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SYMMETRY CLASSES OF TENSORS ASSOCIATED WITH THE SEMI-DIHEDRAL GROUPS SD_{8n}

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Abstract. We discuss the existence of an orthogonal basis consisting of decomposable vectors for all symmetry classes of tensors associated with semi-dihedral groups SD_{8n}. In particular, a necessary and sufficient condition for the existence of such a basis associated with SD_{8n} and degree two characters is given.

1. Introduction. Let V be an n-dimensional complex inner product space and G be a permutation group on m elements. Let χ be any irreducible character of G. For any $\sigma \in G$, define the operator

$$P_\sigma : \bigotimes_1^m V \to \bigotimes_1^m V$$

by

$$P_\sigma(v_1 \otimes \cdots \otimes v_m) = (v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(m)}). \tag{1.1}$$

The symmetry class of tensors associated with G and χ is the image of the symmetry operator

$$T(G,\chi) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) P_\sigma, \tag{1.2}$$

and it is denoted by $V_\chi^m(G)$. We say that the tensor $T(G,\chi)(v_1 \otimes \cdots \otimes v_m)$ is a *decomposable symmetrized tensor*, and we denote it by $v_1 * \cdots * v_m$.

The inner product on V induces an inner product on $V_\chi(G)$ which satisfies

$$\langle v_1 * \cdots * v_m, u_1 * \cdots * u_m \rangle = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^m \langle v_i, u_{\sigma(i)} \rangle.$$

Let Γ_n^m be the set of all sequences $\alpha = (\alpha_1, \ldots, \alpha_m)$, with $1 \le \alpha_i \le n$. Define the action of G on Γ_n^m by

$$\sigma.\alpha = (\alpha_{\sigma^{-1}(1)}, \ldots, \alpha_{\sigma^{-1}(m)}).$$

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Let $O(\alpha) = \{\sigma.\alpha \mid \sigma \in G\}$ be the orbit of α. We write $\alpha \sim \beta$ if α and β belong to the same orbit in Γ_n^m. Let Δ be a system of distinct representatives of the orbits. We denote by G_α the *stabilizer subgroup* of α, i.e., $G_\alpha = \{\sigma \in G \mid \sigma.\alpha = \alpha\}$. Define

$$\Omega = \Big\{\alpha \in \Gamma_n^m \;\Big|\; \sum_{\sigma \in G_\alpha} \chi(\sigma) \neq 0\Big\},$$

and put $\overline{\Delta} = \Delta \cap \Omega$.

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of V, and denote by e_α^* the tensor $e_{\alpha_1} * \cdots * e_{\alpha_m}$. We have

$$\langle e_\alpha^*, e_\beta^* \rangle = \begin{cases} 0 & \text{if } \alpha \nsim \beta, \\ \dfrac{\chi(1)}{|G|} \displaystyle\sum_{\sigma \in G_\beta} \chi(\sigma h^{-1}) & \text{if } \alpha = h.\beta. \end{cases}$$

In particular, for $\sigma_1, \sigma_2 \in G$ and $\gamma \in \overline{\Delta}$ we obtain

$$\langle e_{\sigma_1.\gamma}^*, e_{\sigma_2.\gamma}^* \rangle = \frac{\chi(1)}{|G|} \sum_{x \in \sigma_2 G_\gamma \sigma_1^{-1}} \chi(x). \tag{1.3}$$

Moreover, $e_\alpha^* \neq 0$ if and only if $\alpha \in \Omega$.

For $\alpha \in \overline{\Delta}$, $V_\alpha^* = \langle e_{\sigma.\alpha}^* : \sigma \in G \rangle$ is called the *orbital subspace* of $V_\chi(G)$. It follows that

$$V_\chi(G) = \bigoplus_{\alpha \in \overline{\Delta}} V_\alpha^*$$

is an orthogonal direct sum. In [9] it is proved that

$$\dim V_\alpha^* \;=\; \frac{\chi(1)}{|G_\alpha|} \sum_{\sigma \in G_\alpha} \chi(\sigma). \tag{1.4}$$

Thus we deduce that if χ is a linear character, then $\dim V_\alpha^* = 1$ and in this case the set

$$\{e_\alpha^* \mid \alpha \in \overline{\Delta}\}$$

is an orthogonal basis of $V_\chi(G)$.

