COLLOQUIUM MATHEMATICUM

VOL. 138 2015 NO. 2

THE PERIODICITY CONJECTURE FOR BLOCKS OF GROUP ALGEBRAS

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Abstract. We describe the representation-infinite blocks B of the group algebras KG of finite groups G over algebraically closed fields K for which all simple modules are periodic with respect to the action of the syzygy operators. In particular, we prove that all such blocks B are periodic algebras of period 4. This confirms the periodicity conjecture for blocks of group algebras.

1. Introduction and the main results. Throughout this article, K will denote a fixed algebraically closed field. By an *algebra* we mean an associative, finite-dimensional K-algebra with identity, and we denote by $\operatorname{mod} A$ the category of finite-dimensional right A-modules. An algebra A is called *self-injective* if A_A is an injective module, or equivalently, the projective modules in $\operatorname{mod} A$ are injective. A prominent class of self-injective algebras is formed by the *symmetric algebras* A for which there exists an associative, non-degenerate, symmetric, K-bilinear form $(-,-) : A \times A \to K$.

Given a module M in the module category $\operatorname{mod} A$, its *syzygy* is defined to be the kernel $\Omega_A(M)$ of a minimal projective cover $P_A(M) \to M$ of M in $\operatorname{mod} A$. The module M in $\operatorname{mod} A$ is said to be *periodic* if $\Omega_A^n(M) \cong M$ for some $n \geq 1$, and if so the minimal such n is called the *period* of M. Further, the category of finite-dimensional A-A-bimodules over an algebra A is canonically equivalent to the module category $\operatorname{mod} A^e$ of the enveloping algebra $A^e = A^{\mathrm{op}} \otimes_K A$ of A. Then the algebra A is called a *periodic algebra* if A is a periodic module in $\operatorname{mod} A^e$. It is known that any periodic algebra A is self-injective, and that every module M in $\operatorname{mod} A$ without non-zero projective direct summands is periodic. It is conjectured that the following should be true:

$$\boxed{\text{simple modules in } \operatorname{mod} A \text{ are periodic}} \Rightarrow \boxed{A \text{ is a periodic algebra}}$$

This is known as the *periodicity conjecture*, and is an exciting open problem.

2010 *Mathematics Subject Classification*: Primary 16D50, 16E30, 20C20; Secondary 16G60, 16G70, 20C05.

Key words and phrases: syzygy, periodic module, periodic algebra, block, generalized quaternion group.

DOI: 10.4064/cm138-2-12

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It has been proved by Green, Snashall and Solberg [24] that if all simple modules in $\operatorname{mod} A$ are periodic then A is self-injective and some syzygy $\Omega^n_{A^e}(A)$ is isomorphic to a twisted bimodule ${}_1A_\sigma$ where σ is some K-algebra automorphism of A. However, it is not clear at all whether such an automorphism σ has finite order. We also mention that Dugas proved in [13] that an arbitrary indecomposable self-injective algebra of finite representation type, which is not simple, is a periodic algebra. In particular, this proves the periodicity conjecture for any algebra of finite representation type. Recently, it has been proved by Białkowski, Erdmann and Skowroński [7] that the periodicity conjecture holds for all algebras of polynomial growth. We also mention that to prove these results it was important that the periodicity of algebras is invariant under derived equivalences (see [21, Theorem 2.9] and [37, Corollary 2.3]).

The main aim of this article is to describe the representation-infinite blocks of group algebras of finite groups over algebraically closed fields for which all simple modules are periodic, and to prove that all these blocks are periodic algebras.

