COLLOQUIUM MATHEMATICUM

VOL. 138 2015 NO. 2

WEAK PRECOMPACTNESS AND PROPERTY (V^*) IN SPACES OF COMPACT OPERATORS

BY

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Abstract. We give sufficient conditions for subsets of compact operators to be weakly precompact. Let $L_{w^*}(E^*, F)$ (resp. $K_{w^*}(E^*, F)$) denote the set of all w^*-w continuous (resp. w^*-w continuous compact) operators from E^* to F.

We prove that if H is a subset of $K_{w^*}(E^*, F)$ such that $H(x^*)$ is relatively weakly compact for each $x^* \in E^*$ and $H^*(y^*)$ is weakly precompact for each $y^* \in F^*$, then H is weakly precompact. We also prove the following results:

If E has property (wV^*) and F has property (V^*), then $K_{w^*}(E^*, F)$ has property (wV^*).

Suppose that $L_{w^*}(E^*, F) = K_{w^*}(E^*, F)$. Then $K_{w^*}(E^*, F)$ has property (V^*) if and only if E and F have property (V^*).

1. Introduction. In this paper weak precompactness and relative weak compactness in spaces of compact operators are used to investigate whether the spaces $K_{w^*}(E^*, F)$ and $E \otimes_\epsilon F$ have property (wV^*) (resp. (V^*)) when E and F have property (wV^*) (resp. (V^*)).

Our results are organized as follows. First we give sufficient conditions for subsets of compact operators to be weakly precompact and relatively weakly compact. These results are used to study whether the spaces $K_{w^*}(E^*, F)$ and $E \otimes_\epsilon F$ have property (wV^*) (resp. (V^*)), when E and F have the respective property. Next we give some applications to the spaces $\ell_1 \otimes_\epsilon E$, $(N_1(E, F))^*$, and $L_1(\mu) \otimes_\epsilon E$. Finally, we prove that in some cases, if $K_{w^*}(E^*, F)$ has property (V^*), then $L_{w^*}(E^*, F) = K_{w^*}(E^*, F)$. Our results generalize some results from [17] and [25].

2. Definitions and notation. Throughout this paper, X, Y, E, and F will denote Banach spaces. The unit ball of X will be denoted by B_X, and X^* will stand for the continuous linear dual of X. The closed linear span of a sequence (x_n) in X will be written $[x_n]$. The space X *embeds* in Y (in symbols $X \hookrightarrow Y$) if X is isomorphic to a closed subspace of Y. An *operator* $T : X \to Y$ will be a continuous and linear function. The operator $T : X \to Y$

2010 *Mathematics Subject Classification*: Primary 46B20, 46B28; Secondary 28B05.
Key words and phrases: property (V^*), property (wV^*), spaces of operators, compact operators, w^*-w compact operators, weakly precompact sets.

DOI: 10.4064/cm138-2-10

is *completely continuous* (or Dunford–Pettis) if it maps weakly convergent sequences to convergent sequences. The set of all operators, weakly compact operators, and compact operators from X to Y will be denoted by $L(X,Y)$, $W(X,Y)$, and $K(X,Y)$. The set of all w^*-w continuous (resp. w^*-w continuous compact) operators from X^* to Y will be denoted by $L_{w^*}(X^*,Y)$ (resp. $K_{w^*}(X^*,Y)$). The injective (resp. projective) tensor product of two Banach spaces X and Y will be denoted by $X \otimes_\epsilon Y$ (resp. $X \otimes_\pi Y$). The space $X \otimes_\epsilon Y$ can be embedded into the space $K_{w^*}(X^*,Y)$, by identifying $x \otimes y$ with the rank one operator $x^* \mapsto \langle x^*, x\rangle\, y$, and $X \otimes_\epsilon Y = K_{w^*}(X^*,Y)$ if X or Y has the approximation property. We recall the following well-known isometries [34, p. 60]:

(1) $L_{w^*}(E^*,F) \simeq L_{w^*}(F^*,E)$ and $K_{w^*}(E^*,F) \simeq K_{w^*}(F^*,E)$ $(T \mapsto T^*)$.
(2) $W(E,F) \simeq L_{w^*}(E^{**},F)$ and $K(E,F) \simeq K_{w^*}(E^{**},F)$ $(T \mapsto T^{**})$.

The reader should consult [14] and [35] for further details of these tensor products.

For $1 < p < \infty$, let p' denote the conjugate of p.

A Banach space X has the *approximation property* if for each norm compact subset M of X and $\epsilon > 0$, there is a finite rank operator $T : X \to X$ such that $\|T(x) - x\| < \epsilon$ for all $x \in M$. If in addition T can be found with $\|T\| \le 1$, then X is said to have the *metric approximation property.* $C(K)$ spaces, c_0, ℓ_p, $1 \le p < \infty$, $L_p(\mu)$ (μ any measure), $1 \le p < \infty$, and their duals have the metric approximation property [14], [35].

A subset S of X is said to be *weakly precompact* provided that every bounded sequence from S has a weakly Cauchy subsequence [3]. The operator $T : X \to Y$ is weakly precompact (or almost weakly compact) if $T(B_X)$ is weakly precompact.

A series $\sum x_n$ in X is said to be *weakly unconditionally convergent* (wuc) if for every $x^* \in X^*$, the series $\sum |x^*(x_n)|$ is convergent. Equivalently, $\sum x_n$ is wuc if $\{\sum_{n\in A} x_n : A \subseteq \mathbb{N},\ A \text{ finite}\}$ is bounded. An operator $T : X \to Y$ is *unconditionally converging* if it maps weakly unconditionally convergent series to unconditionally convergent ones.