A basis which consists of decomposable symmetrized tensors e_α^* is called an *orthogonal $*$-basis*. If χ is not linear, it is possible that $V_\chi(G)$ has no orthogonal $*$-basis. The reader can find further information about the symmetry classes of tensors in [1–8], [10–11], [13–15] and [17].

In this paper we discuss the existence of an orthogonal basis consisting of decomposable vectors for all symmetry classes of tensors associated with semi-dihedral groups SD_{8n}.

2. Semi-dihedral groups SD_{8n}. The presentation for SD_{8n} for $n \geq 2$ is given by

$$SD_{8n} = \langle a, b \mid a^{4n} = b^2 = 1,\ bab = a^{2n-1} \rangle,$$

where the embedding of SD_{8n} into the symmetric group S_{4n} is given by $T(a)(t) := \overline{t+1}$ and $T(b)(t) := \overline{(2n-1)t}$, where $\overline{m}$ is the remainder of m divided by $4n$.

Definition 2.1. Define

$$\begin{aligned} C_1 &:= \{0, 2, 4, \ldots, 2n\}, \\ C_2 &:= \{1, 3, 5, \ldots, n\} \cup \{2n+1, 2n+3, 2n+5, \ldots, 3n\}, \\ C^{\dagger}_{\text{even}} &:= \{2, 4, \ldots, 2n-2\}, \\ C^{\dagger}_{\text{odd}} &= \{1, 3, 5, \ldots, 2[n/2]-1, 2n+1, 2n+3, \ldots, 2[3n/2]-1\}. \end{aligned}$$

We define two-dimensional representations, for each natural number h and $\omega = e^{\frac{i\pi}{2n}}$:

$$\rho^h(a^r) = \begin{pmatrix} \omega^{hr} & 0 \\ 0 & \omega^{(2n-1)hr} \end{pmatrix} \quad \text{and} \quad \rho^h(ba^r) = \begin{pmatrix} 0 & \omega^{(2n-1)hr} \\ \omega^{hr} & 0 \end{pmatrix}, \tag{2.1}$$

for each $r \in \{1, 2, \ldots, 4n\}$.

Denote $\chi_h = \mathrm{Tr}(\rho^h)$. The non-linear irreducible complex characters of SD_{8n} are the characters χ_h where $h \in C^{\dagger}_{\text{even}}$ or $h \in C^{\dagger}_{\text{odd}}$. Since the numbers of conjugacy classes of SD_{8n} are different for n even ($2n+3$ classes) and n odd ($2n+6$ classes), we consider the corresponding two non-linear character tables separately.

Table I. The non-linear character table for SD_{8n}, n even

Conjugacy classes → Characters ↓	$[a^r]$, $r \in C_1$	$[a^r]$, $r \in C^{\dagger}_{\text{odd}}$	$[b]$	$[ba]$
χ_h, $h \in C^{\dagger}_{\text{even}}$	$2\cos\left(\frac{hr\pi}{2n}\right)$	$2\cos\left(\frac{hr\pi}{2n}\right)$	0	0
χ_h, $h \in C^{\dagger}_{\text{odd}}$	$2\cos\left(\frac{hr\pi}{2n}\right)$	$2i\sin\left(\frac{hr\pi}{2n}\right)$	0	0

Table II. The non-linear character table for SD_{8n}, n odd

Conjugacy classes → Characters ↓	$[a^r]$, $r \in C_1$	$[a^r]$, $r \in C_2$	$[b]$	$[ba]$	$[ba^2]$	$[ba^3]$
χ_h, $h \in C^{\dagger}_{\text{even}}$	$2\cos\left(\frac{hr\pi}{2n}\right)$	$2\cos\left(\frac{hr\pi}{2n}\right)$	0	0	0	0
χ_h, $h \in C^{\dagger}_{\text{odd}}$	$2\cos\left(\frac{hr\pi}{2n}\right)$	$2i\sin\left(\frac{hr\pi}{2n}\right)$	0	0	0	0

3. Existence of an orthogonal basis for the symmetry classes of tensors associated with SD_{8n}. In this section we study the existence of an orthogonal basis for the symmetry classes of tensors associated with SD_{8n}. As explained in the introduction, if χ is a linear character of G then the symmetry class of tensors associated with G and χ has an orthogonal basis. Therefore we will concentrate on non-linear irreducible complex characters of SD_{8n}, i.e. the characters χ_h where $h \in C^{\dagger}_{\text{even}}$ or $h \in C^{\dagger}_{\text{odd}}$.