Let G be a finite group, K the fixed algebraically closed field, p the characteristic of K, and B a block of the group algebra KG. Recall that then $KG = B \oplus C$, where B and C are two-sided ideals of KG and B is indecomposable as an algebra. In particular, B is a symmetric algebra. We note that if $\operatorname{mod} B$ admits a periodic module, then B, and hence KG, is not semisimple, and consequently p divides the order $|G|$ of G, by Maschke's Theorem. Therefore, assume that p is a prime number dividing $|G|$. By general theory, the representation type of B is controlled by the *defect group* $D = D_B$ of B, which is a minimal subgroup of G such that every module M in $\operatorname{mod} B$ is a direct summand of a module of the form $N \otimes_{KD} KG$ for a module N in $\operatorname{mod} KD$. The defect groups of B form one conjugacy class of p-subgroups of G (see [3], [33] for more details). In particular, if the trivial module K is a B-module, then D_B is a Sylow p-subgroup of G. The analogue of Maschke's Theorem asserts that B is semisimple if and only if D_B is trivial. More generally, the analogue of Higman's Theorem asserts that B is of finite representation type if and only if D_B is a cyclic group. By the remarkable theorem of Dade, Janusz and Kupisch [11], [27], [28], [29] every block B of KG which is of finite representation type but not simple (so D_B is a non-trivial cyclic group) is Morita equivalent to a Brauer tree algebra. Moreover, by a result of Rickard [36] every Brauer tree algebra is derived equivalent to a symmetric Nakayama algebra, and hence is a periodic algebra (see [18]). Therefore, it remains to investigate the blocks B of infinite representation type having all simple modules periodic.

The following theorem is the main result of the paper.

Theorem 1.1. *Let B be a block of infinite representation type of the group algebra KG of a finite group G over an algebraically closed field K. The following statements are equivalent:*

(i) *All simple modules in* $\operatorname{mod} B$ *are periodic.*
(ii) *K has characteristic* 2 *and D_B is a generalized quaternion* 2*-group.*
(iii) *B is a periodic algebra.*

Recall that the *generalized quaternion* 2*-groups* are the groups

$$Q_{2^{m+2}} = \langle x, y \mid x^{2^m} = y^2,\, y^4 = 1,\, xyx = y\rangle$$

of orders 2^{m+2}, for $m \geq 1$.

Motivated by the representation theory of blocks of group algebras with defects being generalized quaternion 2-groups, Erdmann introduced in [15], [16] (see also [17]) algebras of quaternion type (for algebraically closed fields of any characteristic) and proved that they are Morita equivalent to algebras belonging to 12 families of symmetric algebras defined by quivers and relations. With the exception of cases of small dimension (described in [20, Proposition 5.4]), these algebras are tame of non-polynomial growth, and called algebras of *pure quaternion type*. Applying the derived equivalence classification of algebras of pure quaternion type by Holm [26], Erdmann and Skowroński proved in [20, Theorem 5.9] that all these algebras are periodic of period 4. Moreover, all blocks of group algebras whose defect groups are generalized quaternion 2-groups are algebras of pure quaternion type (see [17]).

Then we obtain the following direct consequence of Theorem 1.1 and Proposition 2.3 (proved in Section 2).

Corollary 1.2. *Let B be a block of infinite representation type of the group algebra KG of a finite group G over an algebraically closed field K such that all simple modules in* $\operatorname{mod} B$ *are periodic. Then:*

(i) *B is a periodic algebra of period* 4.
(ii) *Every simple module in* $\operatorname{mod} B$ *is periodic of period* 4.

From the remarkable Tame and Wild Theorem of Drozd [12] (see also [10]) the class of finite-dimensional algebras over an algebraically closed field K may be divided into two disjoint classes. The first class is formed by the tame algebras for which the indecomposable modules occur in each dimension in a finite number of discrete and a finite number of one-parameter families. The second class is formed by the wild algebras whose representation theory comprises the representation theories of all finite-dimensional algebras over K. Accordingly, we may realistically hope to classify the indecomposable finite-dimensional modules only for the tame algebras (see [38, Chapter XIX] for details and related discussion). We also note that a prominent class of tame algebras is formed by the algebras of finite representation type (representation-finite algebras) for which there are only finitely many isomorphism classes of indecomposable modules.

Then we obtain the following consequence of Theorem 1.1, the main results of [15], [16], and the tameness of all algebras of pure quaternion type established in [26].

COROLLARY 1.3. *Let B be a block of the group algebra KG of a finite group G over an algebraically closed field K such that all simple modules in* $\operatorname{mod} B$ *are periodic. Then B is a tame algebra.*

We would like to point out that there are many wild periodic algebras. For example, with the exception of few cases of small dimension, the preprojective algebras of Dynkin type, or more generally the deformed preprojective algebras of generalized Dynkin type, are wild periodic algebras (see [6], [22], [20, Theorem 3.7]).

