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ℓ^2-HOMOLOGY AND PLANAR GRAPHS

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Abstract. In his 1930 paper, Kuratowski proves that a finite graph Γ is planar if and only if it does not contain a subgraph that is homeomorphic to K_5, the complete graph on five vertices, or $K_{3,3}$, the complete bipartite graph on six vertices. This result is also attributed to Pontryagin. In this paper we present an ℓ^2-homological method for detecting non-planar graphs. More specifically, we view a graph Γ as the nerve of a related Coxeter system and construct the associated Davis complex, Σ_Γ. We then use a result of the author regarding the (reduced) ℓ^2-homology of Coxeter groups to prove that if Γ is planar, then the orbihedral Euler characteristic of Σ_Γ/W_Γ is non-positive. This method not only implies as subcases the classical inequalities relating the number of vertices V and edges E of a planar graph (that is, $E \leq 3V-6$ or $E \leq 2V-4$ for triangle-free graphs), but it is stronger in that it detects non-planar graphs in instances the classical inequalities do not.

1. Introduction. Let S be a finite set of generators. A *Coxeter matrix* on S is a symmetric $S \times S$ matrix $M = (m_{st})$ with entries in $\mathbb{N} \cup \{\infty\}$ such that each diagonal entry is 1 and each off-diagonal entry is ≥ 2. The matrix M gives a presentation of an associated *Coxeter group* W:

$$W = \langle S \mid (st)^{m_{st}} = 1 \text{ for each pair } (s,t) \text{ with } m_{st} \neq \infty\rangle. \tag{1.1}$$

The pair (W,S) is called a *Coxeter system*. Denote by L the nerve of (W,S). It is a simplicial complex with vertex set S; the precise definition will be given in Section 2. In several papers (e.g., [1], [2], and [3]), M. Davis describes a construction which associates to any Coxeter system (W,S) a simplicial complex $\Sigma(W,S)$, or simply Σ when the Coxeter system is clear, on which W acts properly and cocompactly. The two salient features of Σ are that (1) it is contractible, and (2) it admits a cellulation under which the nerve of each vertex is L. It follows that if L is a triangulation of $\mathbb{S}^{n-1}$, then Σ is an aspherical n-manifold. Hence, there is a variation of Singer's Conjecture, originally regarding the (reduced) ℓ^2-homology of aspherical manifolds, for such Coxeter groups.

CONJECTURE 1.1 (Singer's Conjecture for Coxeter groups). *Let (W,S) be a Coxeter group such that its nerve, L, is a triangulation of $\mathbb{S}^{n-1}$. Then $\mathcal{H}_i(\Sigma_L) = 0$ for all $i \neq n/2$.*

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For details on ℓ^2-homology theory, see [3], [4] and [5]. Conjecture 1.1 holds for elementary reasons in dimensions 1 and 2. In [8], Lott and Lück prove that the general statement of Singer's Conjecture holds for those aspherical 3-manifolds for which Thurston's Geometrization Conjecture is true. (Hence, by Perelman, for all aspherical 3-manifolds.) Thurston proved in [11] that the Geometrization Conjecture holds for Haken 3-manifolds; and in [4], Davis and Okun show that when (W, S) is *right-angled* (this means that generators either commute, or have no relation), Davis' construction yields examples of Haken 3-manifolds. Thus, they show that Thurston's Geometrization Conjecture holds for closed aspherical 3-manifolds arising as quotient spaces of right-angled Davis complexes. Also in [4], Davis and Okun prove that if Conjecture 1.1 holds for right-angled Coxeter systems in some odd dimension n, then it also holds in dimension $n + 1$. They prove directly that Conjecture 1.1 holds for right-angled systems in dimension 3, and thus by the previous statement, also in dimension 4. In [10], the author proves that Conjecture 1.1 holds for arbitrary Coxeter systems with nerve $\mathbb{S}^2$, and in [9], that Conjecture 1.1 holds for $n = 4$ if (W, S) is even (that is, for $s \neq t$, m_{st} is either even or infinite) and if the nerve is a flag triangulation of $\mathbb{S}^3$.

Much of [4] is devoted to variations of Conjecture 1.1, including the following generalization.