Remark 3.1. Let ν_2 be the 2-adic valuation, that is, $\nu_2\big(\frac{2^k m}{n}\big) = k$ for m and n odd. Then the condition $\nu_2\big(\frac{h}{2n}\big) < 0$ means that every power of 2 that divides h also divides n.

Lemma 3.2. *Let $G := SD_{8n}$ and H be a subgroup of G. Then there is a natural number r, $0 \le r < 4n$, such that either $H = \langle a^r \rangle$, or $\langle a^r \rangle \subsetneqq H$ and $H \cap \langle a \rangle = \langle a^r \rangle$. In the latter case we have $|H| \ge 2|\langle a^r \rangle|$.*

Proof. This is straightforward. ∎

Lemma 3.3. *Suppose $\chi = \chi_h$. If r is defined by $G_\alpha \cap \langle a \rangle = \langle a^r \rangle$ and $l = 4n/\gcd(4n, r)$, then*

$$\sum_{g \in G_\alpha} \chi(g) = \begin{cases} 2l & \text{if } rh \equiv 0 \pmod{4n}, \\ 0 & \text{if } rh \not\equiv 0 \pmod{4n}, \end{cases}$$

and for $\alpha \in \overline{\Delta}$, we have $rh \equiv 0 \pmod{4n}$.

Proof. Since G_α is a subgroup of G, using Lemma 3.2 there is a natural number r, $0 \le r < 4n$, such that either $G_\alpha = \langle a^r \rangle$ or $\langle a^r \rangle < G_\alpha$. Using Table I, we find that χ vanishes outside $\langle a \rangle$, therefore

$$\sum_{g \in G_\alpha} \chi(g) = \sum_{t=1}^{l} \chi(a^{tr}) = 2\sum_{t=1}^{l} \cos\left(\frac{trh\pi}{2n}\right) = \begin{cases} 2l, & rh \equiv 0 \pmod{4n}, \\ 0, & rh \not\equiv 0 \pmod{4n}. \end{cases}$$

Also if $rh \not\equiv 0 \pmod{4n}$, then $\sum_{g \in G_\alpha} \chi(g) = 0$, which shows $\alpha \notin \overline{\Delta}$. ∎

Lemma 3.4. *Let $1 \le h < 2n$ and let ν_2 be the 2-adic valuation. Then there exist t_1, t_2, $0 \le t_1$, $t_2 < 4n$, such that $\cos\big(\frac{(t_1 - t_2)h\pi}{2n}\big) = 0$ if and only if $\nu_2\big(\frac{h}{2n}\big) < 0$.*

Theorem 3.5. *Let $G = SD_{8n}$ be a subgroup of S_{4n}, denote $\chi = \chi_h$ for $h \in C^{\dagger}_{\text{even}}$, and assume $d = \dim V \ge 2$. Then $V_\chi(G)$ has an orthogonal $*$-basis if and only $\nu_2\big(\frac{h}{2n}\big) < 0$.*

Proof. It is enough to prove that for any $\alpha \in \overline{\Delta}$ the orbital subspace V^*_α has an orthogonal $*$-basis if $\nu_2(\frac{h}{2n}) < 0$. Let $\nu_2(\frac{h}{2n}) < 0$ and assume $\alpha \in \overline{\Delta}$. By Lemma 3.2, either $G_\alpha = \langle a^r \rangle$ or $\langle a^r \rangle < G_\alpha$. Let $l = 4n/\gcd(4n, r)$. Now we consider two cases.