For a finite group G and a positive integer n, denote by $H^n(G,\mathbb{Z}) = \operatorname{Ext}^n_{\mathbb{Z}G}(\mathbb{Z},\mathbb{Z})$ the nth cohomology group of G with coefficients in the trivial $\mathbb{Z}G$-module $\mathbb{Z}$. A finite group G is called *periodic* if there exists a positive integer d such that $H^n(G,\mathbb{Z}) \cong H^{n+d}(G,\mathbb{Z})$ for all $n \geq 1$. Then it has been proved by Artin and Tate (unpublished, see [9, Theorem XII.11.6] for details) that a finite group G is periodic if and only if for any prime p dividing the order of G, the Sylow p-subgroups of G are either cyclic or generalized quaternion 2-groups, and if and only if every abelian subgroup of G is cyclic. We note that by a result of Swan [42] the periodic groups are exactly all finite groups acting freely on finite CW-complexes homotopically equivalent to spheres. We also mention that all finite groups whose abelian subgroups are all cyclic have been classified by Zassenhaus [43] (solvable groups case) and Suzuki [41] (general case), and a complete list of such groups consists of six families (see [1, Chapter IV] for details on these groups).

As another consequence of Theorem 1.1 we obtain the following characterization of periodic groups.

COROLLARY 1.4. *Let G be a finite group. The following statements are equivalent:*

(i) *G is periodic.*
(ii) *For any algebraically closed field K, every non-projective simple right KG-module is periodic.*

For basic background on the relevant representation theory we refer to [3], [4], [17], [21], [33], [39], [40].

2. Projective resolutions. Let A be an algebra, and $e_1, \dots, e_m$ a set of pairwise orthogonal primitive idempotents of A such that $P_1 = e_1A, \dots, P_m$

$= e_m A$ form a complete set of pairwise non-isomorphic indecomposable projective right A-modules. Then $e_i \otimes e_j$ for $i, j \in \{1, \ldots, m\}$ is a set of pairwise orthogonal primitive idempotents of the enveloping algebra $A^e = A^{\mathrm{op}} \otimes_K A$ such that $P(i,j) = (e_i \otimes e_j)A^e = Ae_i \otimes e_j A$, for $i, j \in \{1, \ldots, m\}$, form a complete set of pairwise non-isomorphic indecomposable projective right A^e-modules (see [40, Proposition IV.11.3]). We also note that $S_i = e_i A / e_i \operatorname{rad} A$ for $i \in \{1, \ldots, m\}$ give a complete set of pairwise non-isomorphic simple right A-modules.

The following result by Happel [25, Lemma 1.5] (based on [9, Corollary 4.4]) describes the terms of a minimal projective resolution of A in $\operatorname{mod} A^e$.

Proposition 2.1. *Let A be an algebra. Then there is in* $\operatorname{mod} A^e$ *a minimal projective resolution of A of the form*

$$\cdots \to \mathbb{P}_n \xrightarrow{d_n} \mathbb{P}_{n-1} \to \cdots \to \mathbb{P}_1 \xrightarrow{d_1} \mathbb{P}_0 \xrightarrow{d_0} A \to 0,$$

where

$$\mathbb{P}_n = \bigoplus_{1 \le i,j \le m} P(i,j)^{\dim_K \operatorname{Ext}_A^n(S_i, S_j)} \quad \textit{for any } n \in \mathbb{N}.$$

We note that in particular we obtain $\operatorname{pd}_{A^e} A = \operatorname{gl.dim} A$. Hence, A is a projective module in $\operatorname{mod} A^e$ if and only if A is a semisimple algebra.

We also recall an important property of the syzygy bimodules of an algebra (see [40, Lemma IV.11.16]).

Lemma 2.2. *Let A be an algebra. For any positive integer n, the module $\Omega^n_{A^e}(A)$ is projective as a left A-module and projective as a right A-module.*

The following fact has been mentioned in [24, proof of Theorem 1.4] as a consequence of Proposition 2.1. We will provide an explicit proof.

