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P_λ-SETS AND SKELETAL MAPPINGS

BY

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Abstract. We prove that if the topology on the set Seq of all finite sequences of natural numbers is determined by P_λ-filters and $\lambda \leq \mathfrak{b}$, then Seq is a P_λ-set in its Čech–Stone compactification. This improves some results of Simon and of Juhász and Szymański. As a corollary we obtain a generalization of a result of Burke concerning skeletal maps and we partially answer a question of his.

1. Definitions and basic construction. Let Seq be the set of all finite sequences of natural numbers,

$$\mathrm{Seq} = \omega^{<\omega} = \bigcup \{ {}^n\omega : n < \omega \}.$$

Seq is a tree with the natural order defined as follows: for all $x, y \in \mathrm{Seq}$ we declare

$$s \leq t \;\Leftrightarrow\; t{\restriction}\mathrm{dom}(s) = s.$$

For every $s \in \mathrm{Seq}$ we write

$$[s, \to) = \{ t \in X \colon s \leq t \}.$$

The set of all immediate successors of an element $s \in \mathrm{Seq}$ is denoted by

$$\mathrm{succ}(s) = \{ t \in \mathrm{Seq} \colon t \text{ is minimal in } \{ t \in \mathrm{Seq} \colon t > s \} \}.$$

Hence $\mathrm{succ}(s) = \{ s^\frown n \colon n \in \omega \}$, where $s^\frown n$ denotes the concatenation of s and n.

Now, for every $s \in \mathrm{Seq}$ we pick a free filter $\mathcal{F}_s \subseteq \mathcal{P}(\mathrm{succ}(s))$. We assume that every filter contains the Fréchet filter. We shall identify the set of all immediate successors of s with the set ω. Therefore, every filter on the set of all immediate successors is considered here as a filter on ω.

For every indexed collection $\mathfrak{F} = (\mathcal{F}_t \colon t \in \mathrm{Seq})$ of filters we define the $\mathfrak{F}$-topology on Seq as follows:

Definition 1.1. A set $U \subseteq \mathrm{Seq}$ is open in the $\mathfrak{F}$-topology on Seq whenever

$$(\forall s \in U)(\exists F \in \mathcal{F}_s)(F \subseteq U).$$

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The $\mathfrak{F}$-topologies on Seq were introduced by Szymański [12] and Trnková [13] and studied e.g. in [2], [5], [7], [8], [14]. A review of $\mathfrak{F}$-topologies and their generalizations can be found in [1].

For every $s \in \mathrm{Seq}$ and every $\Phi \in \prod\{\mathcal{F}_t \colon t \in [s, \to)\}$ we consider the set

$$U(s, \Phi) = \bigcup\{U_n(s, \Phi) \colon n < \omega\},$$

where

$$U_0(s, \Phi) = \{s\},$$

and

$$U_{n+1}(s, \Phi) = U_n(s, \Phi) \cup \bigcup\{\Phi(t) \colon t \in U_n(s, \Phi)\} \quad \text{for every } n < \omega.$$

The following lemma first appeared in [2].

LEMMA 1.2. *For every indexed family* $\mathfrak{F} = (\mathcal{F}_t \colon t \in \mathrm{Seq})$ *of filters the family of sets*

$$\mathcal{B}(\mathfrak{F}) = \Big\{U(s, \Phi) \colon s \in \mathrm{Seq} \text{ and } \Phi \in \prod\{\mathcal{F}_t \colon t \in [s, \to)\}\Big\}$$

is a base for the $\mathfrak{F}$*-topology on* Seq *and consists of clopen sets. Consequently, the* $\mathfrak{F}$*-topology on* Seq *is a zero-dimensional Hausdorff topology.*

In fact Seq endowed with the $\mathfrak{F}$-topology is normal since it is countable and regular. In particular, one can consider the Čech–Stone compactification of Seq endowed with the $\mathfrak{F}$-topology.

The next two lemmas are easy to verify.