LEMMA 1.2 ([4, Lemma 9.2.3]). *Suppose (W, S) is a right-angled Coxeter system with nerve L, a flag triangulation of $\mathbb{S}^2$. Let A be a full subcomplex of L. Then*

$$\mathcal{H}_i(W\varSigma_A) = 0 \quad \text{for } i > 1.$$

Here, $\varSigma_A$ is the Davis complex associated to the Coxeter system (W_A, A^0), where W_A is the subgroup of W generated by vertices in A, with nerve A. It is a subcomplex of $\varSigma$. Finally, $W\varSigma_A$ is the collection of all the W-translates of $\varSigma_A$ in $\varSigma$.

Lemma 1.2 is the key to what Davis and Okun call "a complicated proof of the classical fact that $K_{3,3}$ is not planar," where $K_{3,3}$ denotes the complete bipartite graph on six vertices (See Section 11.4.1 of [4]). We outline that argument in Section 3. The key observation there is that $K_{3,3}$ can correspond to the nerve of a right-angled Coxeter system. We generalize this to non-right-angled systems by labeling the edges of a graph $\varGamma$ with integers ≥ 2 in such a way that (the labeled) $\varGamma$ defines, and is the nerve of, a Coxeter system with generators corresponding to the vertices of $\varGamma$. We call such a labeling *metric flag*. (See Sections 2 and 4.) The purpose of this paper is to use ℓ^2-homological methods to prove the following classification of planar graphs.

COROLLARY 1.3 (see Corollary 4.8). *Let $\varGamma$ be a simple, connected graph, with $V > 2$ vertices. If $\varGamma$ admits a metric flag labeling where n_e (an inte-*

ger ≥ 2) *is the label on the edge e with*

$$1-\frac{V}{2}+\Big(\sum_{edges\, e}\frac{1}{n_e}\Big)\frac{1}{2}>0,$$

then Γ *is not planar.*

The key step for us is proving a result analogous to Lemma 1.2, but for subcomplexes of arbitrary Coxeter systems.

Main Theorem (see Theorem 4.5). *Let* (W,S) *be a Coxeter system with nerve* L*, a triangulation of* $\mathbb{S}^2$*. Let* A *be a full subcomplex of* L *with right-angled complement. Then*

$$\mathcal{H}_i(W\Sigma_A)=0 \quad \textit{for } i>1.$$

The subcomplex A having a "right-angled complement" means that for generators s and t, the Coxeter relation $m_{st}\neq 2$ nor ∞ implies that the vertices corresponding to s and t are both in A.

2. The Davis complex. Let (W,S) be a Coxeter system. Given a subset U of S, define W_U to be the subgroup of W generated by the elements of U. A subset T of S is *spherical* if W_T is a finite subgroup of W. In this case, we will also say that the subgroup W_T is spherical. Denote by $\mathcal{S}$ the poset of spherical subsets of S, partially ordered by inclusion. Given a subset V of S, let $\mathcal{S}_{\geq V}:=\{T\in\mathcal{S}\mid V\subseteq T\}$. Similar definitions exist for $<,>,\leq$. For any $w\in W$ and $T\in\mathcal{S}$, we call the coset wW_T a *spherical coset*. We will denote by $W\mathcal{S}$ the poset of all spherical cosets.

Let $K=|\mathcal{S}|$, the geometric realization of the poset $\mathcal{S}$. It is a finite simplicial complex. Denote by $\Sigma(W,S)$, or simply Σ when the system is clear, the geometric realization of the poset $W\mathcal{S}$. This is the *Davis complex*. The natural action of W on $W\mathcal{S}$ induces a simplicial action of W on Σ which is proper and cocompact. Observe that K includes naturally into Σ via the map induced by $T\to W_T$. So we view K as a subcomplex of Σ, and note that K is a strict fundamental domain for the action of W on Σ.

The poset $\mathcal{S}_{>\emptyset}$ is an abstract simplicial complex. This simply means that if $T\in\mathcal{S}_{>\emptyset}$ and T' is a non-empty subset of T, then $T'\in\mathcal{S}_{>\emptyset}$. Denote this simplicial complex by L and call it the *nerve* of (W,S). The vertex set of L is S and a non-empty subset of vertices T spans a simplex of L if and only if T is spherical.