Case 1. If $\langle a^r \rangle < G_\alpha$, then by Lemma 3.2 we obtain $|G_\alpha| \ge 2l$ where

$$\langle a^r \rangle = \langle a \rangle \cap G_\alpha = \{a^r, a^{2r}, \dots, a^{lr} = 1\}.$$

By (1.4), $|G_\alpha| \geq 2l$ and Lemma 3.3, we have

$$\dim V_\alpha^* = \frac{\chi(1)}{|G_\alpha|} \sum_{\sigma \in G_\alpha} \chi(\sigma) \leq \frac{2}{2l}(2l) = 2.$$

If $\dim V_\alpha^* = 1$, then it is obvious that we have an orthogonal $*$-basis. Let us consider $\dim V_\alpha^* = 2$. Set $\sigma_1 = a^j$, $\sigma_2 = a^i$. Then

$$\sigma_2 G_\alpha \sigma_1^{-1} \cap \langle a \rangle = \{a^{r+i-j}, \dots, a^{lr+i-j}\}.$$

Hence if $\sigma_1 = a^j$, $\sigma_2 = a^i$, by (1.3), we have

$$\begin{aligned}(3.1)\qquad \langle e^*_{\sigma_1.\alpha}, e^*_{\sigma_2.\alpha}\rangle &= \frac{\chi(1)}{|G|} \sum_{x \in \sigma_2 G_\alpha \sigma_1^{-1}} \chi(x) = \frac{2}{8n} \sum_{t=1}^{l} \chi(a^{tr+i-j}) \\ &= \frac{4}{8n} \sum_{t=1}^{l} \cos \frac{(tr+i-j)h\pi}{2n} \\ &= \frac{1}{2n} \sum_{t=1}^{l} \cos\left(\frac{trh\pi}{2n} + \frac{(i-j)h\pi}{2n}\right) \\ &= \frac{1}{2n} \sum_{t=1}^{l} \cos\left(\frac{(i-j)h\pi}{2n}\right) = \frac{l}{2n} \cos\left(\frac{(i-j)h\pi}{2n}\right)\end{aligned}$$

where the penultimate equality is due to an application of Lemma 3.3. By Lemma 3.4, there exist i and j such that

$$\langle e^*_{a^j.\alpha}, e^*_{a^i.\alpha}\rangle = 0,$$

which means that $\{e^*_{\sigma_1.\alpha}, e^*_{\sigma_2.\alpha}\}$ is an orthogonal $*$-basis for V_α^*.

CASE 2. If $G_\alpha = \langle a^r \rangle = \{a^r, a^{2r}, \dots, a^{lr} = 1\}$, then by (1.4) and Lemma 3.3,

$$\dim V_\alpha^* = \frac{\chi(1)}{|G_\alpha|} \sum_{\sigma \in G_\alpha} \chi(\sigma) = \frac{2}{l}(2l) = 4.$$

For any $\sigma_1, \sigma_2 \in G$, we have

$$\sigma_2 G_\alpha \sigma_1{}^{-1} = \begin{cases} \{a^{r+i-j}, a^{2r+i-j}, \dots, a^{lr+i-j}\} & \text{if } \sigma_1 = a^j,\ \sigma_2 = a^i, \\ \{a^{r+i+j(1-2n)}b, a^{2r+i+j(1-2n)}b, \dots, a^{lr+i+j(1-2n)}b\} & \text{if } \sigma_1 = a^j b,\ \sigma_2 = a^i, \\ \{a^{(1-2n)r+i-j}, a^{2r(1-2n)+i-j}, \dots, a^{lr(1-2n)+i-j}\} & \text{if } \sigma_1 = a^j b,\ \sigma_2 = a^i b. \end{cases}$$

If $\sigma_1 = a^j$, $\sigma_2 = a^i$, by (3.1) we have

$$\langle e^*_{\sigma_1.\alpha}, e^*_{\sigma_2.\alpha}\rangle = \frac{l}{2n} \cos\left(\frac{(i-j)h\pi}{2n}\right).$$