LEMMA 1.3. *For every* $s \in \mathrm{Seq}$ *and* $\Phi \in \prod\{\mathcal{F}_t \colon t \in [s, \to)\}$,

$$U(s, \Phi) = \{s\} \cup \bigcup\{U(t, \Phi) \colon t \in \Phi(s)\}.$$

LEMMA 1.4. *Assume that* $s \in \mathrm{Seq}$ *and* $\Phi, \Psi \in \prod\{\mathcal{F}_t \colon t \in [s, \to)\}$. *If* $\Phi(t) \subseteq \Psi(t)$ *for every* $t \in U(s, \Phi)$, *then* $U(s, \Phi) \subseteq U(s, \Psi)$.

As usual, $\mathfrak{b}$ denotes the minimal cardinality of an unbounded subset of ${}^{\omega}\omega$ ordered by the relation $\leq^*$ defined as follows:

$$f \leq^* g \;\Leftrightarrow\; (\exists n < \omega)(\forall k > n)(f(k) \leq g(k))$$

for all $f, g \in {}^{\omega}\omega$. Clearly, $\mathfrak{b} > \omega$. Since Seq is countable we immediately obtain the following:

LEMMA 1.5. *If* $\tau < \mathfrak{b}$ *and* $\{f_\alpha \colon \alpha < \tau\} \subseteq {}^{\mathrm{Seq}}\omega$, *then there exists* $g \in {}^{\mathrm{Seq}}\omega$ *such that*

$$f_\alpha \leq^* g \quad \text{for every } \alpha < \tau.$$

2. P_λ-sets. We use the standard definition of P_λ-filters:

DEFINITION 2.1. A (free) filter $\mathcal{F} \subseteq \mathcal{P}(\omega)$ is called a P_λ-*filter* whenever for every family $\mathcal{R} \subseteq \mathcal{F}$ of size smaller than λ there exists $F \in \mathcal{F}$ such that $F \subseteq^* U$ for every $U \in \mathcal{F}$.

As usual, $F \subseteq^* U$ means that $F \setminus U$ is finite. Hence, all P-filters are just P_{ω_1}-filters. A P_λ-filter which is simultaneously an ultrafilter is called a P_λ-*ultrafilter*. Since non-empty G_δ's in $\beta\mathbb{N} \setminus \mathbb{N}$ have non-empty interior, non-trivial P-filters always exist in ZFC, whereas the existence of P-ultrafilters requires adding some extra assumptions.

Clearly, every set of the form $U(s, \Phi)$ is non-compact. Hence, Seq with the $\mathfrak{F}$-topology is nowhere compact. Therefore, the Čech–Stone remainder

$$\mathrm{Seq}^* = \beta\,\mathrm{Seq} \setminus \mathrm{Seq}$$

is dense in $\beta\,\mathrm{Seq}$. Consequently, every closed subset of $\beta\,\mathrm{Seq}$ contained in Seq^* is nowhere dense in $\beta\,\mathrm{Seq}$.

A subset S of a topological space X is called a P_λ-*set* for $\lambda \geq \omega$ if S is contained in the interior of the intersection of every family of size smaller than λ consisting of open neighborhoods of S.

THEOREM 2.2. *Assume* $\mathfrak{F} = (\mathcal{F}_s \colon s \in \mathrm{Seq})$ *is a collection of* P_λ-*filters and* $\omega < \lambda \leq \mathfrak{b}$. *Then* Seq *endowed with the* $\mathfrak{F}$-*topology is a* P_λ-*set in* $\beta\,\mathrm{Seq}$.

Proof. Assume that $\tau < \lambda$ and $\{U_\alpha \colon \alpha < \tau\}$ is a family of open subsets of the Čech–Stone compactification of Seq with the $\mathfrak{F}$-topology and $\mathrm{Seq} \subseteq U_\alpha$ for every $\alpha < \tau$. We shall show that

$$\mathrm{Seq} \subseteq \mathrm{Int} \bigcap \{U_\alpha \colon \alpha < \tau\}.$$

By Lemma 1.2 and the normality of Seq, the sets of the form $\mathrm{cl}\, U(s, \Phi_s)$ (the closure in $\beta\,\mathrm{Seq}$) form a clopen base at the point $s \in \mathrm{Seq}$. So for every $s \in \mathrm{Seq}$ and every $\alpha < \tau$ there exists $\Psi_s^\alpha \in \prod\{\mathcal{F}_t \colon t \in [s, \rightarrow)\}$ such that

$$s \in U(s, \Psi_s^\alpha) \subseteq \mathrm{cl}\, U(s, \Psi_s^\alpha) \subseteq U_\alpha.$$