Define a labeling on the edges of L by the map $m:\mathrm{Edge}(L)\to\{2,3,\ldots\}$, where $\{s,t\}\mapsto m_{st}$. This labeling accomplishes two things: (1) the Coxeter system (W,S) can be recovered (up to isomorphism) from L, and (2) the 1-skeleton of L inherits a natural piecewise spherical structure in which the edge $\{s,t\}$ has length $\pi-\pi/m_{st}$. The complex L is then a *metric flag* simplicial complex (see [2, Definition I.7.1]). This means that any finite set of vertices, which are pairwise connected by edges, spans a simplex of L if and only if it is possible to find some spherical simplex with the given edge lengths. In other words, L is "metrically determined by its 1-skeleton".

Recall that a simplicial complex L is *flag* if every non-empty, finite set of vertices that are pairwise connected by edges spans a simplex of L. Thus, it is clear that any flag simplicial complex can correspond to the nerve of a right-angled Coxeter system. For the purpose of this paper, we will say that labeled (with integers ≥ 2) simplicial complexes are *metric flag* if they correspond to the labeled nerve of some Coxeter system. We then treat vertices of metric flag simplicial complexes as generators of a corresponding Coxeter system. Moreover, for a metric flag simplicial complex L, we write Σ_L to denote the associated Davis complex.

A cellulation of Σ by Coxeter cells. The complex Σ has a coarser cell structure: its cellulation by "Coxeter cells". (References include [2] and [4].) The features of the Coxeter cellulation are summarized by [2, Proposition 7.3.4]. We point out that under this cellulation the link of each vertex is L. It follows that if L is a triangulation of $\mathbb{S}^{n-1}$, then Σ is a topological n-manifold.

Full subcomplexes. Suppose A is a full subcomplex of L. Then A is the nerve for the subgroup generated by the vertex set of A. We will denote this subgroup by W_A. (This notation is natural since the vertex set of A corresponds to a subset of the generating set S.) Let $\mathcal{S}_A$ denote the poset of the spherical subsets of W_A and let Σ_A denote the Davis complex associated to (W_A, A^0), with fundamental domain K_A. The inclusion $W_A \hookrightarrow W_L$ induces an inclusion of posets $W_A\mathcal{S}_A \hookrightarrow W_L\mathcal{S}_L$ and thus an inclusion of Σ_A as a subcomplex of Σ_L. Note that W_A acts on Σ_A and that if $w \in W_L - W_A$, then Σ_A and $w\Sigma_A$ are disjoint copies of Σ_A in Σ_L. Denote by $W_L\Sigma_A$ the union of all translates of Σ_A in Σ_L.

3. Previous results in ℓ^2-homology. Let L be a metric flag simplicial complex, and let A be a full subcomplex of L. The following notation will be used throughout.

$$\mathfrak{h}_i(L) := \mathcal{H}_i(\Sigma_L), \tag{3.1}$$

$$\mathfrak{h}_i(A) := \mathcal{H}_i(W_L\Sigma_A), \tag{3.2}$$

$$\beta_i(A) := \dim_{W_L}(\mathfrak{h}_i(A)). \tag{3.3}$$

Here $\dim_{W_L}(\mathfrak{h}_i(A))$ is the von Neumann dimension of the Hilbert W_L-module $W_L\Sigma_A$ and $\beta_i(A)$ is the ith ℓ^2-Betti number of $W_L\Sigma_A$. The notation in (3.2) and (3.3) will not lead to confusion since $\dim_{W_L}(W_L\Sigma_A) = \dim_{W_A}(\Sigma_A)$. (see [4] and [5]).

0-dimensional homology. Let Σ_A be the Davis complex constructed from a Coxeter system with nerve A, so W_A acts geometrically on Σ_A. The reduced ℓ^2-homology groups of Σ_A can be identified with the subspace of *harmonic i-cycles* (see [5] or [4]). That is, $x \in \mathfrak{h}_i(A)$ is an i-cycle and i-cocycle. 0-dimensional cocycles of Σ_A must be constant on all vertices of Σ_A. It follows that if W_A is infinite, and therefore the 0-skeleton of Σ_A is infinite, then $\beta_0(A) = 0$.

Singer's Conjecture in dimensions 1 and 2. As mentioned in Section 1, Conjecture 1.1 is true in dimensions 1 and 2. Indeed, let L be $\mathbb{S}^0$ or $\mathbb{S}^1$, the nerve of a Coxeter system (W, S). Then W is infinite and so, as stated above, $\beta_0(L) = 0$. Poincaré duality then implies that the top-dimensional ℓ^2-Betti numbers are also 0.

