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# ON $E$-$S$-SUPPLEMENTED SUBGROUPS OF FINITE GROUPS

BY

CHANGWEN LI (Xuzhou), XUEMEI ZHANG (Yancheng) and
XIAOLAN YI (Hanzhou)

**Abstract.** The major aim of the present paper is to strengthen a nice result of Shemetkov and Skiba which gives some conditions under which every non-Frattini $G$-chief factor of a normal subgroup $E$ of a finite group $G$ is cyclic. As applications, some recent known results are generalized and unified.

**1. Introduction.** All groups considered in this paper will be finite. Most of the notation is standard and can be found in [4] and [13]. $G$ always denotes a group, $|G|$ is the order of $G$, $O_p(G)$ is the maximal normal $p$-subgroup of $G$, and $F^*(G)$ is the generalized Fitting subgroup of $G$, i.e., the product of all normal quasinilpotent subgroups of $G$. The symbol $\mathcal{U}$ denotes the class of all supersoluble groups. Clearly, $\mathcal{U}$ is a saturated formation.

Two subgroups $A$ and $B$ of a group $G$ are said to be *permutable* if $AB = BA$. A subgroup $H$ of $G$ is said to be *$S$-permutable* or *$S$-quasinormal* in $G$ if $H$ permutes with every Sylow subgroup of $G$ (see [6]). There are many interesting generalizations of $S$-permutability in the literature. For example, Ballester-Bolinches and Pedraza-Aguilera [2] called $H$ *$S$-permutably embedded* in $G$ if for each prime $p$ dividing $|H|$, a Sylow $p$-subgroup of $H$ is also a Sylow $p$-subgroup of some $S$-permutable subgroup of $G$. Again, Skiba [18] called $H$ *weakly $S$-permutable* in $G$ if there is a subnormal subgroup $T$ of $G$ such that $G = HT$ and $H \cap T \leq H_{sG}$, where $H_{sG}$ is the subgroup of $H$ generated by all those subgroups of $H$ which are $S$-permutable in $G$.

We introduce the following concept, which covers both weak $S$-permutability and $S$-permutable embeddability.

Definition 1.1. A subgroup $H$ of a subgroup $G$ is said to be *$E$-$S$-supplemented* in $G$ if there exists a subnormal subgroup $T$ of $G$ such that $G = HT$ and $H \cap T \leq H_{eG}$, where $H_{eG}$ denotes the subgroup of $H$ generated by all those subgroups of $H$ which are $S$-permutably embedded in $G$.

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Example 1.2. Suppose that $G = A_5$, the alternating group of degree 5. Then the Sylow 2-subgroups of $G$ are $E$-$S$-supplemented in $G$, but not weakly $S$-permutable in $G$.

Example 1.3. Suppose that $G = S_4$, the symmetric group of degree 4. Consider the subgroup $H = \langle (3, 4) \rangle$. Then $H$ is $E$-$S$-supplemented in $G$, but not $S$-permutably embedded in $G$.

In [17], Skiba improved [15, Theorem 1.4] by replacing non-Frattini $G$-chief factor with $G$-chief factor. In this paper, we further weaken the hypotheses of Skiba's result from weak $S$-permutability to being $E$-$S$-supplemented and get the following theorem.

Theorem 1.4. *Let $E$ be a normal subgroup of a group $G$ and $X \leq E$. Suppose that for every noncyclic Sylow subgroup $P$ of $X$, there is an integer $n_p$ such that $1 < n_p < |P|$ and every subgroup $H$ of $P$ of order $n_p$, as well as every order 4 cyclic subgroup of $P$ (when $n_p = 2$ and $P$ is a nonabelian 2-group), is $E$-$S$-supplemented in $G$. If $X = E$ or $X = F^*(E)$, then every $G$-chief factor of $E$ is cyclic.*

We shall prove Theorem 1.4 in Section 4. The following useful fact is an important stage in that proof.