For every $s \in \mathrm{Seq}$ and every $t \in [s, \rightarrow)$ we set

$$\Phi_s^\alpha(t) = \bigcap \{\Psi_p^\alpha(t) \colon p \leq s\}.$$

Then, by Lemma 1.4, for every $s \in \mathrm{Seq}$ and every $\alpha < \tau$ we have

$$s \in U(s, \Phi_s^\alpha) \subseteq \mathrm{cl}\, U(s, \Phi_s^\alpha) \subseteq U_\alpha, \tag{1}$$

and moreover, for every $p \geq t$,

$$s \leq t \;\Rightarrow\; \Phi_t^\alpha(p) \subseteq \Phi_s^\alpha(p). \tag{2}$$

Since each $\mathcal{F}_s$ is a P_λ-filter, for every $s \in \mathrm{Seq}$ there exists a set $A_s \in \mathcal{F}_s$ such that $A_s \subseteq^* \Phi_s^\alpha(s)$ for all $\alpha < \tau$. Hence, for every $\alpha < \tau$ we can define a function $f_\alpha\colon \mathrm{Seq} \to \omega$ by

$$f_\alpha(s) = \max\{n < \omega\colon s^\frown n \in A_s \setminus \Phi^\alpha_s(s)\} + 1.$$

Since $\tau < \mathfrak{b}$, by Lemma 1.5, we have a function $g\colon \mathrm{Seq} \to \omega$ such that $f_\alpha \le^* g$ for all $\alpha < \tau$. Now we define on Seq a function Φ as follows:

$$\Phi(s) = A_s \cap \{s^\frown n\colon n \ge g(s)\}.$$

Clearly, $\Phi \in \prod\{\mathcal{F}_t\colon t \in \mathrm{Seq}\}$ since $A_s \in \mathcal{F}_s$ and $\mathcal{F}_s$ contains the Fréchet filter.

Claim. *Assume $s \in \mathrm{Seq}$ and $\Psi \in \prod\{\mathcal{F}_t\colon t \in [s,\to)\}$ and $\alpha < \tau$. If*

$$\Psi(t) \subseteq \Phi(t) \quad \textit{and} \quad f_\alpha(t) \le g(t)$$

for every $t \in U(s,\Psi)$, then

$$U(s,\Psi) \subseteq U(s,\Phi^\alpha_s).$$

By Lemma 1.4, to prove the Claim it suffices to show that $\Psi(t) \subseteq \Phi^\alpha_s(t)$ for every $t \in U(s,\Psi)$. But if $t \in U(s,\Psi)$ and $t^\frown n \in \Psi(t) \subseteq \Phi(t)$, then $t^\frown n \in A_t$ and $n \ge g(t) \ge f_\alpha(t)$. Hence, by the definition of f_α we get $t^\frown n \in \Phi^\alpha_t(t)$. Finally, by (2), we obtain $t^\frown n \in \Phi^\alpha_s(t)$, which completes the proof of the claim.

It remains to show that

$$\operatorname{cl} U(s, \Phi{\restriction}[s,\to)) \subseteq U_\alpha \tag{3}$$

for every $s \in \mathrm{Seq}$ and every $\alpha < \tau$. Fix $\alpha < \tau$ and suppose that (3) does not hold for some $s \in {}^m\omega$. We can assume that m is maximal with this property. In fact, since the set $\{t \in \mathrm{Seq}\colon f_\alpha(t) > g(t)\}$ is finite, by the Claim, for sufficiently large $n < \omega$ we have $U(t,\Phi{\restriction}[t,\to)) \subseteq U(t,\Phi^\alpha_t)$ for every $t \in {}^n\omega$. Then by (1) we get $\operatorname{cl} U(t,\Phi{\restriction}[t,\to)) \subseteq U_\alpha$. Thus we can assume that

$$\operatorname{cl} U(s,\Phi{\restriction}[s,\to)) \not\subseteq U_\alpha \quad \text{for some } s \in {}^m\omega$$

and

$$\operatorname{cl} U(t,\Phi{\restriction}[t,\to)) \subseteq U_\alpha \quad \text{for every } t \in {}^{m+1}\omega. \tag{4}$$

