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ON SEQUENTIALLY RAMSEY SETS

BY

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Abstract. We consider sequentially completely Ramsey and sequentially nowhere Ramsey sets on ω^ω with the topology generated by a free filter $\mathcal{F}$ on ω. We prove that if $\mathcal{F}$ is an ultrafilter, then the σ-algebra of Baire sets is the σ-algebra $S_{\mathcal{F}}\mathcal{CR}$ of sequentially completely Ramsey sets. Further we study additivity and cofinality of the σ-ideal $S_{\mathcal{F}}\mathcal{CR}^0$ of sequentially nowhere Ramsey sets. We prove that if $\mathcal{F}$ is a $P(\mathfrak{b})$-ultrafilter then $\mathrm{add}(S_{\mathcal{F}}\mathcal{CR}^0) = \mathfrak{b}$, and if $\mathcal{F}$ is a P-ultrafilter then $\mathrm{cof}(S_{\mathcal{F}}\mathcal{CR}^0)$ is the point π-character of the space $\mathrm{Seq}(\mathcal{F})$.

1. Introduction. Ramsey and completely Ramsey sets (or in other terminology: completely and nowhere Ramsey sets) were studied by many authors (e.g. [L], [GP], [P], [Sz]) in the context of open, Borel and analytic sets ([El], [GP], [Si], [P]) and cardinal coefficients of ideals ([BSh]). In this paper we study *sequentially completely Ramsey* sets ($S_{\mathcal{F}}\mathcal{CR}$) and *sequentially nowhere Ramsey* sets ($S_{\mathcal{F}}\mathcal{CR}^0$) on ω^ω equipped with the topology generated by a free filter $\mathcal{F}$. These notions are generalizations of the notions of completely Ramsey sets ($\mathcal{CR}_{\mathcal{F}}$) and nowhere Ramsey sets ($\mathcal{CR}^0_{\mathcal{F}}$) on $[\omega]^\omega$ (or on $\omega^{\omega\uparrow}$).

Let Seq and ω^ω denote respectively the set of all finite and all infinite sequences of non-negative integers. We will call them *sequences* and *branches* respectively. Note that we have a natural partial order on Seq: if s, t are two sequences then $s \preceq t$ whenever $s = t{\restriction}\mathrm{dom}(s)$.

Let $\mathcal{F}$ be a free filter on ω. We consider the standard topology on Seq generated by all sets of the form

$$U(s,\phi) = \bigcup\{U_n(s,\phi) : n \in \omega\}$$

where $U_0(s,\phi) = \{s\}$ and $U_{n+1}(s,\phi) = \bigcup\{t^\frown\phi(t) : t \in U_n(s,\phi)\}$, s is a sequence and $\phi(t) \in \mathcal{F}$ for each sequence t.

Note that each $U(s,\phi)$ is clopen and Seq endowed with this topology is Lindelöf and normal because of its cardinality. It is also known ([BSz]) that it is extremally disconnected if and only if $\mathcal{F}$ is an ultrafilter.

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We will write $\phi \subseteq \psi$ when $\phi(s) \subseteq \psi(s)$ for each sequence s. Analogously $(\phi \cap \psi)(s) = \phi(s) \cap \phi(s)$ for every sequence s.

LEMMA 1.1. *Let $s, t \in \mathrm{Seq}$ and $\phi, \psi : \mathrm{Seq} \to \mathcal{F}$. Then the following statements hold:*

(1) *If $s \in U(t, \psi)$, then $t \preceq s$ and $U(s, \psi) \subseteq U(t, \psi)$.*
(2) *If $U(s, \phi) \cap U(t, \psi) \neq \emptyset$ then either $s \preceq t$ or $t \preceq s$.*
(3) *If $\mathrm{dom}(s) = \mathrm{dom}(t)$ and $s \neq t$ then $U(s, \phi) \cap U(t, \psi) = \emptyset$.*
(4) *If $\mathrm{dom}(t) \leq \mathrm{dom}(s)$ and $s \notin U(t, \psi)$ then $U(s, \phi) \cap U(t, \psi) = \emptyset$.* ∎

Take a collection $\{\phi_n : \mathrm{Seq} \to \mathcal{F} : n \in \omega\}$ and a sequence s. Then $\{U(s, \phi_n) : n \in \omega\}$ is a *fusion sequence* if $U(s, \phi_{n+1}) \subseteq U(s, \phi_n)$ and $U_k(s, \phi_{n+1}) = U_k(s, \phi_n)$ for every $k \leq n$.