Theorem 1.5. *Let $P$ be a Sylow $p$-subgroup of a group $G$, where $p$ is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Suppose that there is an integer $n_p$ such that $1 < n_p < |P|$ and every subgroup $H$ of $P$ of order $n_p$, and every cyclic subgroup of $P$ of order 4 (if $P$ is a nonabelian 2-group and $n_p = 2$), without a $p$-nilpotent supplement in $G$ is $E$-$S$-supplemented in $G$. Then $G$ is $p$-nilpotent.*

## 2. Preliminaries

Lemma 2.1. *Suppose that $H$ is $S$-permutably embedded in a group $G$, $L \leq G$ and $N \trianglelefteq G$.*

1. *If $H \leq L$, then $H$ is $S$-permutably embedded in $L$.*
2. *The subgroup $HN$ is $S$-permutably embedded in $G$ and $HN/N$ is $S$-permutably embedded in $G/N$.*
3. *If $H$ is a $p$-subgroup of $G$ contained in $O_p(G)$, then $H$ is $S$-permutable in $G$.*

*Proof.* (1) and (2) are from [2, Lemma 1]; (3) is [10, Lemma 2.4]. ∎

Lemma 2.2. *Suppose that $H$ is $E$-$S$-supplemented in a group $G$.*

1. *If $H \leq L \leq G$, then $H$ is $E$-$S$-supplemented in $L$.*
2. *If $N \trianglelefteq G$ and $N \leq H \leq G$, then $H/N$ is $E$-$S$-supplemented in $G/N$.*
3. *If $H$ is a $\pi$-subgroup and $N$ is a normal $\pi'$-subgroup of $G$, then $HN/N$ is $E$-$S$-supplemented in $G/N$.*

(4) *Suppose $H$ is a $p$-group for some prime $p$ and $H \neq H_{eG}$. Then $G$ has a normal subgroup $M$ such that $|G : M| = p$ and $G = HM$.*
(5) *If $H \leq O_p(G)$ for some prime $p$, then $H$ is weakly $S$-permutable in $G$.*

*Proof.* By the hypothesis, there exists a subnormal subgroup $K$ of $G$ such that $G = HK$ and $H \cap K \leq H_{eG}$.

(1) We have

$$L = L \cap HK = H(L \cap K) \quad \text{and} \quad H \cap (L \cap K) = H \cap K \leq H_{eG}.$$

Let $U_1, \dots, U_s$ be all the subgroups of $H$ which are $S$-permutably embedded in $G$. By Lemma 2.1(1), they are $S$-permutably embedded in $L$ and so $H_{eG} \leq H_{eL}$. Obviously, $L \cap K$ is subnormal in $L$. Hence $H$ is $E$-$S$-supplemented in $L$.

(2) We have

$$G/N = HK/N = H/N \cdot NK/N$$

and

$$(H/N) \cap (KN/N) = (H \cap KN)/N = (H \cap K)N/N \leq H_{eG}N/N.$$

Let $U_1, \dots, U_s$ be all the subgroups of $H$ which are $S$-permutably embedded in $G$. By Lemma 2.1(2), $U_1N/N, \dots, U_sN/N$ are $S$-permutably embedded in $G/N$ and so $H_{eG}N/N \leq (H/N)_{e(G/N)}$. Obviously, $KN/N$ is subnormal in $G/N$. Hence $H/N$ is $E$-$S$-supplemented in $G/N$.

(3) Since $(|G : K|, |N|) = 1$, we have $N \leq K$. It is easy to see that

$$G/N = HN/N \cdot KN/N = HN/N \cdot K/N$$

and

$$\begin{aligned}(HN/N) \cap (K/N) = (HN \cap K)/N &= (H \cap K)N/N \leq H_{eG}N/N \\ &\leq (HN/N)_{e(G/N)}.\end{aligned}$$

Obviously, $K/N$ is subnormal in $G/N$. Hence $HN/N$ is $E$-$S$-supplemented in $G/N$.