On the other hand, since $f_\alpha \le^* g$, there exist $t_1,\ldots,t_n \in \Phi(s)$ such that $f_\alpha(t) \le g(t)$ for every $t \in U(s,\Omega)$ where $\Omega \in \prod\{\mathcal{F}_t\colon t \in [s,\to)\}$ is defined by

$$\Omega(t) = \begin{cases} \Phi(s) \setminus \{t_1,\ldots,t_n\} & \text{if } t = s,\\ \Phi(t) & \text{if } t \ne s.\end{cases}$$

Since $\Omega(t) \subseteq \Phi(t)$ for every $t \in U(s,\Omega)$, by the Claim and (1) we get

$$\operatorname{cl} U(s,\Omega) \subseteq U_\alpha.$$

Hence, by Lemma 1.3 and (4) we have

$$\begin{aligned}\operatorname{cl} U(s,\Phi\restriction[s,\to)) &= \operatorname{cl}\Big(\{s\}\cup\bigcup\{U(t,\Phi\restriction[t,\to)\colon t\in\Phi(s)\}\Big)\\ &= \operatorname{cl}\Big(\{s\}\cup\bigcup\{U(t,\Phi\restriction[t,\to))\colon t\in\Phi(s)\setminus\{t_1,\dots,t_n\}\}\Big)\\ &\quad\cup\bigcup\{\operatorname{cl} U(t_i,\Phi\restriction[t_i,\to))\colon i\le n\})\\ &= \operatorname{cl} U(s,\Omega)\cup\bigcup\{\operatorname{cl} U(t_i,\Phi\restriction[t_i,\to)\colon i\le n\}\subseteq U_\alpha.\end{aligned}$$

This contradiction proves (3) and completes the proof. ∎

The following corollaries follow immediately from Theorem 2.2:

Corollary 2.3 (Simon [11]). *If every filter in the collection $\mathfrak{F}=(\mathcal{F}_s\colon s\in \mathrm{Seq})$ is a P-filter, then* Seq *endowed with the $\mathfrak{F}$-topology is a P-set in* $\beta\,\mathrm{Seq}$.

Corollary 2.4 (Juhász and Szymański [6]). *If $\omega<\lambda\le\mathfrak{b}$ and $\mathcal{F}_s=\mathcal{F}$ for every $s\in\mathrm{Seq}$, where $\mathcal{F}$ is a P_λ-ultrafilter, then* Seq *is a P_λ-set in* $\beta\,\mathrm{Seq}$.

Remark 2.5. Both the results of Simon and of Juhász and Szymański have the form of an equivalence. A slight modification of the arguments used in [11] and [6] shows that Theorem 2.2 also yields an equivalence, i.e. one can prove moreover that if Seq is a P_λ-set in $\beta\,\mathrm{Seq}$ then $\mathfrak{F}=(\mathcal{F}_s\colon s\in\mathrm{Seq})$ is a collection of P_λ-filters and $\omega<\lambda\le\mathfrak{b}$.

3. Skeletal mappings. To prove theorems on skeletal mappings we shall use a dual version of Theorem 2.2: if $\mathfrak{F}=(\mathcal{F}_s\colon s\in\mathrm{Seq})$ is a collection of P_λ-filters and $\omega<\lambda\le\mathfrak{b}$, then the union of less than λ closed subsets of Seq^* is a nowhere dense subset of $\beta\,\mathrm{Seq}$. First we recall some definitions.

Definition 3.1. A continuous mapping $f\colon X\to Y$ is *skeletal* whenever $f^{-1}[G]$ is dense in X for every open and dense $G\subseteq Y$.

Equivalently, a mapping $f\colon X\to Y$ is skeletal if $f^{-1}[F]$ is nowhere dense for every nowhere dense closed set $F\subseteq Y$. It is clear that $f[X]$ cannot be nowhere dense in Y for a skeletal mapping $f\colon X\to Y$. So, it can happen that f is skeletal as a map into Y but not as a map into Z where $Z\supseteq Y$. For this reason we prefer to consider skeletal surjections. Also, if $f\colon X\to Y$ is skeletal, then it is also skeletal as a surjection of X onto $f[X]$. In fact, if $F\subseteq f[X]\subseteq Y$ and F is a nowhere dense subset of $f[X]$, then it is also nowhere dense in Y.