Orbihedral Euler characteristic. Since Σ_L is a geometric W-complex, there are only a finite number of W-orbits of cells in Σ_L, and the order of each cell stabilizer is finite. The *orbihedral Euler characteristic* of Σ_L/W, denoted $\chi^{\text{orb}}(\Sigma_L/W)$, is the rational number defined by

$$\chi^{\text{orb}}(\Sigma_L/W) = \chi^{\text{orb}}(K) = \sum_{\sigma} \frac{(-1)^{\dim \sigma}}{|W_\sigma|}, \tag{3.4}$$

where the summation is over the simplices of K, and $|W_\sigma|$ denotes the order of the stabilizer of Σ in W. Then, if the dimension of L is $n-1$, a standard argument (see [5]) proves Atiyah's formula

$$\chi^{\text{orb}}(K) = \sum_{i=0}^{n} (-1)^i \beta_i(L). \tag{3.5}$$

Joins. If $L = L_1 * L_2$, the join of L_1 and L_2, where each edge connecting a vertex of L_1 with a vertex of L_2 is labeled 2, we write $L = L *_2 L_2$ and then $W_L = W_{L_1} \times W_{L_2}$ and $\Sigma_L = \Sigma_{L_1} \times \Sigma_{L_2}$. We may then use the Künneth formula to calculate the (reduced) ℓ^2-homology of Σ_L, and the following equation from [4, Lemma 7.2.4] extends to our situation:

$$\beta_k(L_1 *_2 L_2) = \sum_{i+j=k} \beta_i(L_1)\beta_j(L_2). \tag{3.6}$$

If $L = P *_2 L_2$, where P is one point, then we call L a *right-angled cone*. Since $\Sigma_P = [-1, 1]$, there are no 1-cycles in Σ_P and $\beta_1(P) = 0$. But $\chi^{\text{orb}}(\Sigma_P/W_P) = 1/2$. So by equation (3.5), $\beta_0(P) = 1/2$. Thus, in reference to the right-angled cone L, equation (3.6) implies that

$$\beta_i(L) = \tfrac{1}{2}\beta_i(L_2). \tag{3.7}$$

Kuratowski's $K_{3,3}$ graph. Along with Lemma 1.2, the above gives us enough to prove that $K_{3,3}$ is not planar. Indeed, let P_3 denote 3 disjoint points. Then $K_{3,3} = P_3 *_2 P_3$ is the nerve of a right-angled Coxeter system. If $K_{3,3}$ were a planar graph, it could be embedded as a full subcomplex of a flag triangulation of $\mathbb{S}^2$, where each edge is labeled 2. That is, $K_{3,3}$ is a full-subcomplex of $\mathbb{S}^2$, the nerve of a right-angled Coxeter system. Since $W_{K_{3,3}}$ is infinite, $\beta_0(K_{3,3}) = 0$, and equations (3.4) and (3.5) imply that $\beta_1(P_3) = 1/2$. It then follows from (3.6) that $\beta_2(K_{3,3}) = 1/4$. But this contradicts Lemma 1.2. For details on this proof see [4, Sections 8, 9 and 11].

4. Planar graphs and planar complexes. Now suppose Γ is a simple, connected graph. The idea is similar to that above: We understand Γ as the labeled nerve of some Coxeter system and, if possible, embed Γ as a full subcomplex of a metric flag triangulation of $\mathbb{S}^2$. The problem here is that if Γ contains triangles, or 3-cycles, then we must place restrictions on the labels of these edges in order for Γ itself to be metric flag or for Γ to embed as a full subcomplex of a metric flag simplicial complex. In particular, if r, s and t are vertices of a 3-cycle, then, using the notation from Section 2, we must have the corresponding edge labels satisfy

$$\frac{1}{m_{rs}} + \frac{1}{m_{st}} + \frac{1}{m_{rt}} \leq 1.$$

For then $\{r, s, t\}$ is not a spherical subset of generators and this set does not span a 2-simplex in the nerve of the corresponding Coxeter system. Note that for a given graph, there are many labelings that result in the graph being a metric flag complex. So, our emphasis will not only be on a given graph, but also on a specific labeling of edges. Thus, we say a *labeling* of edges of a graph Γ is *metric flag* if the labeled Γ corresponds to the labeled nerve of a Coxeter system. We observe that if Γ contains 3-cycles, then Γ cannot correspond to the labeled nerve of a right-angled Coxeter system. So, we require the following definition.