(4) If $K = G$, then $H = H \cap K \leq H_{eG} \leq H$, and so $H = H_{eG}$, contrary to the hypotheses. Consequently, $K$ is a proper subgroup of $G$. Hence, $G$ has a proper normal subgroup $B$ such that $K \leq B$. Since $G/B$ is a $p$-group, $G$ has a normal maximal subgroup $M$ such that $|G : M| = p$ and $G = MH$.

(5) This follows from Lemma 2.1(3). ∎

Lemma 2.3 ([21, Lemma 2.2]). *Let $G$ be a group and $p$ a prime dividing $|G|$ with $(|G|, p - 1) = 1$.*

(1) *If $N$ is normal in $G$ of order $p$, then $N$ lies in $Z(G)$.*
(2) *If $G$ has cyclic Sylow $p$-subgroups, then $G$ is $p$-nilpotent.*
(3) *If $M \leq G$ and $|G : M| = p$, then $M \trianglelefteq G$.*

Lemma 2.4. *Let $P$ be a noncyclic Sylow $p$-subgroup of a group $G$, where $p$ is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every maximal subgroup of $P$ has a $p$-nilpotent supplement in $G$, then $G$ is $p$-nilpotent.*

*Proof.* Let $M_1T_1 = G$ where $T_1$ is $p$-nilpotent and $M_1$ is maximal in $P$. We can assume that $T_1 = N_G(H_1)$ for some Hall $p'$-subgroup $H_1$ of $G$. Clearly, $P = M_1(P \cap T_1)$.

Suppose that $P \cap T_1 \neq P$. Then we can choose a maximal subgroup $M_2$ in $P$ containing $P \cap T_1$. By assumption, $G = M_2T_2$ where $T_2$ is $p$-nilpotent. Again, we can assume that $T_2 = N_G(H_2)$ for some Hall $p'$-subgroup $H_2$ of $G$. If $p = 2$, then $H_1$ and $H_2$ are conjugate in $G$ by applying a deep result of Gross. If $p > 2$, then $G$ is a soluble group by the Feit–Thompson Theorem and so $H_1$ and $H_2$ are also conjugate in $G$. Then we have $H_1^x = H_2$ for some $x \in G$. Therefore, $G = M_1T_1 = M_2T_2 = M_2T_1^x = M_2T_1$ and $P = M_2(P \cap T_1) = M_2$, a contradiction.

Hence $P \cap T_1 = P$, which implies the $p$-nilpotency of $G$. ∎

Lemma 2.5 ([15, Lemma 2.6]). *Let $V$ be an $S$-permutable subgroup of order 4 of a group $G$.*

(1) *If $V = A \times B$, where $|A| = |B| = 2$ and $A$ is $S$-permutable in $G$, then $B$ is $S$-permutable in $G$.*
(2) *If $V = \langle x \rangle$ is cyclic, then $\langle x^2 \rangle$ is $S$-permutable in $G$.*

Lemma 2.6. *Let $G$ be a group and $P$ a Sylow $p$-subgroup of $G$, where $p$ is a prime dividing $|G|$ with $(|G|, p-1) = 1$. If every cyclic subgroup of $P$ of prime order or of order 4 (when $P$ is a nonabelian 2-group) without a $p$-nilpotent supplement in $G$ is $E$-$S$-supplemented in $G$, then $G$ is $p$-nilpotent.*

*Proof.* In view of Lemma 2.3(2), this easily follows from the proof of [7, Theorem 3.3]. ∎

Lemma 2.7. *Let $P$ be a Sylow $p$-subgroup of a group $G$, where $p$ is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every maximal subgroup of $P$ without a $p$-nilpotent supplement in $G$ is $E$-$S$-supplemented in $G$, then $G$ is $p$-nilpotent.*

*Proof.* In view of Lemma 2.3(2), this easily follows from the proof of [7, Theorem 3.2]. ∎