3. Weakly precompact subsets of spaces of compact operators. We begin by giving sufficient conditions for a subset of $K_{w^*}(E^*,F)$ to be weakly precompact and relatively weakly compact. Let *wot* denote the weak operator topology on $L(E,F)$: we have $(T_n) \to T$ (*wot*) provided that $\langle T_n(x), y^*\rangle \mapsto \langle T(x), y^*\rangle$ for all $x \in E$ and $y^* \in F^*$.

If H is a subset of $L_{w^*}(E^*,F)$, $x^* \in E^*$, and $y^* \in F^*$, then let $H(x^*) = \{T(x^*) : T \in H\}$ and $H^*(y^*) = \{T^*(y^*) : T \in H\}$. We set off with the following lemma, similar to results in [24].

LEMMA 3.1 ([21, Lemma 4.7]). *Let (T_n) be a sequence of w^*-w continuous compact operators such that $(T_n) \to T$ (wot), where T is w^*-w continuous and compact. Then $(T_n) \to T$ weakly.*

Theorem 1.3 of [25] shows that if H is a subset of $E \otimes_\epsilon F$ satisfying the hypotheses (i) and (ii) in the next theorem, then H is weakly precompact. The following theorem generalizes this result.

THEOREM 3.2. *Let H be a subset of $K_{w^*}(E^*, F)$ such that either*

(i) *$H(x^*)$ is relatively weakly compact for each $x^* \in E^*$, and*
(ii) *$H^*(y^*)$ is weakly precompact for each $y^* \in F^*$,*

or

(i)′ *$H(x^*)$ is weakly precompact for each $x^* \in E^*$, and*
(ii)′ *$H^*(y^*)$ is relatively weakly compact for each $y^* \in F^*$.*

Then H is weakly precompact.

Proof. Suppose H satisfies (i) and (ii). Let (T_n) be a sequence in H, and let S be the closed linear span of $\{T_n(x^*) : x^* \in E^*, n \in \mathbb{N}\}$. The compactness of each T_n implies that S is a separable subspace of F. Therefore (B_{S^*}, w^*) is a compact metric space. Let $A = (y_n^*)$ be a w^*-dense sequence in S^*. By hypotheses, $\{T_n^*(y_k^*) : n \in \mathbb{N}\}$ is weakly precompact for each k. By diagonalization, we may assume that (T_{n_i}) is a subsequence of (T_n) so that $(T_{n_i}^*(y_k^*))_{i=1}^\infty$ is weakly Cauchy for each k. Without loss of generality, we assume that $(T_n^*(y^*))$ is weakly Cauchy for each $y^* \in A$.

For fixed $x^* \in E^*$, the sequence $(T_n(x^*))$ must have a weakly convergent subsequence. Suppose that y_1 and y_2 are weak sequential cluster points of this sequence. Then $y_1, y_2 \in S$. Suppose that $(T_{k(n)}(x^*)) \xrightarrow{w} y_1$, $(T_{p(n)}(x^*)) \xrightarrow{w} y_2$, and $y^* \in A$. Now

$$\begin{aligned} \langle y_1, y^* \rangle &= \lim_n \langle T_{k(n)}(x^*), y^* \rangle = \lim_n \langle x^*, T_{k(n)}^*(y^*) \rangle = \lim_n \langle x^*, T_n^*(y^*) \rangle \\ &= \lim_n \langle x^*, T_{p(n)}^*(y^*) \rangle = \lim_n \langle T_{p(n)}(x^*), y^* \rangle = \langle y_2, y^* \rangle, \end{aligned}$$

and $y_1 = y_2$ since A is w^*-dense in S^*. Therefore for all $x^* \in E^*$, $(T_n(x^*))$ is weakly convergent in S, and so in F. Thus (T_n) is Cauchy in the *wot* of $K_{w^*}(E^*, F)$. Hence for any two subsequences (A_n) and (B_n) of (T_n), $(A_n - B_n) \to 0$ (*wot*). By Lemma 3.1, $(A_n - B_n) \to 0$ weakly; thus (T_n) is weakly Cauchy in $K_{w^*}(E^*, F)$.

Now suppose that H satisfies (i)′ and (ii)′. Consider the subset $H^* = \{T^* : T \in H\}$ of $K_{w^*}(F^*, E)$. The previous argument shows that the set H^* is weakly precompact. Let (T_n) be a sequence in H. Without loss of generality, we can assume that (T_n^*) is weakly Cauchy. Hence $\lim_n \langle T_n^*(y^*), x^* \rangle = \lim_n \langle T_n(x^*), y^* \rangle$ exists for all $x^* \in E^*$, $y^* \in F^*$. Therefore (T_n) is Cauchy in the *wot* of $K_{w^*}(E^*, F)$, and thus weakly Cauchy (as above). ∎

COROLLARY 3.3 ([11, Theorem 1.14]). *If E does not contain a copy of ℓ_1 and F^* has the Radon–Nikodym property, in particular if F is reflexive, then $K_{w^*}(E^*, F)$ does not contain a copy of ℓ_1.*

Proof. Let (T_n) be a sequence in $K_{w^*}(E^*, F)$, $\|T_n\| \le 1$. Let S be defined as in the proof of Theorem 3.2. The compactness of each T_n implies that S is a separable subspace of F. Since F^* has the Radon–Nikodym property, S^* is separable, by [14, Theorem 6, p. 195]. Let $A = (y_n^*)$ be a (norm) dense sequence in S^*.

