1}^{a}(x_i^2+x_iy_i+y_i^2)+2\sum_{j=1}^{b}(u_j^2+u_jv_j+v_j^2)+4\sum_{k=1}^{c}(r_k^2+r_ks_k+s_k^2).$$

For some small values of a,b,c, explicit formulas for $N(a,b,c;n)$ were established, for example, by Alaca, Alaca and Williams [AAW1], Chan and Cooper [CC], Köklüce [K1, K2, K3], Lomadze [L], Xia [X] and Xia and Yao [XY]. In particular, Lomadze [L] proved that for $n\in\mathbb{N}$,

$$R(3,0,0;n)=27G_3(n)-9H_3(n), \tag{1.1}$$

where $G_3(n)$ and $H_3(n)$ are defined by

$$G_3(n)=\sum_{d\in\mathbb{N},\,d\mid n}\left(\frac{-3}{n/d}\right)d^2, \tag{1.2}$$

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$$H_3(n) = \sum_{d\in\mathbb{N},\, d|n} \left(\frac{-3}{d}\right) d^2. \tag{1.3}$$

Let $N \in \mathbb{N}$ and $\gcd(6, N) = 1$. Set $N = \prod_{p|N} p^{\alpha_p}$ be the prime factorization of N. Alaca, Alaca and Williams [AAW2] proved that

$$G_3(N) = N^2 \prod_{P|N} \frac{1 - \left(\frac{-3}{p}\right)^{\alpha_p+1} p^{-2\alpha_p-2}}{1 - \left(\frac{-3}{p}\right)p^{-2}}, \tag{1.4}$$

where $\left(\frac{-3}{k}\right)$ $(k \in \mathbb{N})$ is the Legendre–Jacobi–Kronecker symbol for discriminant -3, that is,

$$\left(\frac{-3}{k}\right) = \begin{cases} 1 & \text{if } k \equiv 1 \pmod 3, \\ -1 & \text{if } k \equiv 2 \pmod 3, \\ 0 & \text{if } k \equiv 0 \pmod 3). \end{cases} \tag{1.5}$$

The aim of this paper is to determine explicit formulas for $N(2,1,0;n)$, $N(2,0,1;n)$, $N(1,2,0;n)$, $N(1,0,2;n)$ and $N(1,1,1;n)$. The main results can be stated as follows.

THEOREM 1.1. *Set $n = 2^\alpha 3^\beta N \in \mathbb{N}$, where $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and* $\gcd(N, 6) = 1$. *Let $N = \prod_{p|N} p^{\alpha_p}$ be the prime factorization of N and let $G_3(N)$ be defined by* (1.4). *Then*

$$N(2,1,0;n) = \frac{3}{5}(2^{2\alpha+3} - 3(-1)^\alpha)\left(3^{2\beta+1} + (-1)^\alpha\left(\frac{-3}{N}\right)\right)G_3(N), \tag{1.6}$$

(1.7)

$$N(2,0,1;n) = \begin{cases} \frac{3}{2}\left(3^{2\beta+1} - \left(\frac{-3}{N}\right)\right)G_3(N) + 9f(n) & \text{if } 2 \nmid n, \\ \frac{9}{5}(3\cdot 2^{2\alpha-1} + (-1)^\alpha)\left(3^{2\beta+1} - (-1)^\alpha\left(\frac{-3}{N}\right)\right)G_3(N) & \text{if } 2 \mid n, \end{cases}$$

(1.8)

$$N(1,2,0;n) = \frac{3}{5}(2^{2\alpha+1} + 3(-1)^\alpha)\left(3^{2\beta+1} - (-1)^\alpha\left(\frac{-3}{N}\right)\right)G_3(N),$$

(1.9)

$$N(1,0,2;n) = \begin{cases} \frac{3}{4}\left(3^{2\beta+1} - \left(\frac{-3}{N}\right)\right)G_3(N) + \frac{9}{2}f(n) & \text{if } 2 \nmid n, \\ \frac{9}{5}(2^{2\alpha-2} + (-1)^\alpha)\left(3^{2\beta+1} - (-1)^\alpha\left(\frac{-3}{N}\right)\right)G_3(N) & \text{if } 2 \mid n, \end{cases}$$

$$N(1,1,1;n) = T(\alpha)\left(3^{2\beta+1} + (-1)^\alpha\left(\frac{-3}{N}\right)\right)G_3(N), \tag{1.10}$$

where

$$T(\alpha) = \tfrac{3}{20}(2^{2\alpha} - 6(-1)^{\alpha})(1 + (-1)^{2^{\alpha}}) + \tfrac{3}{4}(1 - (-1)^{2^{\alpha}}), \tag{1.11}$$

$$f(n) = \frac{1}{2} \sum_{\substack{(x_1,x_2)\in\mathbb{Z}^2 \\ x_1^2+3x_2^2=n}} (x_1^2 - 3x_2^2). \tag{1.12}$$