Proposition 2.3. *Let A be an algebra and S a simple right A-module. Then, for any positive integer n, we have an isomorphism of right A-modules*

$$\Omega^n_A(S) \cong S \otimes_A \Omega^n_{A^e}(A).$$

Proof. Let $e_1, \ldots, e_m$ be a set of pairwise orthogonal primitive idempotents of A such that $S_1 = e_1 A / e_1 \operatorname{rad} A, \ldots, S_m = e_m A / e_m \operatorname{rad} A$ is a complete set of pairwise non-isomorphic simple right A-modules. We may assume that $S = S_r$ for some $r \in \{1, \ldots, m\}$. Moreover, let $P(i,j) = (e_i \otimes e_j)A^e = Ae_i \otimes e_j A$ for $i, j \in \{1, \ldots, m\}$. It follows from Proposition 2.1 that A admits a minimal projective resolution in $\operatorname{mod} A^e$ of the form

$$\cdots \to \mathbb{P}_n \xrightarrow{d_n} \mathbb{P}_{n-1} \to \cdots \to \mathbb{P}_1 \xrightarrow{d_1} \mathbb{P}_0 \xrightarrow{d_0} A \to 0,$$

where

$$\mathbb{P}_n = \bigoplus_{1 \le i,j \le m} P(i,j)^{\dim_K \operatorname{Ext}_A^n(S_i, S_j)}$$

for any $n \in \mathbb{N}$. For $i, j \in \{1, \ldots, m\}$, we have isomorphisms of right A-modules

$$S \otimes_A P(i,j) = S_r \otimes_A (Ae_i \otimes_K e_j A) \cong (e_j A)^{\dim_K (S_r \otimes_A Ae_i)},$$

and $S_r \otimes_A Ae_i = 0$ for $r \neq i$, and $\dim_K S_i \otimes_A Ae_i = \dim_K (e_i A e_i / e_i \operatorname{rad} Ae_i) = 1$, because K is algebraically closed. Then, applying Proposition 2.1 and Lemma 2.2 (see also the proof of [40, Proposition IV.11.17]), we conclude that

$$\begin{aligned} \cdots \to S \otimes_A \mathbb{P}_n \xrightarrow{1 \otimes_A d_n} S \otimes_A \mathbb{P}_{n-1} \to \cdots \to S \otimes_A \mathbb{P}_1 & \\ \xrightarrow{1 \otimes_A d_1} S \otimes_A \mathbb{P}_0 \xrightarrow{1 \otimes_A d_0} S \otimes_A A \to 0 & \end{aligned}$$

is a projective resolution of $S = S \otimes_A A$ in $\operatorname{mod} A$. Moreover, a projective right A-module $e_j A$ occurs in $S \otimes_A \mathbb{P}_n$ with multiplicity $\dim_K \operatorname{Ext}^n_A(S, S_j)$. This shows that it is a minimal projective resolution of S in $\operatorname{mod} A$. Fix a positive integer n. It follows from Lemma 2.2 that the canonical exact sequence of A-A-bimodules

$$0 \to \Omega^n_{A^e}(A) \xrightarrow{u_{n-1}} \mathbb{P}_{n-1} \xrightarrow{d_{n-1}} \Omega^{n-1}_{A^e}(A) \to 0,$$

with u_{n-1} the canonical embedding, splits as an exact sequence of left A-modules and as an exact sequence of right A-modules. Hence the induced exact sequence of right A-modules

$$0 \to S \otimes_A \Omega^n_{A^e}(A) \xrightarrow{1 \otimes_A u_{n-1}} S \otimes_A \mathbb{P}_{n-1} \xrightarrow{1 \otimes_A d_{n-1}} S \otimes_A \Omega^{n-1}_{A^e}(A) \to 0$$

splits. Therefore, we obtain isomorphisms of right A-modules

$$\Omega^n_A(S) \cong \operatorname{Ker}(1 \otimes d_{n-1}) = \operatorname{Im}(1 \otimes u_{n-1}) \cong S \otimes_A \Omega^n_{A^e}(A). \ ■$$

We obtain the following immediate consequence of the above proposition.