Lemma 2.8 ([14, Lemma A]). *If $P$ is an $S$-quasinormal $p$-subgroup of a group $G$ for some prime $p$, then $N_G(P) \geq O^p(G)$.*

Lemma 2.9 ([16, Theorem C]). *Let $E$ be a normal subgroup of a group $G$. If every $G$-chief factor of $F^*(E)$ is cyclic, then every $G$-chief factor of $E$ is cyclic.*

LEMMA 2.10 ([3, IV, 3.11]). *If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two saturated formations such that $\mathcal{F}_1 \subseteq \mathcal{F}_2$, then $Z_{\mathcal{F}_1}(G) \leq Z_{\mathcal{F}_2}(G)$.*

**3. Proof of Theorem 1.5.** Suppose that the theorem is false and let $G$ be a counterexample of minimal order. We will derive a contradiction in several steps.

(1) $n_p > p$.

This follows from Lemma 2.6.

(2) $|P|/n_p > p$.

This follows from Lemma 2.7.

(3) *$G$ has no subgroup of index $p$.*

Suppose that $G$ has a subgroup $M$ such that $|G : M| = p$. Then $M$ satisfies the hypotheses of the theorem by Step (2) and Lemma 2.2(1). The choice of $G$ guarantees that $M$ is $p$-nilpotent. By Lemma 2.3(3), $M \trianglelefteq G$. It follows that $G$ is $p$-nilpotent, a contradiction.

(4) *If $H$ is a subgroup of $P$ with $|H| = n_p$, then either $H$ has a $p$-nilpotent supplement in $G$, or $H = H_{eG}$.*

Let $H < P$ with $|H| = n_p$. Assume that $H$ has no $p$-nilpotent supplement in $G$. If $H \neq H_{eG}$, then we may assume $G$ has a normal subgroup $M$ such that $|G : M| = p$ and $G = HM$ by Lemma 2.2(4), contrary to Step (3).

(5) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, Lemma 2.2(3) guarantees that $G/O_{p'}(G)$ satisfies the hypotheses of the theorem. Thus $G/O_{p'}(G)$ is $p$-nilpotent by the choice of $G$. Then $G$ is $p$-nilpotent, a contradiction.

(6) *If $H$ is a subgroup of $P$ with $|H| = n_p$, then either $H$ has a $p$-nilpotent supplement in $G$, or $H$ is $S$-permutable in $G$.*

Let $H$ be a subgroup of $P$ of order $n_p$ without a $p$-nilpotent supplement in $G$. By Step (4), $H = H_{eG}$. Let $U_1, \ldots, U_s$ be all the nontrivial subgroups of $H$ which are $S$-permutably embedded in $G$. For every $i \in \{1, \ldots, s\}$, there is an $S$-permutable subgroup $K_i$ of $G$ such that $U_i$ is a Sylow $p$-subgroup of $K_i$. Obviously, $K_i \neq G$. Suppose that for some $i \in \{1, \ldots, s\}$, we have $G = PK_i$. Then $|G : K_i|$ is $p$-power. From the $S$-permutability of $K_i$, we get $K_i \lhd \lhd G$. It follows that $G$ has a normal subgroup of index $p$, contrary to Step (3). Thus, for all $i \in \{1, \ldots, s\}$, we have $G > PK_i$. Then $PK_i$ satisfies the hypotheses of the theorem by Lemma 2.2(1). From the choice of $G$, $PK_i$ is $p$-nilpotent and so $K_i$ is $p$-nilpotent. Let $K_{ip'}$ be a normal $p$-complement of $K_i$. By Step (5), $K_{ip'} \leq O_{p'}(G) = 1$, which shows that $U_i = K_i$, and so $H$ is $S$-permutable in $G$.