DEFINITION 4.1. We say a full subcomplex A of a metric flag simplicial complex L has a *right-angled complement* if the label on all edges not in A is 2.

The following two lemmas will be used in the set-up and proof of our main theorem.

LEMMA 4.2. *Let L be a metric flag simplicial complex, and $A \subseteq L$ a full subcomplex with a right-angled complement. Let B be a full subcomplex of L such that $A \subseteq B$ and let $v \in B - A$ be a vertex. Then B_v, the link of v in B, is a full subcomplex of L.*

Proof. Let T be a subset of vertices contained in B_v and the vertex set of a simplex σ of L. Then T defines a spherical subset of the corresponding

Coxeter system. Since the elements of T are in B_v, v commutes with each vertex of T. Thus $T \cup \{v\}$ is a spherical subset and therefore σ is in B_v. ∎

LEMMA 4.3. *Let L be a metric flag triangulation of $\mathbb{S}^1$, and A a full subcomplex of L. Then $\beta_i(A) = 0$ for $i > 1$.*

Proof. Consider the long exact sequence of the pair $(\Sigma_L, W\Sigma_A)$:

$$0 \to \mathfrak{h}_2(A) \to \mathfrak{h}_2(L) \to \mathfrak{h}_2(L, A) \to \cdots$$

Since Conjecture 1.1 is true in dimension 2, $\mathfrak{h}_2(L) = 0$ and exactness implies the result. ∎

For convenience, we restate the relevant result from [10] needed to prove our main result, Theorem 4.5.

THEOREM 4.4 (see Corollary 4.4 of [10]). *Let L be a metric flag triangulation of $\mathbb{S}^2$. Then*

$$\mathfrak{h}_i(L) = 0 \quad \textit{for all } i.$$

THEOREM 4.5. *Let L be a metric flag triangulation of $\mathbb{S}^2$, and $A \subseteq L$ a full subcomplex with right-angled complement. Then*

$$\beta_i(A) = 0 \quad \textit{for } i > 1.$$

Proof. Let B be a full subcomplex of L such that $A \subseteq B \subseteq L$. We induct on the number of vertices of $L - B$, the case $L = B$ being given by Theorem 4.4. Assume $\mathfrak{h}_i(B) = 0$ for $i > 1$. Let v be a vertex of $B - A$ and set $B' = B - v$. Then $B = B' \cup C_2B_v$ where B_v (by Lemma 4.2) and B' are full subcomplexes. We have the following Mayer–Vietoris sequence:

$$\cdots \to \mathfrak{h}_i(B_v) \to \mathfrak{h}_i(B') \oplus \mathfrak{h}_i(C_2B_v) \to \mathfrak{h}_i(B) \to \cdots.$$

Observe that B_v is a full subcomplex of L_v, the link of v in L, a metric flag triangulation of $\mathbb{S}^1$. So Lemma 4.3 implies $\mathfrak{h}_i(B_v) = 0$ for $i > 1$. Thus, by (3.7), $\mathfrak{h}_i(C_2B_v) = 0$ for $i > 1$. It follows from exactness that $\mathfrak{h}_i(B') = 0$. ∎

Planar complexes. Consider a connected, metric flag complex A of dimension ≤ 2. If A is planar, then it can be embedded as a subcomplex of the 2-sphere. In this case, our goal is to attain a flag triangulation of $\mathbb{S}^2$ with A as a full subcomplex. To that end, we introduce a new vertex in the interior of each complementary region, and cone off the boundary of each region. Now, it could be the case that an n-cycle in A, $n \geq 4$, in which non-adjacent vertices are connected by an edge in A, bounds a complementary region in $\mathbb{S}^2$. In this case, coning off this n-cycle as above and labeling the cone edges with 2's results in a non-metric flag triangulation of $\mathbb{S}^2$. So, after coning off the boundary of each complementary region, we take the barycentric subdivision of each coned region, though to keep A intact as a full subcomplex, we do not subdivide edges included in A. Finally, we label each new edge with 2 and obtain a metric flag triangulation of $\mathbb{S}^2$ in which every edge not in A is labeled 2, i.e. A has a right-angled complement. Thus, we have the following restatement of Theorem 4.5 (cf. [4, Theorem 11.4.1]).