COROLLARY 2.4. *Let A be a periodic indecomposable algebra of period d. Then every simple module in* $\operatorname{mod} A$ *is periodic of period dividing d.*

We shall use the following application of this result.

COROLLARY 2.5. *Let A be a representation-infinite, symmetric, indecomposable periodic algebra of period* 4. *Then every simple module in* $\operatorname{mod} A$ *is periodic of period* 4.

Proof. It follows from Corollary 2.4 that every simple module in $\operatorname{mod} A$ is periodic of period dividing 4. However, if the period of a simple module in $\operatorname{mod} A$ is 1 or 2, then the indecomposability and symmetry of A force A to be of finite representation type. Hence the claim follows. ■

3. Complexity of modules. We say that an $\mathbb{N}$-graded K-vector space $V = \bigoplus_{n \in \mathbb{N}} V_n$ has *polynomial growth* if there are a non-negative integer c and a non-zero real constant μ with $\dim_K V_n \leq \mu n^{c-1}$ for $n \gg 0$. If these exist then the smallest such c is denoted by $\gamma(V)$ and is called the *rate of growth* of V. If V is not of polynomial growth then we set $\gamma(V)=\infty$.

Let A be a self-injective algebra and M a module in $\operatorname{mod} A$. Consider a minimal projective resolution

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

of M in $\operatorname{mod} A$. If $\gamma(\bigoplus_{n\in\mathbb{N}} P_n) < \infty$, then following Alperin [2] we set $c_A(M) = \gamma(\bigoplus_{n\in\mathbb{N}} P_n)$ and call it the *complexity* of M. Observe that $c_A(M)=0$ if and only if M is projective. Moreover, if M is periodic, then $c_A(M)=1$.

The following fact is a consequence of the horseshoe lemma (see [4, Lemma 2.5.1]).

Lemma 3.1. *Let A be a self-injective algebra such that all simple modules in* $\operatorname{mod} A$ *are periodic. Then every non-projective indecomposable module in* $\operatorname{mod} A$ *has complexity* 1.

For a self-injective algebra A, we denote by Γ_A^s the stable Auslander–Reiten quiver of A, obtained from the Auslander–Reiten quiver Γ_A of A by removing the projective modules and the arrows attached to them.

Theorem 3.2. *Let A be a symmetric algebra with Γ_A^s having no component of type $\mathbb{Z}\mathbb{A}_\infty$. The following statements are equivalent:*

(i) *All simple modules in* $\operatorname{mod} A$ *are periodic.*
(ii) *All non-projective indecomposable modules in* $\operatorname{mod} A$ *are periodic.*

Proof. The implication (ii)⇒(i) holds trivially. Assume that the statement (i) holds. Let M be a non-projective indecomposable module in $\operatorname{mod} A$. It follows from Lemma 3.1 that $c_A(M)=1$. Since A is a symmetric algebra, the Auslander–Reiten translate $\tau_A(M)$ of M is isomorphic to $\Omega_A^2(M)$ (see [40, Corollary IV.8.3]). Hence $c_A(M)=1$ implies that there is a common bound on the dimensions of indecomposable modules in the τ_A-orbit of M in Γ_A. Then it follows from [31, Theorem 3.9] that the component $\mathcal{C}$ of Γ_A^s containing M is either a stable tube $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ for some $r\geq 1$, or has the form $\mathbb{Z}\mathbb{A}_\infty$. Thus $\mathcal{C}$ is a stable tube $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$, by the assumption on A. But then $\Omega_A^{2r}(M)\cong\tau_A^r(M)\cong M$, and consequently M is periodic. Therefore, (i) implies (ii). ■

We would like to mention that there are symmetric algebras having indecomposable non-periodic finite-dimensional modules of complexity 1. For a non-zero element q of K, Liu and Schulz considered in [32] a symmetric algebra $\Lambda(q)$ of dimension 8 which is given by generators x_0,x_1,x_2 and relations $x_i^2=0$ and $x_{i+1}x_i+qx_ix_{i+1}=0$ for $i=0,1,2$ (where $x_3=x_0$). Moreover, for any q which is not a root of unity, they constructed a 4-dimensional cyclic non-periodic right $\Lambda(q)$-module M such that all syzygies $\Omega^n_{\Lambda(q)}(M)$ are 4-dimensional, and hence $c_{\Lambda(q)}(M)=1$.