(7) *Suppose $N$ is a minimal normal subgroup of $G$ contained in $P$. Then $|N| \leq n_p$.*

Suppose that $|N| > n_p$. Since $N \leq O_p(G)$, $N$ is elementary abelian. If a subgroup $H$ of $N$ of order $n_p$ has a $p$-nilpotent supplement $T$ in $G$, then $G = HT = NT$. Hence $N \cap T \trianglelefteq G$. By the minimality of $N$, either $N \cap T = 1$ or $N \cap T = N$. If $N \cap T = 1$, then $N = N \cap HT = H(N \cap T) = H$, a contradiction. Thus $N \cap T = N$ and $G = NT = T$, also a contradiction. Hence we may chose a subgroup $H$ of $N$ of order $n_p$ such that $H \trianglelefteq P$. In view of Lemma 2.8, $H \trianglelefteq PO^p(G) = G$, contrary to the minimality of $N$.

(8) *Suppose that $p = 2$, $|P|/n_p > 2$ and some subgroup $H$ of $P$ of order 4 has a 2-nilpotent supplement $T$ in $G$. Then $H$ is not cyclic, $G/T_G \cong A_4$, no subgroup of $H$ of order 2 is $S$-permutable in $G$, and $T_G$ is a 2-group.*

In view of Step (3), $|G : T| = 4$. By considering the permutation representation of $G/T_G$ on the right cosets of $T/T_G$ one can see that $G/T_G$ is isomorphic to some subgroup of the symmetric group $S_4$. But since $G$ does not have a subgroup $M$ with $|G : M| = 2$ by Step (3), we have $G/T_G \cong A_4$. It follows that $H \cong HT_G/T_G$ is not cyclic. Since $O_{2'}(G) = 1$ by Step (5), we deduce that $O_{2'}(T_G) = 1$. Hence $T_G$ is a 2-group. Suppose that some subgroup $V$ of $H$ of order 2 is $S$-permutable in $G$ and let $Q$ be a Sylow 3-subgroup of $T$. Then $V \leq N_G(Q)$. On the other hand, since $T$ is 2-nilpotent and $|T| = 2^n 3$, we have $T \leq N_G(Q)$. Hence $|G : N_G(Q)| = 2$, a contradiction.

(9) *If $P$ is a nonabelian 2-group and $|P|/n_p > 2$, then $n_p > 4$.*

Since $P$ is a nonabelian 2-group, it has a cyclic subgroup $H = \langle x \rangle$ of order 4. Suppose that $n_p = 4$. Then from Step (6) and $|P|/n_p > 2$ we know that every subgroup of $P$ of order 4 without a 2-nilpotent supplement in $G$ is $S$-permutable in $G$. Hence in view of Step (8), $H$ is $S$-permutable in $G$. Then by Lemma 2.5(2), $\langle x^2 \rangle$ is $S$-permutable in $G$. Now note that if $G$ has a subgroup $V = A \times B$ of order 4, where $|A| = 2$ and $A$ is $S$-permutable in $G$, then $V$ and $B$ are $S$-permutable in $G$ by Step (8) and Lemma 2.5(1). Therefore some subgroup $Z$ of $Z(P)$ with $|Z| = 2$ is $S$-permutable in $G$. Hence every subgroup of $P$ of order 2 is $S$-permutable in $G$, which contradicts Step (1).

(10) *If $N$ is an abelian minimal normal subgroup of $G$ contained in $P$, then the hypotheses of Theorem 1.5 are still true for $G/N$.*

If either $p > 2$ and $|N| < n_p$, or $p = 2$ and $2|N| < n_p$, this is clear. So let $p > 2$ and $|N| = n_p$, or $p = 2$ and $|N| \in \{n_p, n_p/2\}$. By Step (6) every subgroup $H$ of $P$ of order $n_p$ without a $p$-nilpotent supplement in $G$ is $S$-permutable in $G$. Moreover, in view of Step (1), $n_p > p$. Suppose that

$|N| = n_p$. Then $N$ is noncyclic and hence every subgroup of $G$ containing $N$ is noncyclic. Let $N \leq K \leq P$, where $|K : N| = p$. Since $K$ is noncyclic, it has a maximal subgroup $L \neq N$. If $L$ or $N$ has a $p$-nilpotent supplement in $G$, then $K$ does. Otherwise, $K = LN$ is $S$-permutable in $G$, as it is the product of two subgroups $S$-permutable in $G$. Thus if either $p > 2$ or $P/N$ is abelian, the hypotheses of the theorem are true for $G/N$ by Lemma 2.2.