PROPOSITION 1.2 (Fusion Lemma). *If $\{U(s, \phi_n) : n \in \omega\}$ is a fusion sequence then $\bigcap\{U(s, \phi_n) : n \in \omega\}$ is open.*

Proof. Set $U = \bigcap\{U(s, \phi_n) : n \in \omega\}$. Of course U is not empty. Assume that ψ is such that $\psi(t) = \bigcap_{k \leq n+1} \phi_k(t)$ for each t which satisfies $\mathrm{dom}(s) + n = \mathrm{dom}(t)$. To prove the statement it is enough to check that $U(s, \psi) = U$. By the definition of ψ it suffices to show that $U \subseteq U(s, \psi)$.

So assume that $t \in U(s, \psi)$ whenever $t \in U$ and $\mathrm{dom}(t) = \mathrm{dom}(s) + n$ for some integer $n > 0$. If $t_1 \in U$ is such that $\mathrm{dom}(t_1) = \mathrm{dom}(s) + n + 1$ then there exists a sequence $t \in U$ such that $t \prec t_1$ and $\mathrm{dom}(t) = \mathrm{dom}(t_1) - 1$. So $t \in U(s, \psi)$ by the inductive assumption and $t_1 \in t^\frown\psi(t)$ by the choice of ψ. ∎

Let $s \in \mathrm{Seq}$ and $\phi : \mathrm{Seq} \to \mathcal{F}$ be given. We define the set of all branches of $U(s, \phi)$ as follows:

$$[U(s, \phi)] = \{f \in \omega^\omega : \forall n \in \omega \ (f{\restriction}(\mathrm{dom}(s) + n) \in U_n(s, \phi))\}.$$

LEMMA 1.3. *For any sequences s and t:*

(1) $U(s, \phi) \subseteq U(t, \psi) \Rightarrow [U(s, \phi)] \subseteq [U(t, \psi)]$.
(2) $[U(s, \psi)] = \bigcup\{[U(t, \phi)] : t \in U_n(s, \phi)\}$. ∎

LEMMA 1.4. *The family of all sets of branches is a base of a topology on the set ω^ω.*

Proof. Note that every branch $f \in \omega^\omega$ is a member of $[U(\emptyset, \phi_\omega)]$ where $\mathrm{Seq} = \phi_\omega^{-1}[\{\omega\}]$. Further, the statement is a consequence of filter properties. ∎

We will consider ω^ω to be equipped with the topology defined in Lemma 1.4 for the rest part of this paper.

2. The classes $S_{\mathcal{F}}\mathcal{CR}$ and $S_{\mathcal{F}}\mathcal{CR}^0$. A set $M \subseteq \omega^\omega$ is *sequentially completely Ramsey* if for every $U(s,\phi)$ there exists $\psi \subseteq \phi$ such that either $[U(t,\psi)] \subseteq M$ or $[U(t,\psi)] \cap M = \emptyset$. If for every $U(s,\phi)$ there exists $\psi \subseteq \phi$ which satisfies the latter condition then $M \subseteq \omega^\omega$ is *sequentially nowhere Ramsey.* The families of all sequentially completely Ramsey sets and all sequentially nowhere Ramsey sets will be denoted by $S_{\mathcal{F}}\mathcal{CR}$ and $S_{\mathcal{F}}\mathcal{CR}^0$ respectively.

LEMMA 2.1. *$S_{\mathcal{F}}\mathcal{CR}^0$ is the ideal of all nowhere dense sets.* ∎

LEMMA 2.2. *Let $\phi : \mathrm{Seq} \to \mathcal{F}$. Then $[U(s,\phi)] \in S_{\mathcal{F}}\mathcal{CR}$ for any sequence s.*

Proof. Assume $[U(t,\psi)] \cap [U(s,\phi)] \neq \emptyset$. Then by Lemma 1.1, $t \prec s$ or $s \prec t$. In the first case we put $\lambda(t) = \psi(t) \setminus s(\mathrm{dom}(t))$, and $\lambda(u) = \psi(u)$ for $u \neq t$. Then $U(s,\phi) \cap U(t,\lambda) = \emptyset$ and $[U(s,\phi)] \cap [U(t,\lambda)] = \emptyset$.