By Rosenthal's ℓ_1 theorem, $(T_n^*(y_k^*))$ has a weakly Cauchy subsequence in E for each k. By diagonalization, we may assume that (T_{n_i}) is a subsequence of (T_n) so that $(T_{n_i}^*(y_k^*))_{i=1}^\infty$ is weakly Cauchy for each k. Without loss of generality, we assume that $(T_n^*(y^*))$ is weakly Cauchy for each $y^* \in A$. Then $\lim \langle T_n^*(y^*), x^*\rangle = \lim \langle T_n(x^*), y^*\rangle$ exists for all $x^* \in E^*$, $y^* \in A$. Since A is dense in S^*, $(T_n(x^*))$ is weakly Cauchy in S, and thus in F, for all $x^* \in E^*$. Then (T_n) is Cauchy in the *wot* of $K_{w^*}(E^*, F)$. As in the proof of Theorem 3.2, (T_n) is weakly Cauchy in $K_{w^*}(E^*, F)$. By Rosenthal's ℓ_1 theorem, $K_{w^*}(E^*, F)$ does not contain a copy of ℓ_1.

We note that if F is reflexive, then F^* has the Radon–Nikodym property, by [14, Corollary 4, p. 82]. ∎

The following result (first obtained in [21]) gives an extension of Corollary 2 of [24], where E and F are assumed to be reflexive. We present a different proof based on Theorem 3.2.

THEOREM 3.4 ([21, Theorem 4.8]). *Suppose $L_{w^*}(E^*, F) = K_{w^*}(E^*, F)$. Let H be a subset of $K_{w^*}(E^*, F)$ such that*

(i) *$H(x^*)$ is relatively weakly compact for each $x^* \in E^*$, and*
(ii) *$H^*(y^*)$ is relatively weakly compact for each $y^* \in F^*$.*

Then H is relatively weakly compact.

Proof. Let (T_n) be a sequence in H. By Theorem 3.2, H is weakly precompact. Without loss of generality, assume that (T_n) is weakly Cauchy. For each $x^* \in E^*$, the sequence $(T_n(x^*))$ is relatively weakly compact and weakly Cauchy in F, hence weakly convergent. Define $T : E^* \to F$ by

$$T(x^*) = w\text{-}\lim T_n(x^*)$$

for $x^* \in E^*$. Let $y^* \in F^*$ and consider $T^*(y^*)$. The sequence $(T_n^*(y^*))$ is relatively weakly compact and weakly Cauchy in E, and hence weakly convergent. Suppose that $(T_n^*(y^*)) \xrightarrow{w} x \in E$, and let $x^* \in E^*$. Then $\langle T(x^*), y^*\rangle = \lim_n \langle T_n(x^*), y^*\rangle = \lim_n \langle x^*, T_n^*(y^*)\rangle = \langle x^*, x\rangle$. Therefore $T^*(y^*) = x$, $T^*(F^*) \subseteq E$, and T is w^*-w continuous. Then T is compact by assumption. By Lemma 3.1, $(T_n) \to T$ weakly. ∎

REMARK. If $L_{w^*}(E^*, F) = K_{w^*}(E^*, F)$, then a subset H of $K_{w^*}(E^*, F)$ is relatively weakly compact if and only if conditions (i) and (ii) of the previous theorem hold.

4. Properties (V^*) and (wV^*) in tensor products. A bounded subset A of X (resp. of X^*) is called a V^*-*subset* of X (resp. a V-*subset* of X^*) provided that

$$\lim_n (\sup\{|x_n^*(x)| : x \in A\}) = 0$$
$$(\text{resp. } \lim_n (\sup\{|x^*(x_n)| : x^* \in A\}) = 0)$$

for each wuc series $\sum x_n^*$ in X^* (resp. wuc series $\sum x_n$ in X).

In his fundamental paper [29], Pełczyński introduced property (V) and property (V^*). The Banach space X has *property* (V) (resp. (V^*)) if every V-subset of X^* (resp. V^*-subset of X) is relatively weakly compact. The following results were also established in [29]: $C(K)$ spaces have property (V); L_1-spaces have property (V^*); reflexive Banach spaces have both properties (V) and (V^*); the Banach space X has property (V) if and only if every unconditionally converging operator T from X to any Banach space Y is weakly compact; every closed subspace of a Banach space with property (V^*) has property (V^*); if X has property (V^*), then X is weakly sequentially complete. We note that property (V^*) is stable under isomorphisms.

The Banach space X has property *weak* (V^*) (wV^*) if every V^*-subset of X is weakly precompact [5]. The Banach space X has property (V^*) if X is weakly sequentially complete and X has property (wV^*). If X does not contain a copy of ℓ_1, then X has property (wV^*), by Rosenthal's theorem [12, Ch. XI]. In particular, c_0 has property (wV^*), but it does not have property (V^*). The Banach space X has property (wV^*) (resp. (V^*)) if and only if every operator $T : Y \to X$ with unconditionally converging adjoint is weakly precompact (resp. weakly compact) [21, Theorem 3.10].