Next suppose that $P/N$ is a nonabelian 2-group. Then $P$ is nonabelian, so $n_p > 4$ by Step (9). Let $N \leq K \leq V$ where $|V : N| = 4$ and $|V : K| = 2$. Let $K_1$ be a maximal subgroup of $V$ such that $V = K_1K$. Suppose that $K_1$ is cyclic. Then $N \nleq K_1$, so $V = K_1N$, which implies $|N| = 4$. But then $n_p = 4$, which contradicts Step (9). Hence $K_1$ is noncyclic and hence as above one can show that $K_1$ either is $S$-permutable in $G$ or has a 2-nilpotent supplement in $G$. Therefore every subgroup of $P/N$ of order 2 or 4 without a $p$-nilpotent supplement in $G/N$ is $S$-permutable in $G/N$.

Finally, suppose that $n_p = 2|N|$. If $|N| > 2$, then as above one can show that every subgroup of $P/N$ of order 2 or 4 (if $P/N$ is nonabelian) without a 2-nilpotent supplement in $G/N$ is $S$-permutable in $G/N$. Now, suppose that $|N| = 2$ and $P/N$ is nonabelian. Then $P$ is nonabelian and $n_p = 4$, which contradicts Step (9).

(11) $O_p(G) = 1$.

If $O_p(G) \neq 1$, then $G/O_p(G)$ is $p$-nilpotent by Step (10). This means that $G$ has a subgroup of index $p$, contrary to Step (3).

(12) *If $L$ is a minimal normal subgroup of $G$, then $L$ is not $p$-nilpotent.*

Assume that $L$ is $p$-nilpotent. Let $L_{p'}$ be the normal $p$-complement of $L$. Since $L_{p'}$ *char* $L \trianglelefteq G$, we have $L_{p'} \trianglelefteq G$ and so $L_{p'} \leq O_{p'}(G) = 1$ by Step (5). It follows that $L$ is a $p$-group and so $L \leq O_p(G) = 1$ by Step (11), a contradiction.

(13) *If $L$ is a minimal normal subgroup of $G$, then $G = LP$.*

Obviously, $LP$ satisfies the hypotheses of the theorem. If $LP < G$, then the choice of $G$ implies that $LP$ is $p$-nilpotent. It follows that $L$ is $p$-nilpotent, contrary to Step (12).

(14) *$G$ is a nonabelian simple group.*

Take a minimal normal subgroup $L$ of $G$. If $L < G$, then by Step (13), $G = LP$. Then $G$ has a subgroup of index $p$, contrary to Step (3). Thus $G = L$ is simple.

(15) *The final contradiction.*

If every subgroup $H$ of $P$ of order $n_p$ has a $p$-nilpotent supplement in $G$, then every maximal subgroup of $P$ has a $p$-nilpotent supplement in $G$. It follows that $G$ is $p$-nilpotent by Lemma 2.4, a contradiction. Thus, there is a subgroup $R$ of $P$ of order $n_p$ that is $S$-permutable in $G$ by Step (6). The subnormality of $R$ implies that $G$ is not simple, contrary to Step (14).

## 4. Proof of Theorem 1.4

Case I: $X = E$. Suppose that the theorem is false and consider a counterexample $(G, E)$ for which $|G|\,|E|$ is minimal. Let $P$ be a Sylow $p$-subgroup of $E$, where $p$ is the smallest prime dividing $|E|$.

(1) *If $K$ is a Hall subgroup of $E$, the hypotheses of Theorem 1.4 are still true for $(K, K)$. Moreover, if $K$ is normal in $G$, then the hypotheses also hold for $(G, K)$ and for $(G/K, E/K)$.*

This follows directly from Lemma 2.2.