The second case is a simple consequence of the filter properties. Namely if $\lambda(u) = \phi(u) \cap \psi(u)$ for every $u \in U(t,\psi)$ then $[U(t,\lambda)] \subseteq [U(s,\phi)]$. ∎

Till the end of the paper, we assume that $\mathcal{F}$ is an ultrafilter.

PROPOSITION 2.3. *Let $M \subseteq \omega^\omega$ and $U(s,\phi)$ be given. Then either*

(1) *there exists a function $\psi \subseteq \phi$ such that $[U(s,\psi)] \subseteq M$, or*
(2) *there exists a function $\psi \subseteq \phi$ such that $[U(t,\lambda)] \not\subseteq M$ for each $t \in U(s,\psi)$ and every λ with $\lambda \subseteq \psi$.*

Proof. Assume that (1) does not hold. We shall construct a fusion sequence $\{U(s,\psi_n) : n \in \omega\}$ such that for every n there is no $t \in U(s,\psi_n)$ and no $\lambda \subseteq \psi_n$ with $[U(t,\lambda)] \subseteq M$.

Take $\psi_0 = \phi$ and assume that we have defined $U(s,\psi_0),\ldots,U(s,\psi_{n-1})$ so that the above statement is true. If $t \in U_{n-1}(s,\psi_{n-1})$ we denote by X_t the set of all $m \in \psi_{n-1}(t)$ such that there exists a function λ with $U(t^\frown m,\lambda) \subseteq U(t^\frown m,\psi_{n-1})$ and $[U(t^\frown m,\lambda)] \subseteq M$.

Then either X_t or $\psi_{n-1}(t) \setminus X_t$ is in $\mathcal{F}$. In the first case there would be a function $\lambda \subseteq \psi_{n-1}$ such that $[U(t,\lambda)] \subseteq M$, contradicting the inductive assumption. So $\psi_{n-1}(t) \setminus X_t \in \mathcal{F}$. To finish the construction we define

$$\psi_n(t) = \begin{cases} \psi_{n-1}(t) \setminus X_t, & t \in U_{n-1}(s,\psi_{n-1}),\\ \psi_{n-1}(t), & \text{other } t.\end{cases}$$

By the Fusion Lemma we are done. ∎

PROPOSITION 2.4.

$$S_{\mathcal{F}}\mathcal{CR}^0 = \{M \subseteq \omega^\omega : \forall U(s,\phi)\ \exists \psi \subseteq \phi\ ([U(s,\psi)] \cap M = \emptyset)\}.$$

Proof. This follows directly from the previous proposition. ∎

Recall ([En]) that M is *nowhere dense* if for every non-empty open set U there exists a non-empty open set $V \subseteq U$ such that $V \cap M = \emptyset$. Hence every nowhere dense set in ω^ω is an $S_{\mathcal{F}}\mathcal{CR}^0$-set.

PROPOSITION 2.5. *$S_{\mathcal{F}}\mathcal{CR}^0$ is the σ-ideal of nowhere dense sets.*

Proof. Consider a family $\{M_n : n \in \omega\}$ of $S_{\mathcal{F}}\mathcal{CR}^0$-sets.

Let $M = \bigcup\{M_n : n \in \omega\}$ and take an arbitrary sequence s and a function ϕ. We shall define a fusion sequence $\{U(s,\phi_n) : n \in \omega\}$ such that $\phi_n \subseteq \phi$ and $[U(s,\phi_n)] \cap M_n = \emptyset$ for every n. We choose $U(s,\phi_0)$ already by the definition of $S_{\mathcal{F}}\mathcal{CR}^0$-set with respect to M_0. Assume $U(s,\phi_0),\ldots,U(s,\phi_n)$ are already defined. Then we take a function ψ_t for every $t \in U_n(s,\phi_n)$ such that $\psi_t \subseteq \phi_n$ and $[U(t,\psi_t)] \cap M_{n+1} = \emptyset$. We put $\phi_{n+1}(u) = \phi_n(u)$ if $\mathrm{dom}(u) < \mathrm{dom}(s) + n$ and $\phi_{n+1}(u) = \psi_{u\restriction(\mathrm{dom}(s)+n)}(u)$ if $\mathrm{dom}(u) \geq \mathrm{dom}(s) + n$. Note that if $f \in [U(s,\phi_{n+1})]$ then there exists $t \in U_n(s,\phi_n)$ such that $f \in [U(t,\phi_{n+1})]$. So by the Fusion Lemma there exists a function ψ such that $U(s,\psi) \subseteq U(s,\phi_n)$ for each $n \in \omega$ and $[U(s,\psi)] \cap M = \emptyset$. ∎

Summarizing the foregoing results, every set $A \subseteq \omega^\omega$ with Baire property is the union of an open set U and an $S_{\mathcal{F}}\mathcal{CR}^0$-set M_0. We shall show that if $\mathcal{F}$ is an ultrafilter, the class $S_{\mathcal{F}}\mathcal{CR}$ coincides with the class of Baire sets.