From (1.7) and (1.9), we obtain the following corollary:

COROLLARY 1.2. *If $n \geq 1$ is an odd integer, then*

$$N(2,0,1;n) = 2N(1,0,2;n). \tag{1.13}$$

2. The (p,k)-parametrization of 2-dimensional theta functions. The aim of this section is to recall the (p,k)-parametrization of three 2-dimensional theta functions due to Alaca, Alaca and Williams [AAW1].

Jonathan and Peter Borwein [BB] introduced the following three 2-dimensional theta functions:

$$a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}, \tag{2.1}$$

$$b(q) := \sum_{m,n=-\infty}^{\infty} \omega^{m-n} q^{m^2+mn+n^2}, \tag{2.2}$$

$$c(q) := \sum_{m,n=-\infty}^{\infty} q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2}, \tag{2.3}$$

where $\omega = e^{2\pi i/3}$. Alaca, Alaca and Williams [AAW1] defined

$$p = p(q) := \frac{\varphi^2(q) - \varphi^2(q^3)}{2\varphi^2(q^3)}, \tag{2.4}$$

$$k = k(q) := \frac{\varphi^3(q^3)}{\varphi(q)}, \tag{2.5}$$

where

$$\varphi(q) := \sum_{n\in\mathbb{Z}} q^{n^2}. \tag{2.6}$$

Alaca, Alaca and Williams [AAW1] also established the parametric representations of $a(q^m)$, $b(q^m)$ and $c(q^m)$ $(m \in \{1,2,4\})$ in terms of p and k. Theorems 1, 2, and 4 in [AAW1] state:

$$a(q) = (1 + 4p + p^2)k, \tag{2.7}$$

$$a(q^2) = (1 + p + p^2)k, \tag{2.8}$$

$$a(q^4) = (1 + p - p^2/2)k, \tag{2.9}$$

$$b(q) = 2^{-1/3}(1-p)^{4/3}(1+2p)^{1/3}(2+p)^{1/3}k, \tag{2.10}$$

$$b(q^2) = 2^{-2/3}(1-p)^{2/3}(1+2p)^{2/3}(2+p)^{2/3}k, \tag{2.11}$$

$$b(q^4) = 2^{-4/3}(1-p)^{1/3}(1+2p)^{1/3}(2+p)^{4/3}k, \tag{2.12}$$

$$c(q) = 2^{-1/3}3p^{1/3}(1+p)^{4/3}k, \tag{2.13}$$

$$c(q^2) = 2^{-2/3}3p^{2/3}(1+p)^{2/3}k, \tag{2.14}$$

$$c(q^4) = 2^{-4/3}3p^{4/3}(1+p)^{1/3}k. \tag{2.15}$$

3. Some identities involving $a(q)$, $b(q)$ and $c(q)$. In this section, we establish five identities involving $a(q)$, $b(q)$ and $c(q)$ by employing the (p,k)-parametrization of $a(q)$, $b(q)$ and $c(q)$. Those identities are used to prove the main results of this paper.

THEOREM 3.1. *We have*

$$a^2(q)a(q^2) = -\tfrac{1}{3}b^3(q) + \tfrac{4}{3}b^3(q^2) + \tfrac{1}{3}c^3(q) + \tfrac{4}{3}c^3(q^2), \tag{3.1}$$

$$a^2(q)a(q^4) = \tfrac{1}{6}b^3(q) - \tfrac{1}{2}b^3(q^2) + \tfrac{4}{3}b^3(q^4) + \tfrac{1}{6}c^3(q) + \tfrac{1}{2}c^3(q^2) + \tfrac{4}{3}c^3(q^4) + 9f(q), \tag{3.2}$$

$$a(q)a^2(q^2) = \tfrac{1}{3}b^3(q) + \tfrac{2}{3}b^3(q^2) + \tfrac{1}{3}c^3(q) - \tfrac{2}{3}c^3(q^2), \tag{3.3}$$