In this section we consider properties (V^*) and (wV^*) in spaces of compact operators. First, we study these properties in the space $K_{w^*}(E^*, F)$. We note that there are examples of Banach spaces E and F such that $K_{w^*}(E^*, F)$ has property (V^*) (resp. (wV^*)). If $1 \le q < p < \infty$, then $L(\ell_p, \ell_q) = K(\ell_p, \ell_q)$ (by a result of Pitt [32]), and this space is reflexive if moreover $q > 1$ (see [24]). Hence $K(\ell_p, \ell_q) \simeq K_{w^*}(\ell_p^{**}, \ell_q) = L_{w^*}(\ell_p^{**}, \ell_q)$ has property (V^*), and the spaces $E = \ell_p^*$ and $F = \ell_q$ are as desired. Further, if E does not contain a copy of ℓ_1 and F is reflexive, then $K_{w^*}(E^*, F)$ does not contain a copy of ℓ_1 (by Corollary 3.3), hence $K_{w^*}(E^*, F)$ has property (wV^*) (by Rosenthal's ℓ_1 theorem).

If $K_{w^*}(E^*, F)$ has property (V^*) (resp. (wV^*)), then E and F have it too, since property (V^*) (resp. (wV^*)) is inherited by closed subspaces. We also note that if $K_{w^*}(E^*, F)$ has property (V^*), then $c_0 \not\hookrightarrow K_{w^*}(E^*, F)$, since c_0 does not have property (V^*). The space $K_{w^*}(\ell_2, \ell_2) = \ell_2 \otimes_\epsilon \ell_2$ does not have property (V^*) even though ℓ_2 has it, since $c_0 \hookrightarrow K_{w^*}(\ell_2, \ell_2)$ (by [20, Theorem 20]). Similarly, for $1 \le r < \infty$ and $1 \le s < \infty$, $L_r[0,1] \otimes_\epsilon L_s[0,1]$ does not have property (V^*), since it contains $\ell_2 \otimes_\epsilon \ell_2$ as a closed subspace. See a generalization of these results in Theorem 4.10 and its consequences. This is in contrast with the setting of the following Theorem 4.1(ii) and Theorem 4.3.

Theorem 4.1 ([17]).

(i) *If E has property (wV^*) and F has property (V^*), or if E has property (V^*) and F has property (wV^*), then $K_{w^*}(E^*, F)$, in particular $E \otimes_\epsilon F$, has property (wV^*). If moreover E or F contains a copy of ℓ_1, then $K_{w^*}(E^*, F)$ contains a complemented copy of ℓ_1.*

(ii) *$K_{w^*}(E^*, F)$ has property (V^*) if and only if it is weakly sequentially complete and E and F have property (V^*).*

Proof. (i) Suppose E has property (wV^*) and F has property (V^*). Let H be a V^*-subset of $K_{w^*}(E^*, F)$. For fixed $x^* \in E^*$, the map $T \mapsto T(x^*)$ is a bounded operator from $K_{w^*}(E^*, F)$ into F. It is easily verified that continuous linear images of V^*-sets are V^*-sets. Then $H(x^*)$ is a V^*-subset of F, hence relatively weakly compact. For fixed $y^* \in F^*$, the map $T \mapsto T^*(y^*)$ is a bounded operator from $K_{w^*}(E^*, F)$ into E. Then $H^*(y^*)$ is a V^*-subset of E, hence weakly precompact. By Theorem 3.2, H is weakly precompact.

Suppose E has property (V^*) and F has property (wV^*). By the previous argument and the isometry $K_{w^*}(E^*, F) \simeq K_{w^*}(F^*, E)$, $K_{w^*}(E^*, F)$ has property (wV^*).

Since a closed subspace of a space with property (wV^*) has the same property, $E \otimes_\epsilon F$ has property (wV^*).

If $\ell_1 \hookrightarrow E$ or $\ell_1 \hookrightarrow F$, then $\ell_1 \hookrightarrow K_{w^*}(E^*, F)$ (since E and F embed in the rank one operators from E^* to F, and thus in $K_{w^*}(E^*, F)$). Then $\ell_1 \overset{c}{\hookrightarrow} K_{w^*}(E^*, F)$, by [5, Corollary 1.6].

(ii) Suppose $K_{w^*}(E^*, F)$ has property (V^*). Then $K_{w^*}(E^*, F)$ is weakly sequentially complete [29], and E and F have property (V^*).

Conversely, suppose E and F have property (V^*). Let H be a V^*-subset of $K_{w^*}(E^*, F)$. By (i), H is weakly precompact. If moreover $K_{w^*}(E^*, F)$ is weakly sequentially complete, then H is relatively weakly compact. ∎

Example 4.2. The following example shows that there are Banach spaces E and F such that $K_{w^*}(E^*, F)$ has property (wV^*) and does not have property (V^*). The space $F = c_0$ has property (wV^*) and does not have property (V^*). More generally, let K be an infinite compact Hausdorff space which is scattered, and let $F = C(K)$. A topological space S is called *scattered* (or *dispersed*) if every nonempty closed subset of S has an isolated point [38]. Then $\ell_1 \not\hookrightarrow C(K)$ (see [30]), $C(K)$ has property (wV^*) and does not have property (V^*) (since $c_0 \hookrightarrow C(K)$; see [10]). Let E have property (V^*). Then $K_{w^*}(E^*, C(K)) \simeq C(K, E)$ has property (wV^*) (by Theorem 4.1(i)) and does not have property (V^*) (since $C(K)$ does not have property (V^*)).