PROPOSITION 2.6.

$$S_{\mathcal{F}}\mathcal{CR} = \{M \subseteq \omega^\omega : \forall U(s,\phi)\ \exists \psi \subseteq \phi \\ ([U(s,\psi)] \cap M = \emptyset \vee [U(s,\psi)] \subseteq M)\}.$$

Proof. Let $M \in S_{\mathcal{F}}\mathcal{CR}$. Then there exists an $S_{\mathcal{F}}\mathcal{CR}^0$-set M_0 and a $U \in \mathcal{T}_p$ such that $M = M_0 \cup U$. Let $U(s,\phi)$ be given. By Proposition 2.3 there exists a function $\psi \subseteq \phi$ such that $[U(s,\psi)] \cap M_0 = \emptyset$. If $[U(s,\psi)] \cap U = \emptyset$ then we are done. Assume otherwise and suppose that there is no $\psi' \subseteq \psi$ such that $[U(s,\psi')] \subseteq M$. Since U is open, there exists a sequence t and a function λ such that $[U(t,\lambda)] \subseteq [U(s,\psi)] \cap U$, contrary to Proposition 2.1. ∎

PROPOSITION 2.7. *$M \subseteq \omega^\omega$ is a $S_{\mathcal{F}}\mathcal{CR}$-set if and only if M is a Baire set.*

Proof. If M is $S_{\mathcal{F}}\mathcal{CR}$-set then it is not hard to see that $M_0 = M \setminus \mathrm{Int}\, M$ is an $S_{\mathcal{F}}\mathcal{CR}^0$-set.

Assume now M is a Baire set. Then we can find an open set U and a $S_{\mathcal{F}}\mathcal{CR}^0$-set N such that $M = U \cup N$. Let $U(s,\phi)$ be such that $[U(s,\phi)] \cap N = \emptyset$ and $[U(s,\phi)] \cap U \neq \emptyset$. By Proposition 2.3, either (1) there exists a function ψ_1 such that $[U(s,\psi_1)] \subseteq U$, or (2) there exists a function ψ_2 such that $[U(t,\psi_2)] \not\subseteq U$ for every $t \in U(s,\psi_2)$. If (1) holds we are done since $U \subseteq M$. If (2) holds, then we get a contradiction, because $U \cap [U(s,\phi)]$ is open and non-empty. ∎

LEMMA 2.8. *$S_{\mathcal{F}}\mathcal{CR}$ is a σ-field of sets.*

Proof. This follows from the previous proposition and Proposition 2.5. ∎

3. Cardinal invariants. Let us recall some cardinal coefficients of non-trivial ideals $\mathcal{I} \subseteq \omega^\omega$ containing all singletons (see e.g. [BSh]):

$$\begin{aligned}\operatorname{add}(\mathcal{I}) &= \min\Big\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \wedge \bigcup \mathcal{F} \notin \mathcal{I}\Big\},\\ \operatorname{cov}(\mathcal{I}) &= \min\Big\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \wedge \bigcup \mathcal{F} = \omega^\omega\Big\},\\ \operatorname{non}(\mathcal{I}) &= \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \wedge \mathcal{F} \notin \mathcal{I}\}\\ \operatorname{cof}(\mathcal{I}) &= \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \wedge \mathcal{F} \text{ is a base of } \mathcal{I}\},\end{aligned}$$

here $\mathcal{F}$ is a *base* of $\mathcal{I}$ if for each $A \in \mathcal{I}$ there exists $B \in \mathcal{F}$ such that $A \subseteq B$. These cardinals are referred to as the *additivity, covering, uniformity* and *cofinality* of $\mathcal{I}$. Observe that $S_{\mathcal{F}}\mathcal{CR}^0$ contains all singletons and hence $\bigcup S_{\mathcal{F}}\mathcal{CR}^0 = \omega^\omega$. So

$$\operatorname{add}(S_{\mathcal{F}}\mathcal{CR}^0) \le \min(\operatorname{non}(S_{\mathcal{F}}\mathcal{CR}^0), \operatorname{cov}(S_{\mathcal{F}}\mathcal{CR}^0))$$

and

$$\operatorname{cof}(S_{\mathcal{F}}\mathcal{CR}^0) \ge \max(\operatorname{non}(S_{\mathcal{F}}\mathcal{CR}^0), \operatorname{cov}(S_{\mathcal{F}}\mathcal{CR}^0)).$$