(2) *If $K$ is a nonidentity normal Hall subgroup of $E$, then $K = E$.*

Since $K$ is a characteristic subgroup of $E$, it is normal in $G$ and by Step (1) the hypotheses are still true for $(G/K, E/K)$ and $(G, K)$. If $K \neq E$, the minimal choice of $(G, E)$ implies that $E/K \leq Z_{\mathcal{U}}(G/K)$ and $K \leq Z_{\mathcal{U}}(G)$. Hence $E \leq Z_{\mathcal{U}}(G)$, a contradiction.

(3) *If $E \neq P$, then $E$ is not $p$-nilpotent.*

Indeed, if $E$ is $p$-nilpotent, then by Step (2), $p$ does not divide $|E|$, contrary to the choice of $p$.

(4) *$P$ is not cyclic.*

This follows from Step (3) and Lemma 2.3(2).

(5) $E = P$.

By Lemma 2.2(1), every subgroup $H$ of $P$ of order $n_p$, as well as every order 4 cyclic subgroup of $P$ (when $n_p = 2$ and $P$ is a nonabelian 2-group), is $E$-$S$-supplemented in $E$. By Theorem 1.5, $E$ is $p$-nilpotent. By Step (3), $E = P$.

(6) *Every subgroup $H$ of $P$ of order $n_p$, as well as every order 4 cyclic subgroup of $P$ (when $n_p = 2$ and $P$ is a nonabelian 2-group), is weakly $S$-permutable in $E$.*

This follows from Lemma 2.2(5).

(7) *The final contradiction.*

By the Theorem in [17], each $G$-chief factor below $E$ is cyclic, a contradiction.

Case II: $X = F^*(E)$. The proof in the case $X = E$ shows that $F^*(E) \leq Z_{\mathcal{U}}(G)$, which implies that $E \leq Z_{\mathcal{U}}(G)$ by Lemma 2.9.

**5. Some applications.** From Theorem 1.5, we obtain the following statement.

THEOREM 5.1. *Let $P$ be a Sylow $p$-subgroup of a group $G$, where $p$ is the smallest prime divisor of $|G|$. Suppose that there is an integer $n_p$ such that $1 < n_p < |P|$ and every subgroup $H$ of $P$ of order $n_p$, and every cyclic subgroup of $P$ of order $4$ (if $P$ is a nonabelian $2$-group and $n_p = 2$), without a $p$-nilpotent supplement in $G$ is $E$-$S$-supplemented in $G$. Then $G$ is $p$-nilpotent.*

COROLLARY 5.2 ([11, Theorem 3.2]). *Let $p$ be a prime dividing the order of a group $G$ with $(|G|, p-1) = 1$. If there exists a Sylow $p$-subgroup $P$ of $G$ such that every maximal subgroup of $P$ is weakly $s$-permutable in $G$, then $G$ is $p$-nilpotent.*

COROLLARY 5.3 ([2, Theorem 1]). *Let $P$ be a Sylow $p$-subgroup of a group $G$, where $p$ is the smallest prime divisor of $|G|$. If every maximal subgroup of $P$ is $S$-permutably embedded in $G$, then $G$ is $p$-nilpotent.*

From [19], we know that a subgroup $H$ of a group $G$ is $c$-normal in $G$ if $G$ has a normal subgroup $T$ such that $G = HT$ and $H \cap T \leq H_G$.