If $\sigma_1 = a^j b$, $\sigma_2 = a^i$, we have
$$\langle e^*_{\sigma_1.\alpha}, e^*_{\sigma_2.\alpha}\rangle = 0,$$
and for $\sigma_1 = a^j b$, $\sigma_2 = a^i b$, we have
$$\begin{aligned}\langle e^*_{\sigma_1.\alpha}, e^*_{\sigma_2.\alpha}\rangle &= \frac{\chi(1)}{|G|}\sum_{x\in\sigma_2 G_\gamma \sigma_1^{-1}} \chi(x) = \frac{2}{8n}\sum_{t=1}^{l}\chi(a^{tr(1-2n)+i-j})\\ &= \frac{4}{8n}\sum_{t=1}^{l}\cos\Big(\frac{(tr(1-2n)+i-j)h\pi}{2n}\Big)\\ &= \frac{1}{2n}\sum_{t=1}^{l}\cos\Big(\frac{trh\pi}{2n}+\frac{(i-j)h\pi}{2n}-trh\pi\Big)\\ &= \frac{1}{2n}\sum_{t=1}^{l}\cos\Big(\frac{(i-j)h\pi}{2n}\Big) = \frac{l}{2n}\cos\Big(\frac{(i-j)h\pi}{2n}\Big)\end{aligned}$$
where the penultimate equality uses Lemma 3.3. Therefore
$$\langle e^*_{\sigma_1.\alpha}, e^*_{\sigma_2.\alpha}\rangle = \begin{cases}\frac{l}{2n}\cos\Big(\frac{(i-j)h\pi}{2n}\Big), & \sigma_1 = a^j,\ \sigma_2 = a^i,\\ 0 & \sigma_1 = a^j b,\ \sigma_2 = a^i,\\ \frac{l}{2n}\cos\Big(\frac{(i-j)h\pi}{2n}\Big), & \sigma_1 = a^j b,\ \sigma_2 = a^i b.\end{cases}$$
In view of Lemma 3.4, if $\nu_2\big(\frac{h}{2n}\big) < 0$, there exist t_1, t_2, $0 \le t_1, t_2 < 4n$ such that $\cos\big(\frac{(t_1-t_2)h\pi}{2n}\big) = 0$. Put
$$S = \{a^{t_1}.\alpha, a^{t_2}.\alpha, a^{t_1}b.\alpha, a^{t_2}b.\alpha\} \subseteq \varGamma^m_n.$$
Then for every $\alpha, \beta \in S$ and $\alpha \ne \beta$ we have
$$\langle e^*_\alpha, e^*_\beta\rangle = 0.$$
But $\dim V^*_\alpha = 4$; hence $\{e^*_\xi \mid \xi \in S\}$ is an orthogonal $*$-basis for V^*_α.

Conversely, assume that $V_\chi(G)$ has an orthogonal basis of decomposable symmetrized tensors. Then since $V_\chi(G) = \bigoplus_{\alpha\in\overline{\varDelta}} V^*_\alpha$ for all $\alpha \in \overline{\varDelta}$, the orbital subspace V^*_α has an orthogonal basis of decomposable symmetrized tensors. Using [17, p. 642], we can choose $\alpha \in \varGamma^m_n$ such that $a^t \notin G_\alpha$ for $1 \le t < 4n$. Thus for such α we have either $G_\alpha = \{1\}$ or $G_\alpha = \{1, a^t b, a^{-(2n-1)t}b\}$ for some $1 \le t < 4n$, since if $G_\alpha \ne \{1\}$ and $a^{t_1}b, a^{t_2}b \in G_\alpha$, then
$$a^{t_1}b.a^{t_2}b = a^{t_1}b.ba^{(2n-1)t_2} = a^{t_1+(2n-1)t_2} \in G_\alpha,$$
which shows that $t_1 = -(2n-1)t_2$.