THEOREM 4.3. *Suppose $L_{w^*}(E^*, F) = K_{w^*}(E^*, F)$. The following statements are equivalent:*

(i) *E and F have property (V^*).*
(ii) *$K_{w^*}(E^*, F)$ has property (V^*).*
(iii) *$E \otimes_\epsilon F$ has property (V^*).*

Proof. Suppose E and F have property (V^*). Let H be a V^*-subset of $K_{w^*}(E^*, F)$. By Theorem 3.4 and the proof of Theorem 4.1(i), H is relatively weakly compact. Then $K_{w^*}(E^*, F)$, and thus $E \otimes_\epsilon F$, has property (V^*). That (iii) implies (i) is clear. ∎

REMARK. In [22, Theorem 3.10] it is proved that if E and F are weakly sequentially complete and $L_{w^*}(E^*, F) = K_{w^*}(E^*, F)$, then $K_{w^*}(E^*, F)$ is weakly sequentially complete. We note that Theorem 4.3 can be proved using this result and Theorem 4.1(ii).

EXAMPLE 4.4. The space $L_1(\mu)$, where μ is a finite measure, has property (V^*) (see [29]). Suppose that E has property (wV^*). It is known that $L_1(\mu) \otimes_\epsilon E \simeq K_{w^*}(E^*, L_1(\mu))$ ([25], [14, Theorem 5, p. 224], [35]). This space has property (wV^*) by Theorem 4.1(i). By Theorem 4.3 it also has property (V^*) if moreover E has property (V^*) and if $L_{w^*}(E^*, L_1(\mu)) = K_{w^*}(E^*, L_1(\mu))$. These two conditions are satisfied if E is a quotient of ℓ_p, $1 < p < 2$, by [33, Theorem A.2, p. 206]. The last equality holds if either E or $L_1(\mu)$ has the Schur property (we recall that $L_1(\mu)$ has the *Schur property* if and only if the measure μ is purely atomic [9]).

COROLLARY 4.5. *Suppose that E has property (V^*) and F is an infinite-dimensional space with property (V^*) and the Schur property. Then $L_{w^*}(E^*, F) = K_{w^*}(E^*, F)$ has property (V^*) and it contains a complemented copy of ℓ_1. If moreover E has the Schur property, then $L_{w^*}(E^*, F)$ has the Schur property.*

Proof. By definition, $L_{w^*}(E^*, F) = W_{w^*}(E^*, F)$, and $W_{w^*}(E^*, F) = K_{w^*}(E^*, F)$ since F has the Schur property. Further, $\ell_1 \hookrightarrow F$. The first assertion of the theorem follows from Theorem 4.3 and the end of Theorem 4.1(i). The last assertion follows from [27], [36]. ∎

EXAMPLE 4.6. Since ℓ_1 has the Schur property and property (V^*), $\ell_1 \otimes_\epsilon E$ has property (V^*) if and only if E has property (V^*) (by the first part of

Corollary 4.5). Recall that $\ell_1 \otimes_\epsilon E \simeq \ell_1[E]$ is the Banach space of all unconditionally convergent series in E with the norm $\|(x_n)\| = \sup\{\sum |x^*(x_n)| : x^* \in B_{E^*}\}$ (see [35]).

Observation 1. If E^* has property (V^*) and F has property (V^*) (resp. (wV^*)), then every operator $T : E \to F$ is weakly compact (resp. weakly precompact). To see this, let $T : E \to F$ be an operator. Since E^* has property (V^*), E^* does not contain a copy of c_0. Then $T^* : F^* \to E^*$ is unconditionally converging ([4], [12, p. 54]). Since F has property (V^*) (resp. (wV^*)), T is weakly compact (resp. weakly precompact), by [21, Theorem 3.10].

We will need the following version of Corollary 4.5, replacing F by F^* and E by E^*.

Corollary 4.7. *Suppose E^* has property (V^*) and F is an infinite-dimensional space such that F^* has property (V^*) and the Schur property. Then $L(E, F^*) = K(E, F^*)$ has property (V^*) and it contains a complemented copy of ℓ_1. If moreover E^* has the Schur property, then $L(E, F^*)$ has the Schur property.*

Proof. By Observation 1, we have $L(E, F^*) = W(E, F^*)$. Taking into account the isometries (2) from Section 2, the first statements follow from Corollary 4.5 applied to E^* instead of E, and F^* instead of F. ∎

A Banach space X has the *Dunford–Pettis property* (DPP) if every weakly compact operator T with domain X is completely continuous. Schur spaces, $C(K)$ spaces, and $L_1(\mu)$ spaces have the DPP. The reader can check [12], [13], [14], and [2] for a guide to the extensive classical literature dealing with the DPP. We recall [13] that F^* has the Schur property as soon as F has the DPP and does not contain ℓ_1. For the definition of $\mathcal{L}_\infty$-spaces and $\mathcal{L}_1$-spaces we refer the reader to [6].

Example 4.8. A. Pełczyński [29] showed that if E has property (V), then E^* has property (V^*), and he asked if the converse is true. Let Y be the Bourgain–Delbaen space so that Y is a separable $\mathcal{L}_\infty$-space, Y is somewhat reflexive, $c_0 \not\hookrightarrow Y$, and $Y^* \simeq \ell_1$ (see [7]). Then Y does not have property (V), since $c_0 \not\hookrightarrow Y$ and Y is not reflexive (by [29, Proposition 8]), and Y^* has property (V^*). Hence Y provides a first counterexample to Pełczyński's question.