Skeletal surjections were introduced in the class of Hausdorff spaces by Mioduszewski and Rudolf [9]. In [3] skeletal mappings are called *nowhere thin*.

The equivalences in the following easy proposition are immediate consequences of the definition of skeletal mappings. It will be needed later.

PROPOSITION 3.2. *Assume $f\colon X \to Y$ is a continuous surjection, f is a closed mapping and X is a regular space. Then the following conditions are equivalent:*

(1) *f is skeletal,*
(2) *for every non-empty open set $U \subseteq X$, $\operatorname{Int} f[U] \neq \emptyset$,*
(3) *for every non-empty open set $U \subseteq X$ there exists a non-empty open set $V \subseteq U$ such that $f[V]$ is open,*
(4) *for every dense set $D \subseteq Y$, $f^{-1}[D]$ is dense in X.*

Proof. (1)⇒(2). Suppose $\operatorname{Int} f[U] = \emptyset$. We can choose an open set $V \subseteq U$ such that $\emptyset \neq \operatorname{cl} V \subseteq U$. Then $F = f[\operatorname{cl} V]$ is a nowhere dense subset of Y and $V \subseteq f^{-1}[F]$, a contradiction.

(2)⇒(3). Since $\operatorname{Int} f[U] \neq \emptyset$, there exists a non-empty open set $W \subseteq f[U]$. We set $V = U \cap f^{-1}[W]$. Clearly, V is non-empty, open and $f[V] = W$.

(3)⇒(4). If $f^{-1}[D]$ is not dense in X, then there exists a non-empty open set $V \subseteq X$ such that $f^{-1}[D] \cap V = \emptyset$. We can assume that $f[V]$ is open. Since $D \cap f[V] = \emptyset$, D cannot be dense.

The remaining implication is obvious. ∎

Recall that a set $G \subseteq X$ is *regular closed* if $G = \operatorname{cl} \operatorname{Int} G$. We get the following corollaries:

COROLLARY 3.3. *If $f\colon X \to Y$ is a skeletal closed surjection, X is a regular space and $G \subseteq X$ is regular closed in X, then $f[G]$ is regular closed in Y.*

Proof. Suppose there exists an open set $U \subseteq Y$ such that $U \cap f[G] \neq \emptyset$ but $U \cap \operatorname{Int} f[G] = \emptyset$. Then $V = f^{-1}[U] \cap \operatorname{Int} G$ is non-empty and open but $\operatorname{Int} f[V] = \emptyset$, contradicting Proposition 3.2(2). ∎

COROLLARY 3.4. *If $f\colon X \to Y$ is a skeletal closed surjection, X is a regular space and $G \subseteq X$ is regular closed in X, then the restriction $f{\restriction}G\colon G \to f[G]$ is skeletal.*

Proof. Assume $U \subseteq f[G]$ is an open and dense subset of $f[G]$. By the previous corollary, $f[G]$ is regular closed and $V = (U \cap \operatorname{Int} f[G]) \cup (Y \setminus f[G])$ is dense and open in Y. Therefore, $f^{-1}[V]$ is dense and open in X, and in particular $(f{\restriction}G)^{-1}[U] = f^{-1}[U] \cap G \supseteq f^{-1}[V] \cap G$ is dense and open in G. ∎

EXAMPLE 3.5. Every zero-dimensional compact space admits a skeletal mapping onto a compactification of a discrete space. In fact, if $\mathcal{U}$ is an infinite, maximal disjoint family of clopen subsets of a zero-dimensional compact space X, then the quotient mapping determined by the closed partition

$$\{\{U\}\colon U\in\mathcal{U}\}\cup\left\{X\setminus\bigcup\mathcal{U}\right\}$$

is a skeletal mapping onto the one-point compactification of the discrete space of cardinality $|\mathcal{U}|$.