THEOREM 4.6. *Let A be a metric flag complex of dimension ≤ 2. Suppose A is planar (that is, it can be embedded as a subcomplex of the 2-sphere). Then*

$$\beta_2(A) = 0.$$

Proof. By Mayer–Vietoris, we may assume A is connected. Take the embedding of A in $\mathbb{S}^2$ described above; then the result follows from the proof of Theorem 4.5. ■

COROLLARY 4.7. *Suppose Γ is a planar, metric flag, simple graph, not a single edge nor a single vertex. Let W_Γ denote the corresponding Coxeter group and Σ_Γ the corresponding Davis complex with fundamental domain K_Γ. Then $\chi^{\mathrm{orb}}(K_\Gamma) \leq 0$.*

Proof. Since W_Γ is infinite, we know $\beta_0(\Gamma) = 0$. By Theorem 4.6, we know $\beta_2(\Gamma) = 0$. Thus, the result follows from Atiyah's formula (3.5). ■

Planar graphs. We do have specific calculations of χ^{orb} in the case described in Corollary 4.7. Indeed, consider a metric flag labeling of a graph Γ with V vertices and E edges in which n_e is the label on the edge e. Let Σ_Γ denote the corresponding Davis complex with fundamental domain K_Γ, and consider the simplicial decomposition of K_Γ in which simplices correspond to linearly ordered (with respect to containment) chains of spherical subsets. Then K_Γ has one 0-simplex with trivial stabilizer, corresponding to the empty set, V 0-simplices with stabilizers of order 2, and for each edge e, a 0-simplex with a stabilizer of order $2n_e$. Moreover, K_Γ has $E+V$ 1-simplices with trivial stabilizers, each corresponding to chains of the form $\emptyset \subset \{r\}$ or $\emptyset \subset \{r,s\}$, where $r \neq s$ are vertices Γ, and $2E$ 1-simplices with stabilizers of order 2, corresponding to chains of the form $\{r\} \subset \{r,s\}$, where $r \neq s$ are vertices of Γ. Finally, K_Γ has $2E$ 2-simplices with trivial stabilizers corresponding to chains of the form $\emptyset \subset \{r\} \subset \{r,s\}$, where $r \neq s$ are vertices of Γ. Hence

$$\begin{aligned}(4.1)\qquad \chi^{\mathrm{orb}}(K_\Gamma) &= \left(1 + \frac{V}{2} + \left(\sum_e \frac{1}{n_e}\right)\frac{1}{2}\right) - \left(V + E + \frac{2E}{2}\right) + (2E)\\ &= 1 - \frac{V}{2} + \left(\sum_e \frac{1}{n_e}\right)\frac{1}{2}.\end{aligned}$$

So, using this formula with the contrapositive of Corollary 4.7, we have the following test for detecting non-planar graphs.

COROLLARY 4.8. *Let Γ be a simple, connected graph with $V > 2$ vertices. If Γ admits a metric flag labeling where n_e (an integer ≥ 2) is the label on* *the edge e with*

$$1 - \frac{V}{2} + \Bigg(\sum_{edges\, e} \frac{1}{n_e} \Bigg) \frac{1}{2} > 0,$$

then Γ is not planar.

Corollary 4.8 does detect that both of Kuratowski's graphs are non-planar. Indeed, if Γ is the complete bipartite graph on six vertices, we can label each edge with 2 and we get $\chi^{\mathrm{orb}}(K_\Gamma) = 1 - \frac{6}{2} + \frac{9}{4} = \frac{1}{4}$. If Γ is the complete graph on five vertices, we can use a uniform labeling with 3's and we have $\chi^{\mathrm{orb}}(K_\Gamma) = 1 - \frac{5}{2} + \frac{10}{6} = \frac{1}{6}$.

Of course, the straight application of Corollary 4.7 to equation (4.1) is that, for a planar graph, all metric flag labelings satisfy $1 - V/2 + \sum(1/n_e)/2 \leq 0$. Now note that for any graph Γ, a uniform labeling of each edge with 3's is metric flag, and if Γ contains no 3-cycles, then a uniform labeling with 2's is metric flag. So the classical inequalities relating the number of edges and vertices of a planar graph follow from Corollary 4.8.