$$a(q)a^2(q^4) = \tfrac{1}{12}b^3(q) + \tfrac{1}{4}b^3(q^2) + \tfrac{2}{3}b^3(q^4) + \tfrac{1}{12}c^3(q) - \tfrac{1}{4}c^3(q^2) + \tfrac{2}{3}c^3(q^4) + \tfrac{9}{2}f(q), \tag{3.4}$$

$$a(q)a(q^2)a(q^4) = -\tfrac{1}{6}b^3(q) - \tfrac{1}{6}b^3(q^2) + \tfrac{4}{3}b^3(q^4) + \tfrac{1}{6}c^3(q) - \tfrac{1}{6}c^3(q^2) - \tfrac{4}{3}c^3(q^4), \tag{3.5}$$

where $f(q)$ is defined by

$$\sum_{n=0}^{\infty} f(n)q^n := f(q) = q\prod_{n=1}^{\infty}(1-q^{2n})^3(1-q^{6n})^3. \tag{3.6}$$

Proof. We just prove (3.2). The rest can be proved similarly. Alaca, Alaca and Williams [AAW2] established the representations of $q^{j/24}\prod_{n=1}^{\infty}(1-q^{nj})$ ($j \in \{2,6\}$) in terms of p and k. It follows from [AAW2, (2.11) and (2.14)] that

$$q^{1/12}\prod_{n=1}^{\infty}(1-q^{2n}) = 2^{-1/3}p^{1/12}(1-p)^{1/4}(1+p)^{1/12}(1+2p)^{1/4}(2+p)^{1/4}k^{1/2}, \tag{3.7}$$

$$q^{1/4}\prod_{n=1}^{\infty}(1-q^{6n}) = 2^{-1/3}p^{1/4}(1-p)^{1/12}(1+p)^{1/4}(1+2p)^{1/12}(2+p)^{1/12}k^{1/2}. \tag{3.8}$$

By (3.6)–(3),

$$f(q) = \frac{p(1-p)(1+p)(1+2p)(2+p)}{4}k^3. \tag{3.9}$$

In view of (2.10)–(2.15) and (3.9), it is easy to check that

$$\begin{aligned}\tfrac{1}{6}b^3(q) - \tfrac{1}{2}b^3(q^2) + \tfrac{4}{3}b^3(q^4) + \tfrac{1}{6}c^3(q) + \tfrac{1}{2}c^3(q^2) + \tfrac{4}{3}c^3(q^4) + 9f(q) \\ = \big(1 + 9p + \tfrac{51}{2}p^2 + 22p^3 - 3p^5 - \tfrac{1}{2}p^6\big)k^3.\end{aligned} \tag{3.10}$$

From (2.7) and (2.9), we have

$$a^2(q)a(q^4) = \big(1 + 9p + \tfrac{51}{2}p^2 + 22p^3 - 3p^5 - \tfrac{1}{2}p^6\big)k^3. \tag{3.11}$$

Identity (3.2) follows from (3.10) and (3.11). ■

4. Proof of Theorem 1.1. In this section, we present a proof of Theorem 1.1 by employing Theorem 3.1. We deduce (1.7) from (3.2). The rest can be proved similarly.

By the definition of $N(a,b,c;n)$ and (2.1), it is easy to see that the generating function of $N(a,b,c;n)$ is

$$1 + \sum_{n=1}^{\infty} N(a,b,c;n)q^n = a^a(q)a^b(q^2)a^c(q^4). \tag{4.1}$$

Setting $a = 2$, $b = 0$ and $c = 1$ in (4.1) and using (3.2), we obtain

$$\begin{aligned}1 + \sum_{n=1}^{\infty} N(2,0,1;n)q^n = \tfrac{1}{6}b^3(q) - \tfrac{1}{2}b^3(q^2) + \tfrac{4}{3}b^3(q^4) + \tfrac{1}{6}c^3(q) \\ + \tfrac{1}{2}c^3(q^2) + \tfrac{4}{3}c^3(q^4) + 9f(q).\end{aligned} \tag{4.2}$$