Suppose E^* has property (V^*). Then $E \otimes_\pi Y$ does not have property (V) (since Y does not have property (V)) and $L(E, Y^*) \simeq (E \otimes_\pi Y)^*$ has property (V^*) (by Corollary 4.7 applied to $F = Y$). Hence $E \otimes_\pi Y$ is another counterexample to Pełczyński's question.

Besides Y^*, there are dual spaces Z^* which have both property (V^*) and the Schur property but are not isomorphic to ℓ_1: take for Z the spaces constructed by Hagler [23] and Talagrand [39].

Corollary 4.9. *Suppose that $W(E^*, F^*) = K(E^*, F^*)$ and both E^{**}, F^* have property (V^*). Then $L(E^*, F^*) = K(E^*, F^*)$ and this space has property (V^*). The dual of the space of all nuclear operators $N_1(E, F)$ also has property (V^*), hence it does not contain ℓ_∞; consequently, ℓ_1 is not complemented in $N_1(E, F)$.*

Proof. By the isometries (2) in Section 2, $K(E^*, F^*) \simeq K_{w^*}(E^{***}, F^*)$ and $W(E^*, F^*) \simeq L_{w^*}(E^{***}, F^*)$. By Theorem 4.3 applied to E^{**} instead of E and to F^* instead of F, $K(E^*, F^*)$ has property (V^*). By Observation 1 applied to E^* instead of E and to F^* instead of F, $L(E^*, F^*) = W(E^*, F^*)$; hence by assumption $L(E^*, F^*) = K(E^*, F^*)$. Note that $L(E^*, F^*) \simeq (E^* \otimes_\pi F)^*$. It is known that $N_1(E, F)$ is a quotient of $E^* \otimes_\pi F$ [35, p. 41]. Hence the dual of $N_1(E, F)$ is a closed subspace of $(E^* \otimes_\pi F)^*$, so it inherits property (V^*) from $(E^* \otimes_\pi F)^* \simeq K(E^*, F^*)$. ∎

Recall that $N_1(E, F) = E^* \otimes_\pi F$ if E^* or F has the approximation property.

One can find numerous references to papers which study the embedability of c_0 in spaces of operators in [15], [16], [18], [26], [24], [19], and [20]. Specifically, we note that [15], [16], and [26] point out that if the Banach space E has an unconditional finite-dimensional decomposition, then $c_0 \hookrightarrow K(E, E)$.

A bounded subset A of E is called a *limited subset* of E if each w^*-null sequence in E^* tends to 0 uniformly on A. Furthermore, a Banach space E has the *Gelfand–Phillips property* if any limited subset of E is relatively compact. See [8] and [37] for discussions of limited sets.

The following theorem and its corollaries are essentially contained in [20]. Theorem 20 of [20] can be rephrased in terms of property (V^*); the assumption there that either $(R(g_i))$ or $(S(g_i^*))$ is a basic sequence can be removed. Theorem 4.10 below generalizes the example showing that $\ell_2 \otimes_\epsilon \ell_2$ does not have property (V^*) given just before Theorem 4.1.

Theorem 4.10. *Let E and F be Banach spaces satisfying the following assumption: there exists a Banach space G with an unconditional basis (g_n) and biorthogonal coefficients (g_n^*), and two operators $R : G \to F$ and $S : G^* \to E$ such that $(R(g_i))$ and $(S(g_i^*))$ are seminormalized sequences. Then c_0 embeds in $E \otimes_\epsilon F$. Thus $E \otimes_\epsilon F$ does not have property (V^*).*

Moreover, if $(R(g_i))$ is basic and F has the Gelfand–Phillips property (or $(S(g_i^*))$ is basic and E has the Gelfand–Phillips property), then $K_{w^*}(E^*, F)$ contains a complemented copy of c_0.

Proof. Suppose that $p \le \|R(g_n)\| \le q$ and $p \le \|S(g_n^*)\| \le q$ for all n. Let $S(g_n^*) \otimes R(g_n) \in K_{w^*}(E^*, F)$, $\langle S(g_n^*) \otimes R(g_n), x^*\rangle = x^*(S(g_n^*))R(g_n)$, $x^* \in E^*$. Let $H = E \otimes_\epsilon F$.

If $g \in G$, then $\sum g_n^*(g) g_n$ converges unconditionally to g. By the Uniform Boundedness Principle, $\{\sum_{n\in A} g_n^* \otimes g_n : A \subseteq \mathbb{N}, A \text{ finite}\}$ is bounded in $L(G,G)$. Then $\sum g_n^* \otimes g_n$ is wuc. On the other hand, S and R induce an operator $S \otimes_\epsilon R : G^* \otimes_\epsilon G \to E \otimes_\epsilon F$, which maps $(g_n^* \otimes g_n)$ into $(S(g_n^*) \otimes R(g_n))$ [14, Chapter VIII]. Then $\sum S(g_n^*) \otimes R(g_n)$ is wuc in $E \otimes_\epsilon F$. Now, $\inf \|S(g_n^*) \otimes R(g_n)\| \geq p^2$. Since $(S(g_n^*) \otimes R(g_n))$ is weakly null and not norm null, by the Bessaga–Pełczyński principle ([4], [12]) we can assume without loss of generality that $(S(g_n^*) \otimes R(g_n))$ is basic. Let (e_n) denote the canonical unit vector basis of c_0. By a result of Bessaga and Pełczyński ([4, Lemma 3, p. 160], [12, p. 45]), we have $(S(g_n^*) \otimes R(g_n)) \sim (e_n)$, and thus $c_0 \hookrightarrow H$.