We also mention that there are symmetric algebras A such that, for any non-projective indecomposable module M in $\operatorname{mod} A$, $c_A(M) = \infty$. For example, this is the case for all symmetric algebras of wild tilted type (see [19, Theorem 9.4]).

Let now K be of characteristic $p > 0$, G a finite group whose order is divisible by p, and $A = KG$. Recall that the *p-rank* of G is the maximal positive integer r such that G has a subgroup isomorphic to an elementary abelian p-group $\mathbb{Z}_p^r = \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ (r times). By Evens' Theorem [23], if M is any non-projective indecomposable module in $\operatorname{mod} A$, then $H^*(G, M)$ is a noetherian module over the ring $H^*(G, K)$, and the latter is a finitely generated graded K-algebra. Then we have the following consequence of this result and a result due to Quillen [34], [35] (see [19, Theorem 9.10]).

THEOREM 3.3. *Let K be of characteristic $p > 0$, G a finite group whose order is divisible by p, and M a non-projective indecomposable module in* $\operatorname{mod} KG$. *Then the complexity $c_{KG}(M)$ exists and is bounded by the p-rank of G.*

The following fact due to Eisenbud [14] is well known (see also [5, Theorem 5.10.4] for a proof due to Carlson).

THEOREM 3.4. *Let K be of characteristic $p > 0$, G a finite group whose order is divisible by p, and M a non-projective indecomposable module in* $\operatorname{mod} KG$. *Then $c_{KG}(M) = 1$ if and only if M is a periodic module.*

The following proposition will be essential for the proof of Theorem 1.1.

PROPOSITION 3.5. *Let K be of characteristic $p > 0$, D a p-group, and assume that K is a periodic module in* $\operatorname{mod} KD$. *Then either D is a cyclic group or $p = 2$ and D is a generalized quaternion 2-group.*

Proof. Since D is a p-group, K is the only simple module in $\operatorname{mod} KD$ [4, Lemma 3.14.1]. Then it follows from Lemma 3.1 that every module in $\operatorname{mod} KD$ has complexity at most 1. We claim that D does not have a non-cyclic elementary abelian p-subgroup. Assume, for the contrary, that D contains a subgroup Q isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. Then the trivial module K in KQ has complexity 2. This follows from [8, Proposition 7.5] where a minimal projective resolution for the trivial module K over the group algebra of an elementary abelian p-group over K was constructed. By Green's indecomposablity theorem (see [33, Chapter 4, Section 7]), tensoring a minimal projective resolution of the module K in $\operatorname{mod} KQ$ by $- \otimes_{KQ} KD$, we obtain a minimal projective resolution of the module $K \otimes_{KQ} KD$ in $\operatorname{mod} KD$. This means that the indecomposable module $K \otimes_{KQ} KD$ in $\operatorname{mod} KD$ has complexity at least 2, a contradiction. Therefore, the required claim follows. Then we conclude that either D is cyclic, or $p = 2$ and D is a generalized quaternion 2-group (see [9, Theorem XII.11.6] or [30, Theorem 5.3.7]). ∎

4. Proof of Theorem 1.1. Let K be of characteristic $p > 0$ and G a finite group whose order is divisible by p. Recall that a *vertex* of an indecomposable module M in $\operatorname{mod} KG$ is a minimal p-subgroup Q of G (unique up to conjugation) such that M is a direct summand of $M \otimes_{KQ} KG$ (see [3, III.9]).

We need the following lemma.

LEMMA 4.1. *Let K be of characteristic $p > 0$, G a finite group whose order is divisible by p, and B a block of KG with a defect group D which is normal in G. Then every simple module in* $\operatorname{mod} B$ *is a direct summand of $K \otimes_{KD} KG$ and has vertex D.*

Proof. It is well known that a normal p-subgroup of G acts trivially on any simple module in $\operatorname{mod} KG$, and that it is contained in its vertex (see [33, Theorem 4.7.8]). Hence, for any simple module S in $\operatorname{mod} KD$, the group D acts trivially on S, and D is contained in its vertex. Moreover, a vertex of any module in $\operatorname{mod} B$ is contained in D (see [33, Theorem 5.1.9]). This shows that every simple module S in $\operatorname{mod} B$ has vertex D. Then, since S is trivial as a right KD-module, this implies that S is a direct summand of $K \otimes_{KD} KG$, by definition of a vertex. ∎

Proof of Theorem 1.1. Let B be a block of infinite representation type of the group algebra KG of a finite group G over an algeraically closed field K. Then K is of positive characteristic p dividing the order of G, and the defect group $D = D_B$ of B is a non-cyclic p-subgroup of G.