Alaca, Alaca and Williams [AAW2] proved that

$$27\sum_{n=1}^{\infty} G_3(n)q^n = c^3(q), \tag{4.3}$$

$$1 - 9\sum_{n=1}^{\infty} H_3(n)q^n = b^3(q), \tag{4.4}$$

where $G_3(n)$ and $H_3(n)$ are defined by (1.2) and (1.3), respectively. By means of (3.6) and (4.2)–(4.4),

$$\begin{aligned}1+\sum_{n=1}^{\infty} N(2,0,1;n)q^n &= \frac{1}{6}\Big(1-9\sum_{n=1}^{\infty} H_3(n)q^n\Big) - \frac{1}{2}\Big(1-9\sum_{n=1}^{\infty} H_3(n)q^{2n}\Big) \\ &+ \frac{4}{3}\Big(1 - 9\sum_{n=1}^{\infty} H_3(n)q^{4n}\Big) + \frac{9}{2}\sum_{n=1}^{\infty} G_3(n)q^n \\ &+ \frac{27}{2}\sum_{n=1}^{\infty} G_3(n)q^{2n} + 36\sum_{n=1}^{\infty} G_3(n)q^{4n} + 9\sum_{n=1}^{\infty} f(n)q^n\end{aligned} \tag{4.5}$$

$$= 1 - \frac{3}{2}\sum_{n=1}^{\infty} H_3(n)q^n + \frac{9}{2}\sum_{n=1}^{\infty} H_3(n)q^{2n} - 12\sum_{n=1}^{\infty} H_3(n)q^{4n}$$
$$+ \frac{9}{2}\sum_{n=1}^{\infty} G_3(n)q^n + \frac{27}{2}\sum_{n=1}^{\infty} G_3(n)q^{2n} + 36\sum_{n=1}^{\infty} G_3(n)q^{4n} + 9\sum_{n=1}^{\infty} f(n)q^n,$$

where $f(n)$ is defined by (3.6). Equating the coefficients of q^n on both sides of (4.5), we find that for $n \in \mathbb{N}$,

$$N(2,0,1;n) = -\tfrac{3}{2}H_3(n) + \tfrac{9}{2}H_3(n/2) - 12H_3(n/4) + \tfrac{9}{2}G_3(n) + \tfrac{27}{2}G_3(n/2) + 36G_3(n/4) + 9f(n). \tag{4.6}$$

Set $n = 2^\alpha 3^\beta N \in \mathbb{N}$, where $\alpha, \beta \in \mathbb{N}_0$, $N \in \mathbb{N}$ and $\gcd(N,6) = 1$. It is easy to show that

$$G_3(n) = \tfrac{1}{5}(2^{2\alpha+2} + (-1)^\alpha)3^{2\beta}G_3(N), \tag{4.7}$$

$$H_3(n) = \tfrac{1}{5}(-1)^\alpha(2^{2\alpha+2} + (-1)^\alpha)3^{2\beta}H_3(N), \tag{4.8}$$

$$H_3(N) = \left(\frac{-3}{N}\right)G_3(N). \tag{4.9}$$

Employing (4.6)–(4.9), we deduce that if $\alpha = 0$, then $2 \nmid n$ and

$$\begin{aligned} N(2,0,1;n) &= -\tfrac{3}{2}H_3(n) + \tfrac{9}{2}G_3(n) + 9f(n) \\ &= \frac{3}{2}\left(3^{2\beta+1} - \left(\frac{-3}{N}\right)\right)G_3(N) + 9f(n). \end{aligned} \tag{4.10}$$

Recently, Chan, Cooper and Liaw [CCL] have proved that

$$f(n) = \frac{1}{2}\sum_{\substack{(x_1,x_2)\in\mathbb{Z}^2 \\ x_1^2+3x_2^2=n}} (x_1^2 - 3x_2^2); \tag{4.11}$$

see also Williams [W]. Combining (4.10) and (4.11), we find that if $n \in \mathbb{N}$ and $2 \nmid n$, then

$$N(2,0,1;n) = \frac{3}{2}\left(3^{2\beta+1} - \left(\frac{-3}{N}\right)\right)G_3(N) + \frac{9}{2}\sum_{\substack{(x_1,x_2)\in\mathbb{Z}^2 \\ x_1^2+3x_2^2=n}} (x_1^2 - 3x_2^2). \tag{4.12}$$

By (3.6), it is easy to see that if $2 \mid n$, then

$$f(n) = 0. \tag{4.13}$$

Define

$$g(\alpha) := \begin{cases} 0 & \text{if } \alpha = 1, \\ 1 & \text{if } \alpha \geq 2. \end{cases} \tag{4.14}$$