To prove the last part of the theorem, suppose that F has the Gelfand–Phillips property and $(R(g_n))$ is basic. If $(R(g_n))$ is limited, then $(R(g_n)) \to 0$ since $(R(g_n))$ is relatively compact and the only weak limit of a basic sequence is zero [12, p. 42]. Therefore $(R(g_n))$ is not limited. Without loss of generality we can choose a w^*-null sequence (y_n^*) in F^* so that $\langle y_n^*, R(g_n)\rangle = 1$ for all n. Let (x_n^*) be a sequence in E^* of norm one such that $\langle x_n^*, S(g_n^*)\rangle = \|S(g_n^*)\|$ for all n. For each $T \in K_{w^*}(E^*, F)$, we have $\|T^*(y_n^*)\| \to 0$, since T^* is compact. Hence

$$\langle x_n^* \otimes y_n^*, T\rangle = \langle T(x_n^*), y_n^*\rangle \leq \|T^*(y_n^*)\| \to 0.$$

Then $(x_n^* \otimes y_n^*)$ is a w^*-null sequence in $(K_{w^*}(E^*, F))^*$. Also, we have $\langle x_n^* \otimes y_n^*, S(g_n^*) \otimes R(g_n)\rangle = \|S(g_n^*)\|$; thus $(S(g_n^*) \otimes R(g_n))$ is not limited. By [37, Theorem 1.3.2], c_0 is complemented in $K_{w^*}(E^*, F)$. ■

Corollary 4.11. *Suppose $L_{w^*}(E^*, F) = K_{w^*}(E^*, F)$. If E and F have property (V^*), then either $\ell_p \not\hookrightarrow E$ or $\ell_q \not\hookrightarrow F$, for $1 < p' \leq q < \infty$ (where p and p' are conjugate).*

Proof. Suppose the contrary. By Theorem 4.10 applied to $G = \ell_q$, c_0 embeds in $E \otimes_\epsilon F$, which contradicts Theorem 4.3. ■

Corollary 4.12. *Suppose that $W(E,F) = K(E,F)$. If E^* and F have property (V^*), then either $\ell_1 \not\hookrightarrow E$ or $\ell_2 \not\hookrightarrow F$. If moreover F is a dual space X^*, the condition $\ell_2 \not\hookrightarrow F$ implies $\ell_1 \not\hookrightarrow X$.*

Proof. If $W(E,F) = K(E,F)$ and E^* and F have property (V^*), then by the isometries (2) in Section 2, the assumptions of Theorem 4.3 applied to E^* instead of E are satisfied. Thus, by Theorem 4.3, $K(E,F)$ has property (V^*). Now suppose $\ell_1 \hookrightarrow E$ and $\ell_2 \hookrightarrow F$. Then $L_1 \hookrightarrow E^*$ (see [28]), hence $\ell_2 \hookrightarrow E^*$. Then c_0 embeds in $K(E,F)$. This contradiction proves the first assertion. The last assertion follows again from [28]. ■

The following theorem shows that the assumption that $L_{w^*}(E^*, F) = K_{w^*}(E^*, F)$ considered in Theorem 4.3 is in some cases even necessary.

THEOREM 4.13. *Suppose that $K_{w^*}(E^*, F)$ has property (V^*), and E, F are infinite-dimensional Banach spaces satisfying one of the assumptions:*

(i) *If T is an operator in $L_{w^*}(E^*, F)$, then there is a sequence of operators (T_n) in $K_{w^*}(E^*, F)$ such that for each $x^* \in E^*$, the series $\sum T_n(x^*)$ converges unconditionally to $T(x^*)$.*
(ii) *Either E or F has the metric approximation property.*

Then $L_{w^*}(E^*, F) = K_{w^*}(E^*, F)$.

Proof. Suppose $K_{w^*}(E^*, F)$ has property (V^*).

(i) In this case, $L_{w^*}(E^*, F) \neq K_{w^*}(E^*, F)$ implies that c_0 embeds in $K_{w^*}(E^*, F)$ by [20, Theorem 14], a contradiction.

(ii) Since $K_{w^*}(E^*, F)$ has property (V^*), it is weakly sequentially complete [29]. Then E, F, and $E \otimes_\epsilon F$ are weakly sequentially complete, since subspaces of weakly sequentially complete spaces are weakly sequentially complete. Under assumption (ii), Theorem 2.1 of [25] implies $L_{w^*}(E^*, F) = K_{w^*}(E^*, F)$. Note that $K_{w^*}(E^*, F) = E \otimes_\epsilon F$, since E or F has the approximation property [34, p. 60]. ∎

Assumption (i) of the previous theorem is satisfied, for instance, in the following cases:

(1) E (or F) has an (u.c.e.i.), that is, there is a sequence (A_n) of compact operators from E to E such that $\sum A_n(x)$ converges unconditionally to x for all $x \in E$ (see [19]).
(2) F is complemented in a Banach space Z which has an unconditional Schauder decomposition (Z_n) and $L_{w^*}(E^*, Z_n) = K_{w^*}(E^*, Z_n)$ for each n.