By Proposition 2.5, if $\mathcal{F}$ is an ultrafilter, then

$$\omega < \operatorname{add}(S_{\mathcal{F}}\mathcal{CR}^0).$$

LEMMA 3.1.

(1) *For any function ϕ,*

$$A_\phi = \omega^\omega \setminus \bigcup \{[U(s,\phi)] : s \in \mathrm{Seq}\}$$

is an $S_{\mathcal{F}}\mathcal{CR}^0$-set.

(2) *The collection of all A_ϕ's is a base of $S_{\mathcal{F}}\mathcal{CR}^0$.* ∎

Let us recall that $\mathfrak{b}$ is the minimal size of an unbounded subfamily of ω^ω, and $\mathfrak{d}$ is the minimal size of a dominating subfamily of ω^ω; we write $f \le^\star g$ if $\{n \in \omega : f(n) > g(n)\}$ is finite. It is well known that $\mathfrak{b} \le \mathfrak{d}$.

PROPOSITION 3.2.

$$\operatorname{cov}(S_{\mathcal{F}}\mathcal{CR}^0) \le \mathfrak{b} \le \mathfrak{d} \le \operatorname{non}(S_{\mathcal{F}}\mathcal{CR}^0).$$

Proof. Let $\mathcal{U} \subseteq \omega^\omega$ realize $\mathfrak{b}$. If $f \in \mathcal{U}$ then we put

$$\phi_f(s) = \{n \in \omega : n > f(\operatorname{dom}(s))\}$$

for each sequence s. Of course $\phi_f(s)$ is a member of $\mathcal{F}$ since it is cofinite and by the previous lemma each set of the form A_{ϕ_f} is an $S_{\mathcal{F}}\mathcal{CR}^0$-set. We shall prove that $\bigcup\{A_{\phi_f} : f \in \mathcal{U}\} = \omega^\omega$. Indeed, if g is a branch then there exists a branch f in $\mathcal{U}$ such that $\neg f \leq^\star g$. This means that for every $n \in \omega$ there exists $m \geq n$ such that $g(m) < f(m)$. Then $g \in A_{\phi_f}$.

Assume now that $A \subseteq \omega^\omega$ is not an $S_{\mathcal{F}}\mathcal{CR}^0$-set. Then A dominates ω^ω. Indeed, assume the contrary. Then there exists a branch f such that $\neg f \leq^\star g$ for each g from A. Define ϕ_f as at the beginning of the proof. Then $A_{\phi_f} \supseteq A$, so we get a contradiction, since $S_{\mathcal{F}}\mathcal{CR}^0$ is a proper ideal of sets. ∎

Take a cardinal $\lambda < \omega$. A free filter $\mathcal{F}$ on ω is a $P(\lambda)$-*filter* if for every $\tau < \lambda$ and every subfamily $\{A_\xi : \xi < \tau\}$ of $\mathcal{F}$ there exists an $A \in \mathcal{F}$ such that $A \setminus A_\xi$ is finite for every $\xi < \tau$ (we write briefly $A \subseteq^\star A_\xi$).

Remark 3.3. Note that in the definitions of the numbers $\mathfrak{b}$ and $\mathfrak{d}$ we can replace ω^ω by ω^{Seq}, where $f \prec^\star g$ means that the set of all sequences x such that $f(x) > g(x)$ is finite.

Proposition 3.4. *Let $\lambda > \omega$ be a cardinal and $\mathcal{F}$ be a $P(\lambda)$-ultrafilter on ω. If $\mathfrak{b} \geq \lambda$ then* $\mathrm{add}(S_{\mathcal{F}}\mathcal{CR}^0) \geq \lambda$.