Using (4.6)–(4.9), (4.13) and (4.14), we deduce that if $\alpha \geq 1$, then $2 \mid n$ and

$$\begin{aligned} N(2,0,2;n) &= \tfrac{9}{2}G_3(n) + \tfrac{27}{2}G_3(n/2) + 36G_3(n/4) \\ &\quad - \tfrac{3}{2}H_3(n) + \tfrac{9}{2}H_3(n/2) - 12H_3(n/4) \\ &= \tfrac{9}{10}(2^{2\alpha+2} + (-1)^\alpha)3^{2\beta}G_3(N) \\ &\quad - \tfrac{3}{10}(-1)^\alpha(2^{2\alpha+2} + (-1)^\alpha)\left(\frac{-3}{N}\right)G_3(N) \\ &\quad + \tfrac{27}{10}(2^{2\alpha} - (-1)^\alpha)3^{2\beta}G_3(N) \\ &\quad - \tfrac{9}{10}(-1)^\alpha(2^{2\alpha} - (-1)^\alpha)\left(\frac{-3}{N}\right)G_3(N) \\ &\quad + \tfrac{36}{5}g(\alpha)(2^{2\alpha-2} + (-1)^\alpha)3^{2\beta}G_3(N) \\ &\quad - \tfrac{12}{5}g(\alpha)(-1)^\alpha(2^{2\alpha-2} + (-1)^\alpha)\left(\frac{-3}{N}\right)G_3(N) \\ &= S(\alpha)\left(3^{2\beta+1} - (-1)^\alpha\left(\frac{-3}{N}\right)\right)G_3(N), \end{aligned} \tag{4.15}$$

where

$$\begin{aligned} S(\alpha) &= \tfrac{3}{10}(2^{2\alpha+2} + (-1)^\alpha) + \tfrac{9}{10}(2^{2\alpha} - (-1)^\alpha) \\ &\quad + \tfrac{12}{5}g(\alpha)(2^{2\alpha-2} + (-1)^\alpha). \end{aligned} \tag{4.16}$$

Therefore, by (4.14) and (4.16),

$$\begin{aligned} S(\alpha) &= \begin{cases} 9 & \text{if } \alpha = 1, \\ \tfrac{9}{5}(3 \cdot 2^{2\alpha-1} + (-1)^\alpha) & \text{if } \alpha \geq 2, \end{cases} \\ &= \tfrac{9}{5}(3 \cdot 2^{2\alpha-1} + (-1)^\alpha). \end{aligned} \tag{4.17}$$

It follows from (4.15) and (4.17) that if $1 \mid n$, then

$$N(a,b,c;n) = \tfrac{9}{5}(3 \cdot 2^{2\alpha-1} + (-1)^\alpha)\left(3^{2\beta+1} - (-1)^\alpha\left(\frac{-3}{N}\right)\right)G_3(N). \tag{4.18}$$

Combining (4.12) and (4.18), we arrive at (1.7). ∎

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REFERENCES

[AAW1] A. Alaca, Ş. Alaca and K. S. Williams, *On the two-dimensional theta functions of the Borweins*, Acta Arith. 124 (2006), 177–195.

[AAW2] A. Alaca, Ş. Alaca and K. S. Williams, *Some new theta function identities with applications to sextenary quadratic forms*, J. Combin. Number Theory 1 (2009), 91–100.

[BB] J. M. Borwein and P. B. Borwein, *A cubic counterpart of Jacobi's identity and the AGM*, Trans. Amer. Math. Soc. 323 (1991), 691–701.

[CC] H. H. Chan and S. Cooper, *Powers of theta functions*, Pacific J. Math. 235 (2008), 1–14.

[CCL] H. H. Chan, S. Cooper and W.-C. Liaw, *On* $\eta^3(a\tau)\eta^3(b\tau)$ *with* $a+b=8$, J. Austral. Math. Soc. 84 (2008), 301–313.

[K1] B. Köklüce, *The representation numbers of three octonary quadratic forms*, Int. J. Number Theory 9 (2013), 505–516.

[K2] B. Köklüce, *The representation numbers of certain octonary quadratic forms*, Int. J. Number Theory 9 (2013), 1125–1139.

[K3] B. Köklüce, *Representation numbers of two octonary quadratic forms*, Int. J. Number Theory 9 (2013), 1641–1648.

[L] G. A. Lomadze, *Representation of numbers by sums of the quadratic forms* $x_1^2+x_1x_2+x_2^2$, Acta Arith. 54 (1989), 9–36 (in Russian).

[W] K. S. Williams, *Quadratic forms and a product-to-sum formula*, Acta Arith. 158 (2013), 79–97.

[X] E. X. W. Xia, *On the number of representations of a positive integer by certain quadratic forms*, Colloq. Math. 135 (2014), 139–145.

[XY] E. X. W. Xia and O. X. M. Yao, *On the representations of integers by certain quadratic forms*, Int. J. Number Theory 9 (2013), 189–204.

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