The above example shows in particular that $\beta\,\mathrm{Seq}$ with the $\mathfrak{F}$-topology has a skeletal mapping onto the convergent sequence $\{0\}\cup\{1/n\colon n>0\}$. Hence, $\beta\,\mathrm{Seq}$ is compact and dense in itself but it has a skeletal mapping onto the space in which the set of isolated points is dense. In this connection we also have the following:

THEOREM 3.6. *Assume $\mathfrak{F}=(\mathcal{F}_s\colon s\in\mathrm{Seq})$ is a collection of P_λ-filters where $\omega<\lambda\le\mathfrak{b}$ and* Seq *is endowed with the $\mathfrak{F}$-topology. If a continuous surjection $f\colon\beta\,\mathrm{Seq}\to X$, where X is Hausdorff, is skeletal and $\pi w(X)<\lambda$, then the set of isolated points in X is dense.*

Proof. Suppose the set of isolated points in X is not dense. Then there exists a non-empty regular closed dense in itself set $W\subseteq X$. By Proposition 3.2(3), there exists an open set $U\subseteq f^{-1}[W]$ such that $f[U]$ is open. Then $G=\mathrm{cl}\,U$ is regular closed and, by Corollaries 3.4 and 3.3, the mapping $f{\restriction}G$ is skeletal and maps G onto the regular closed set $f[G]\subseteq W$. On the other hand, $f[G]$ is compact dense in itself, $f[\mathrm{Seq}]$ is countable and $\pi w(Y)<\lambda$. Hence, there exists a dense set $D\subseteq f[G]\setminus f[\mathrm{Seq}]$ of size smaller than λ. Then, by Proposition 3.2(4), $f^{-1}[D]\cap G$ is dense in G. But $f^{-1}[D]\cap\mathrm{Seq}=\emptyset$. Hence, by Theorem 2.2, $f^{-1}[D]$ is a nowhere dense subset of $\beta\,\mathrm{Seq}$. We get a contradiction since G is regular closed in $\beta\,\mathrm{Seq}$. ■

The following simple lemma is a direct consequence of the definition of skeletal mappings.

LEMMA 3.7. *Assume $f\colon X\to Y$ is a continuous surjection and $D\subseteq X$ is dense. Then f is skeletal iff $f{\restriction}D\colon D\to f[D]$ is skeletal.*

Apart from skeletal mappings, Burke [3] also considers mappings with a slightly weaker property.

DEFINITION 3.8. A mapping $f\colon X\to Y$ is called *nowhere constant* if $f^{-1}(y)$ is nowhere dense for every $y\in Y$.

Clearly, if a mapping $f\colon X\to Y$ is skeletal and $f[X]$ is dense in itself, then f is nowhere constant. Example 3.5 shows that in general skeletal mappings need not be nowhere constant. Also, a nowhere constant mapping need not be skeletal.

Burke [3] proved that if X is Tychonoff and there is a nowhere constant continuous function from X into $\mathbb{R}$, and $\pi w(X) < \mathfrak{p}$, then there also exists a skeletal function from X into $\mathbb{R}$. He also asked whether there exists (in ZFC) a Tychonoff space of π-weight $\mathfrak{p}$ which has a nowhere constant mapping into $\mathbb{R}$ but does not have a skeletal mapping into $\mathbb{R}$ [3, Problem 3.12]. We shall give a partial answer to this question.

We now recall some cardinal characteristics of the continuum. The cardinal number $\mathfrak{p}$ is defined as the minimal cardinality of a base $\mathcal{R} \subseteq \mathcal{P}(\omega)$ of a free filter for which there is no infinite set $A \subseteq \omega$ with $A \subseteq^* R$ for all $R \in \mathcal{R}$. The *dominating number* is defined as follows:

$$\mathfrak{d} = \min\{|D|\colon (\forall f \in {}^{\omega}\omega)(\exists g \in D)(f \leq^* g)\}.$$

It is well known (see e.g. van Douwen [4]) that

$$\omega < \mathfrak{p} \leq \mathfrak{b} \leq \mathfrak{d} \leq 2^{\omega}.$$

DEFINITION 3.9. Let Y be a subset of a topological space X. A collection $\mathcal{B}$ of open neighborhoods of Y is a *base* of Y if for every open U such that $Y \subseteq U$ there exists $V \in \mathcal{B}$ with $V \subseteq U$.