Corollary 4.9. *Let Γ be a simple, connected, planar graph with $V > 2$ vertices and E edges. Then $E \leq 3V - 6$. If, moreover, Γ contains no 3-cycles, then $E \leq 2V - 4$.*

Proof. Take a uniform labeling of 3's on the edges of Γ. Then

$$\chi^{\mathrm{orb}}(K_\Gamma) = 1 - \frac{V}{2} + \frac{E}{6} \leq 0,$$

which implies that $E \leq 3V - 6$. If Γ contains no 3-cycles, then take a uniform labeling of 2's on the edges to find that

$$\chi^{\mathrm{orb}}(K_\Gamma) = 1 - \frac{V}{2} + \frac{E}{4} \leq 0,$$

which implies that $E \leq 2V - 4$. ■

A stronger inequality . . . but not too strong. Note that in equation (4.1), increasing any one edge label of Γ decreases $\chi^{\mathrm{orb}}(K_\Gamma)$. So, in the case Γ contains no 3-cycles, it is clear that a labeling of 2's on each edge will give you the largest possible orbihedral Euler characteristic. In other words, the ℓ^2-homological method, i.e. the calculation of the orbihedral Euler characteristic described in Corollary 4.8, is not stronger than the classical inequality $E \leq 2V - 4$, meaning it will not detect non-planar graphs the classical inequality misses.

However, in the case of a graph containing 3-cycles, there are choices that can be made. It is not the case that a uniform labeling of 3's will always lead to the largest possible orbihedral Euler characteristic for the orbifold K_Γ.

Example 4.10. Let Γ be the graph pictured in Figure 1, a member of the Petersen family of graphs. Then Γ does contain 3-cycles and we have

$V = 7$ and $E = 15$. So the classical inequality (or the labeling by 3's) does not detect that Γ is non-planar. However, with the indicated metric flag labeling,

$$\chi^{\mathrm{orb}}(K_\Gamma) = 1 - \frac{7}{2} + \left(\frac{7}{2} + \frac{8}{4}\right)\frac{1}{2} = \frac{1}{4}.$$

So, by Corollary 4.8, we can conclude Γ is not planar.

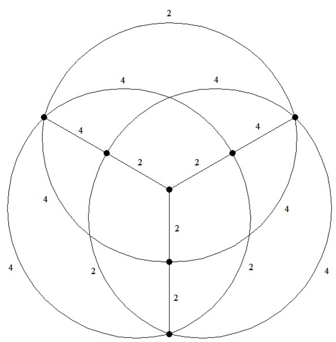

Fig. 1

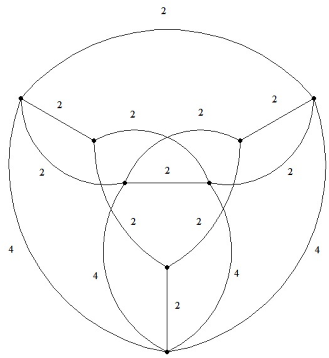

Fig. 2

The Petersen family of graphs provides another example in Figure 2. Here $V = 8$ and $E = 15$, so $E < 3V - 6$, but with the indicated labeling, we again have $\chi^{\mathrm{orb}} = 1/4$ and thus the graph is not planar.

The previous examples do indeed show that the ℓ^2-methods culminating in Corollary 4.8 are stronger than the classical inequality reproved in Corollary 4.9. But there are cases in which non-uniform labelings do not detect a known non-planar graph. Consider the non-planar graph Γ in Figure 3,

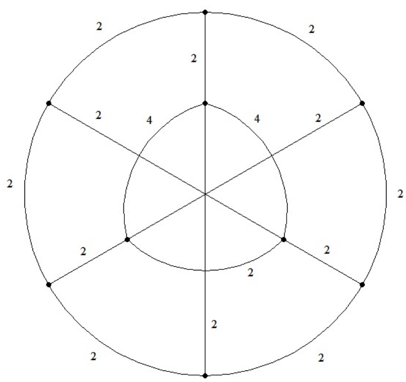

Fig. 3

again a member of the Petersen family of graphs. Since Γ has one 3-cycle, it is clear that the indicated labeling maximizes the orbihedral Euler characteristic of K_Γ. But here $\chi^{\mathrm{orb}}(K_\Gamma) = 0$, and so Corollary 4.8 does not tell us that this graph is non-planar.

Conclusion. As Davis and Okun state in [4], they have found a "complicated proof of the classical fact"[s] that certain (known to be non-planar) graphs are indeed non-planar. However, the ℓ^2-homological methods presented reduce to an accessible and straightforward calculation that provides a stronger test for planarity than the classical inequalities.

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