The space X has (*Rademacher*) *cotype* q for some $2 \le q \le \infty$ if there is a constant C such that for every n and any $x_1, \dots, x_n$ in X,

$$\Big(\sum_{i=1}^{n} \|x_i\|^q\Big)^{1/q} \le C\Big(\int_0^1 \|r_i(t)x_i\|^q\,dt\Big)^{1/q},$$

where (r_n) are the Rademacher functions. A Hilbert space has cotype 2. The dual of $C(K)$, $M(K)$, has cotype 2 (see [1]).

In the next corollary, (i) and (ii) are versions of Theorem 4.13.

COROLLARY 4.14. *Suppose that $K(E, F)$ has property (V^*), and E, F are infinite-dimensional Banach spaces satisfying one of the assumptions:*

(i) *If $T : E \to F$ is a weakly compact operator, then there is a sequence (T_n) in $K(E, F)$ such that for each $x \in X$, the series $\sum T_n(x)$ converges unconditionally to $T(x)$.*
(ii) *Either E^* or F has the metric approximation property.*

(iii) *E is an $\mathcal{L}_\infty$-space and F is a closed subspace of an $\mathcal{L}_1$-space.*
(iv) *$E = C[0,1]$ and F is a space with cotype* 2.

Then $L(E,F) = K(E,F)$.

Proof. Suppose $K(E,F)$ has property (V^*). Then E^* and F have property (V^*), and $L(E,F) = W(E,F)$ (by Observation 1).

(i) Suppose $W(E,F) \neq K(E,F)$. Let $T : E \to F$ be a weakly compact and noncompact operator. Let (T_n) be a sequence as in the hypothesis. By the Uniform Boundedness Principle, $\{\sum_{n\in A} T_n : A \subseteq \mathbb{N}, A \text{ finite}\}$ is bounded in $K(E,F)$. Then $\sum T_n$ is wuc and not unconditionally convergent (since T is noncompact). By a result of Bessaga and Pełczyński [4], $c_0 \hookrightarrow K(E,F)$. This contradiction shows that $W(E,F) = K(E,F)$.

(ii) By Theorem 4.13(ii) and the isometries (2) in Section 2, we have $W(E,F) = K(E,F)$. Therefore $L(E,F) = K(E,F)$.

Suppose (iii) or (iv) holds. It is known that any operator $T : E \to F$ is 2-absolutely summing ([31], [12]), hence it factorizes through a Hilbert space. If $L(E,F) \neq K(E,F)$, then $c_0 \hookrightarrow K(E,F)$, by [18, Remark 3] (or [15, Theorem 4]), and we have a contradiction. ∎

Assumption (i) of the previous corollary is satisfied, for instance, in the following cases:

(1) Either E^* or F has an (u.c.e.i.).
(2) F is complemented in a Banach space Z which has an unconditional Schauder decomposition (Z_n) and $W(E,Z_n) = K(E,Z_n)$ for each n.
(3) E has an unconditional and seminormalized shrinking basis; in particular this holds if E is a reflexive space with an unconditional and seminormalized basis.

We recall that the basis (x_i) of E is *shrinking* if the associated sequence of coordinate functionals (x_i^*) is a basis for E^*. The unit vector bases of c_0 and ℓ_p, $1 < p < \infty$, are shrinking, and the unit vector basis of ℓ_1 is not shrinking.

Example 4.15. For $1 \leq p < q < \infty$, the natural inclusion map $i : \ell_p \to \ell_q$ is weakly compact and not compact. Then $c_0 \hookrightarrow K(\ell_p, \ell_q)$ (by [20, Theorem 14]), and $K(\ell_p,\ell_q)$ does not have property (V^*).

Theorem 4.16. *Assume that $K(E,F^*)$ has property (V^*) and that one of the following assumptions holds:*

(i) *E has the DPP and $\ell_1 \hookrightarrow F$.*
(ii) *E and F have the DPP.*

Then $L(E,F^*) = K(E,F^*)$.

Proof. Suppose $K(E, F^*)$ has property (V^*). Then E^* and F^* have property (V^*). By Observation 1, $L(E, F^*) = W(E, F^*)$.

(i) Since $K(E, F^*)$ has property (V^*), $c_0 \not\hookrightarrow K(E, F^*)$. Hence, by the second part of the proof of Corollary 4.12, either $\ell_1 \not\hookrightarrow E$ or $\ell_2 \not\hookrightarrow F^*$. By the last assumption of Corollary 4.12 and the assumption $\ell_1 \hookrightarrow F$, we obtain $\ell_1 \not\hookrightarrow E$. Since moreover E has the DPP, E^* has the Schur property by [13] or [12, p. 212]. Let $T : E \to F^*$ be a weakly compact operator. Then $T^* : F^{**} \to E^*$ is weakly compact, thus compact, since E^* has the Schur property. Therefore T is compact. Thus $W(E, F^*) = K(E, F^*)$, which proves the result.

(ii) Assume E and F have the DPP. Then $W(E, F^*) = K(E, F^*)$ either by (i) if $\ell_1 \hookrightarrow F$, or because F^* has the Schur property [13] if $\ell_1 \not\hookrightarrow F$. ■

The previous proof shows that if E and F satisfy one of the hypotheses (i), (ii) and if $K(E, F^*)$ does not contain c_0 (as a closed subspace), then $W(E, F^*) = K(E, F^*)$.

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Received 28 May 2014;
revised 28 October 2014 (6275)