The implication (iii)⇒(i) follows from Corollary 2.4. Furthermore, the implication (ii)⇒(iii) follows from [20, Proposition 5.8 and Theorem 5.9] and the fact that periodicity of algebras is invariant under derived equivalences [37], [21]. Therefore, it remains to show that (i) implies (ii).

Assume that all simple modules in $\operatorname{mod} B$ are periodic.

CASE 1. Assume that D is a normal subgroup of G. Then, by Lemma 4.1, all simple modules in $\operatorname{mod} B$ occur as direct summands of the module $K \otimes_{KD} KG$. As well, if a module M in $\operatorname{mod} KG$ is a direct summand of $K \otimes_{KD} KG$, then $\Omega^d_{KG}(M)$ is a direct summand of $\Omega^d_{KG}(K) \otimes_{KD} KG$, for any $d \geq 1$. Let S be a simple module in $\operatorname{mod} B$. Then $\Omega^d_{KG}(S) \cong S$ for some positive integer d. In particular, $\Omega^d_{KG}(S)$ is simple and hence a direct summand of $K \otimes_{KD} KG$, and D acts trivially on $\Omega^d_{KG}(S)$. On the other hand, $\Omega^d_{KG}(S)$ is a direct summand of $\Omega^d_{KD}(K) \otimes_{KD} KG$. This implies that D acts trivially on $\Omega^d_{KD}(K)$. Since $\Omega^d_{KD}(K)$ is indecomposable, we conclude that $\Omega^d_{KD}(K) \cong K$ in $\operatorname{mod} KD$, that is, K is a periodic module in $\operatorname{mod} KD$. Then it follows from Proposition 3.5 (and the assumption on D) that $p = 2$ and D is a generalized quaternion 2-group.

General case. We use general theory to reduce to Case 1. Let $N = N_G(D)$ be the normalizer of D in G. By Brauer's First Main Theorem there is a bijection, called *Brauer correspondence*, between the blocks of KG with defect group D and the blocks of KN with defect group D (see [4, Theorem 6.2.6] or [33, Theorem 5.2.15]). Let b be the block of KN which is the Brauer correspondent of B. Then there is the *Green correspondence* between the indecomposable modules in $\operatorname{mod} B$ with vertex D and the indecomposable modules in $\operatorname{mod} b$ with vertex D (see [3, Theorem 11.1]). It is given as taking direct summands of an induced module from $\operatorname{mod} KN$ to $\operatorname{mod} KG$, or a restricted module from $\operatorname{mod} KG$ to $\operatorname{mod} KN$. As well, induction and restriction takes projective modules to projective modules. This implies that the Green correspondence preserves the complexity of modules. It follows from Lemma 3.1 that all non-projective indecomposable modules in $\operatorname{mod} B$ have complexity 1. Hence we conclude that all non-projective indecomposable modules in $\operatorname{mod} b$ with vertex D have complexity 1. Further, by Lemma 4.1, all simple modules in $\operatorname{mod} b$ are direct summands of the module $K \otimes_{KD} KN$ and have vertex D. This implies that all simple modules in $\operatorname{mod} b$ have complexity 1, and consequently are periodic, by Theorem 3.4. Since D is a normal subgroup of N, and D is a defect group of b, we conclude from Case 1 that $p = 2$ and D is a generalized quaternion 2-group. This completes the proof of Theorem 1.1. ■

Acknowledgments. Both authors gratefully acknowledge support from the research grant DEC-2011/02/A/ST1/00216 of the Polish National Science Center.

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Received 22 October 2014;
revised 5 January 2015 (6413)