The *character* of a (free) filter is the character of the corresponding subset of ω^*, i.e. for every (free) filter $\mathcal{F} \subseteq \mathcal{P}(\omega)$,

$$\chi(\mathcal{F}) = \chi(A_{\mathcal{F}}, \omega^*),$$

where

$$A_{\mathcal{F}} = \bigcap\{\mathrm{cl}_{\beta\mathbb{N}} U : U \in \mathcal{F}\}$$

and

$$\chi(A_{\mathcal{F}}, \omega^{\star}) = \min\{|\mathcal{B}| : \mathcal{B} \text{ is a base of } A_{\mathcal{F}}\}.$$

LEMMA 3.10. *For every $s \in \mathrm{Seq}$ we have*

$$\chi(s, \mathrm{Seq}) = \mathfrak{d} + \chi(\mathcal{F}_s).$$

The proof of the above lemma can be obtained by a slight modification of the proof of [6, Theorem 2].

THEOREM 3.11. *If $\mathfrak{F} = (\mathcal{F}_s\colon s \in \mathrm{Seq})$ is a collection of P-filters of character $\aleph_1$, then* Seq *endowed with the $\mathfrak{F}$-topology is of π-weight $\mathfrak{d}$ and has a nowhere constant mapping into $\mathbb{R}$ but does not have a skeletal mapping into $\mathbb{R}$.*

Proof. Since Seq is countable, the equality $\pi w(\mathrm{Seq}) = \mathfrak{d}$ follows from Lemma 3.10.

To show that there exists a nowhere constant mapping of Seq endowed with the $\mathfrak{F}$-topology into $\mathbb{R}$ we consider a topology on Seq generated by the family

$$\mathcal{B}^* = \Big\{U(s,\Phi)\colon s\in \mathrm{Seq},\ \Phi\in \prod\{\mathcal{F}_t\colon t\in[s,\rightarrow)\},$$
$$|\mathrm{succ}(t)\setminus\Phi(t)|<\omega \text{ for } t\in[s,\rightarrow) \text{ and } |\{t\in\mathrm{Seq}\colon \Phi(t)\neq \mathrm{succ}(t)\}|<\omega\Big\}.$$

It is easy to see that $\mathcal{B}^*$ is countable and generates a zero-dimensional Hausdorff topology on Seq. In fact, if $s,t\in\mathrm{Seq}$ are non-compatible, then $[s,\rightarrow)$ and $[t,\rightarrow)$ are disjoint neighborhoods of s and t, respectively. If $s<t$, then there exists exactly one $u\in\mathrm{succ}(s)$ such that $u\leq t$. Clearly, when $\Psi\in\prod\{\mathcal{F}_t\colon t\in[s,\rightarrow)\}$ is given by the formula

$$\Psi(p)=\begin{cases}\mathrm{succ}(s)\setminus\{u\} & \text{if } p=s,\\ \mathrm{succ}(p) & \text{if } p\neq s,\end{cases}$$

then $U(s,\Psi)$ is clopen and disjoint from $[t,\rightarrow)$.

Moreover, the topology generated by $\mathcal{B}^*$ is nowhere compact and defined on a countable set. Therefore, by a theorem of Sierpiński [10], Seq with the topology generated by $\mathcal{B}^*$ is homeomorphic to the space of rational numbers. Since $\mathcal{B}^*\subseteq\mathcal{B}$, the identity function is a continuous mapping from Seq into $\mathbb{R}$. This mapping is also nowhere constant because it is one-to-one and Seq endowed with the $\mathfrak{F}$-topology is dense in itself.

It remains to show that there is no skeletal mapping from Seq into $\mathbb{R}$. Suppose that there exists a skeletal surjection $f\colon \mathrm{Seq}\to F$, where $F\subseteq\mathbb{R}$. Then, by Lemma 3.7, the Čech–Stone extension $\beta f\colon \beta\,\mathrm{Seq}\to \mathrm{cl}\,F$ is a skeletal surjection of $\beta\,\mathrm{Seq}$ onto the closed set $\mathrm{cl}\,F$. Hence, by Theorem 3.6, the set $\mathrm{cl}\,F$ has a dense set of isolated points. Therefore, $\beta f[\beta\,\mathrm{Seq}]$ is a nowhere dense subset of $\mathbb{R}$ and thus βf cannot be skeletal. This contradiction completes the proof. ∎

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