To prove that $\nu_2\big(\frac{h}{2n}\big) < 0$ is a necessary condition for existence of an orthogonal $*$-basis for $V_\chi(G)$, it is enough to consider the cases $G_\alpha = \{1\}$

and $G_\alpha = \{1, a^t b, a^{-(2n-1)t} b\}$. For both, we have

$$\|e_\alpha^*\|^2 = \frac{\chi(1)}{|G|} \sum_{g \in G_\alpha} \chi(g) \neq 0,$$

so $\alpha \in \overline{\Delta}$. First consider $G_\alpha = \{1\}$. For any $\sigma_1, \sigma_2 \in G$, we have

$$\sigma_2 G_\alpha \sigma_1^{-1} = \begin{cases} \{a^{i-j}\} & \text{if } \sigma_1 = a^j,\ \sigma_2 = a^i, \\ \{a^{i+j(1-2n)} b\} & \text{if } \sigma_1 = a^j b,\ \sigma_2 = a^i, \\ \{a^{(1-2n)i-j}\} & \text{if } \sigma_1 = a^j b,\ \sigma_2 = a^i b. \end{cases}$$

Therefore by (1.3) we have

$$\langle e_{\sigma_1.\alpha}^*, e_{\sigma_2.\alpha}^* \rangle = \begin{cases} \frac{1}{2n} \cos\Big(\frac{(i-j)h\pi}{2n}\Big) & \text{if } \sigma_1 = a^j,\ \sigma_2 = a^i, \\ 0 & \text{if } \sigma_1 = a^j b,\ \sigma_2 = a^i, \\ \frac{1}{2n} \cos\Big(\frac{(i-j)h\pi}{2n}\Big) & \text{if } \sigma_1 = a^j b,\ \sigma_2 = a^i b. \end{cases}$$

Hence $\langle e_{\sigma_1.\alpha}^*, e_{\sigma_2.\alpha}^* \rangle = 0$ implies that there exist t_1 and t_2 such that

$$\cos\Big(\frac{(t_1 - t_2)h\pi}{2n}\Big) = 0,$$

therefore by Lemma 3.4 we get $\nu_2\big(\frac{h}{2n}\big) < 0$.

Now consider $G_\alpha = \{1, a^t b, a^{-(2n-1)t} b\}$. For any $\sigma_1, \sigma_2 \in G$, we have

$$\sigma_2 G_\alpha \sigma_1^{-1} = \begin{cases} \{a^{i-j}, ba^{(2n-1)(j+t)-i}, ba^{(2n-1)(j-(2n-1)t)-i}\} & \text{if } \sigma_1 = a^j,\ \sigma_2 = a^i, \\ \{a^{i+j(1-2n)} b, a^{j+(2n-1)t+i}, a^{j-t+i}\} & \text{if } \sigma_1 = a^j b,\ \sigma_2 = a^i, \\ \{a^{(1-2n)i-j}, a^{j+(2n-1)t+i} b, a^{j-t+i} b\} & \text{if } \sigma_1 = a^j b,\ \sigma_2 = a^i b. \end{cases}$$

Now similar to our previous calculations in this section, we get $\nu_2\big(\frac{h}{2n}\big) < 0$. ∎

REMARK 3.6. In the proof of the necessity part of Theorem 3.5, one can choose $\alpha = (1, 2, \ldots, 2)$. The proof given here shows the stronger statement that the orbital subspace V_α^* has an orthogonal $*$-basis whenever $G_\alpha \cup \langle a \rangle = \{1\}$.

COROLLARY 3.7. *Let $G = SD_{8n}$, n odd, be a subgroup of S_{4n}, denote $\chi = \chi_h$ for $h \in C_{\text{even}}^\dagger$, and assume $d = \dim V \geq 2$. Then $V_\chi(G)$ does not have an orthogonal $*$-basis.*

Proof. Since n is odd we have $\nu_2(\frac{h}{2n}) \geq 0$. Thus Theorem 3.5 implies $V_\chi(G)$ does not have an orthogonal $*$-basis. ∎

THEOREM 3.8. *Let $G = SD_{8n}$ be a subgroup of S_{4n}, denote $\chi = \chi_h$ for $h \in C_{\text{odd}}^\dagger$, and assume $d = \dim V \geq 2$. Then $V_\chi(G)$ does not have an orthogonal $*$-basis.*

Proof. The proof is similar to the proof of Theorem 3.5. Using Table I and Table II we conclude that $\langle e^*_{\sigma 1.\alpha}, e^*_{\sigma 2.\alpha}\rangle \neq 0$ since the imaginary and real parts should both be zero; but $i \sin x$ and $\cos x$ cannot vanish simultaneously. ∎

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