COROLLARY 5.4 ([5, Theorem 3.4]). *Let $P$ be a Sylow $p$-subgroup of a group $G$, where $p$ is the smallest prime divisor of $|G|$. If every maximal subgroup of $P$ is $c$-normal in $G$, then $G$ is $p$-nilpotent.*

COROLLARY 5.5 ([8, Theorem 3.1]). *Let $P$ be a Sylow $p$-subgroup of a group $G$, where $p$ is the smallest prime divisor of $|G|$. If every maximal subgroup of $P$ is weakly $S$-permutably embedded in $G$, then $G$ is $p$-nilpotent.*

THEOREM 5.6. *Let $\mathcal{F}$ be a saturated formation containing all supersoluble groups and let $X \leq E$ be a normal subgroup of a group $G$ such that $G/E \in \mathcal{F}$. Suppose that for every noncyclic Sylow subgroup $P$ of $X$, there is an integer $n_p$ such that $1 < n_p < |P|$ and every subgroup $H$ of $P$ of order $n_p$, as well as every order $4$ cyclic subgroup of $P$ (when $n_p = 2$ and $P$ is a nonabelian $2$-group), is $E$-$S$-supplemented in $G$. If $X$ is either $E$ or $F^*(E)$, then $G \in \mathcal{F}$.*

*Proof.* Since $E \leq Z_{\mathcal{U}}(G)$ by Theorem 1.4 and $Z_{\mathcal{U}}(G) \leq Z_{\mathcal{F}}(G)$ by Lemma 2.10, we have $E \leq Z_{\mathcal{F}}(G)$ and so $G/Z_{\mathcal{F}}(G) \cong (G/E)/(Z_{\mathcal{F}}(G)/E) \in \mathcal{F}$. It follows that $G \in \mathcal{F}$. ∎

COROLLARY 5.7 ([12, Theorem 3.3]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. If there is a normal subgroup $H$ of a group $G$ such that $G/H \in \mathcal{F}$ and every maximal subgroup of any Sylow subgroup of $H$ is $c$-normal in $G$, then $G \in \mathcal{F}$.*

Corollary 5.8 ([12, Theorem 3.9]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Then $G \in \mathcal{F}$ if and only if there is a normal subgroup $H$ of $G$ such that $G/H \in \mathcal{F}$ and all subgroups of $H$ of prime order or of order 4 are c-normal in $G$.*

Corollary 5.9 ([22, Theorem 3.1]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a normal subgroup $H$ such that $G/H \in \mathcal{F}$. If all maximal subgroups of any Sylow subgroup of $F^*(H)$ are c-normal in $G$, then $G \in \mathcal{F}$.*

Corollary 5.10 ([20, Theorem 1]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that $G$ is a group with a solvable normal subgroup $H$ such that $G/H \in \mathcal{F}$. If all maximal subgroups of all Sylow subgroups of $F(H)$ are c-normal in $G$, then $G \in \mathcal{F}$.*

Corollary 5.11 ([9, Theorem 1.1]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let $G$ be a group. If there is a normal subgroup $H$ such that $G/H \in \mathcal{F}$ and all maximal subgroups of any Sylow subgroup of $F^*(H)$ are $S$-permutably embedded in $G$, then $G \in \mathcal{F}$.*

Corollary 5.12 ([9, Theorem 1.2]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let $G$ be a group. If there is a normal subgroup $H$ such that $G/H \in \mathcal{F}$ and all subgroups of $F^*(H)$ of prime order or of order 4 are $S$-permutably embedded in $G$, then $G \in \mathcal{F}$.*

Corollary 5.13 ([1, Theorem 3.3]). *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$ and let $G$ be a group. If there is a normal subgroup $H$ such that $G/H \in \mathcal{F}$ and all maximal subgroups of all Sylow subgroups of $H$ are $S$-quasinormally embedded in $G$, then $G \in \mathcal{F}$.*

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Changwen Li
School of Mathematics and Statistics
Jiangsu Normal University
Xuzhou, 221116, China
E-mail: lcw2000@126.com

Xuemei Zhang
Department of Basic Sciences
Yancheng Institute of Technology
Yancheng, 224051, China
E-mail: xmzhang807@sohu.com

Xiaolan Yi
School of Science
Zhejiang Sci-Tech University
Hanzhou, 310018, China
E-mail: yixiaolan2005@126.com

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