Proof. Consider $\mathfrak{b} \geq \lambda$, $\tau < \lambda$ and let $\{M_\xi : \xi < \tau\}$ be a family of $S_{\mathcal{F}}\mathcal{CR}^0$-sets. Let $U(s,\phi)$ be given. By Proposition 2.4 for every $\xi < \tau$ there exists a subset $U(s,\psi_\xi)$ of $U(s,\phi)$ such that $[U(s,\psi_\xi)] \cap M_\xi = \emptyset$. Since $\mathcal{F}$ is a $P(\lambda)$-filter, there exists an $A \in \mathcal{F}$ such that $A \subseteq^\star \psi_\xi(t)$ for every $t \in \mathrm{Seq}$ and $\xi < \tau$. Put $f_\xi(t) = \max\{n \in \omega : n \in A \setminus \psi_\xi(t)\} + 1$ for $t \in \mathrm{Seq}$. Then there exists a function $G \in \mathrm{Seq}^\omega$ such that $f_\xi \leq^\star G$ for every $\xi < \tau$. We define $\Gamma \in \mathcal{F}^{\mathrm{Seq}}$ by $\Gamma(t) = A \setminus G(t)$ for $t \in \mathrm{Seq}$. Then $[U(s,\Gamma)] \cap \bigcup\{M_\xi : \xi < \tau\} = \emptyset$. Indeed, to see this we set $m(\xi) = \max\{m \in \omega : (\exists t \in {}^m\omega)(f_\xi(t) > G(t))\} + 1$ and consider two cases:

Case 1: $m(\xi) \leq \mathrm{dom}(s)$. Then $f_\xi(t) \leq G(t)$ for every $s \preceq t$. Thus if $s \preceq t$ and $n \in \Gamma(t)$ then $n \in \psi_\xi(t)$, by the definitions of Γ and f_ξ. So by Lemma 1.3, $[U(s,\Gamma)] \subseteq [U(s,\psi_\xi)]$ and $[U(s,\Gamma)] \cap M_\xi = \emptyset$.

Case 2: $m(\xi) > \mathrm{dom}(s)$. Let $n = m(\xi) - \mathrm{dom}(s)$. First note that if $\mathrm{dom}(t) \geq n + \mathrm{dom}(s)$ then $U(t,\Gamma) \subseteq U(t,\psi_\xi)$ just as in Case 1, since $\mathrm{dom}(t) \geq m(\xi)$. Consider now $\mathrm{dom}(t) = \mathrm{dom}(s) + n - 1$. Then, since $\Gamma(t) \subseteq^\star \psi_\xi(t)$, there exist numbers $n_1, \ldots, n_l$ such that $f_\xi(t^\frown n) \leq G(t^\frown n)$ for each $n \notin \{n_1, \ldots, n_l\}$. So if we slightly modify the function Γ setting $\Gamma'(t) = \Gamma(t) \setminus \{n_1, \ldots, n_l\}$ then $U(t,\Gamma') \subseteq U(t,\psi_\xi)$. On the other hand,

$$U(t,\Gamma) = U(t,\Gamma') \cup \bigcup\{U(t^\frown n_i,\Gamma) : i = 1, \ldots, l\}$$

and $\mathrm{dom}(t^\frown n_i) = n + \mathrm{dom}(s)$, so by the inductive assumption $U(t,\Gamma) \subseteq U(t,\psi_\xi)$. Hence we are done. ∎

Corollary 3.5. *If $\mathcal{F}$ is a $P(\mathfrak{b})$-ultrafilter then* $\mathrm{add}(S_{\mathcal{F}}\mathcal{CR}^0) = \mathfrak{b}$.

Question 3.6. *If* $\mathrm{add}(S_{\mathcal{F}}\mathcal{CR}^0) = \lambda > \omega$, *is then $\mathcal{F}$ a $P(\lambda)$-ultrafilter?*

PROPOSITION 3.7.

$$\pi\chi(\mathrm{Seq}(\mathcal{F}), s) \geq \chi(\mathcal{F}) \cdot \mathfrak{d}.$$

Proof (cf. [JSz]). Assume $\mathcal{B}(s)$ is a π-base at s. For every $A \in \mathcal{F}$ we put $\mathbb{A}(t) = A$ for each sequence t. If $A \in \mathcal{F}$ then there exists $U(t, \phi) \in \mathcal{B}(s)$ such that $U(t, \phi) \subseteq U(s, \mathbb{A})$. Hence $\{\phi(t) : U(t, \phi) \in \mathcal{B}(s)\}$ is a base of the ultrafilter $\mathcal{F}$. So we conclude $\chi(\mathcal{F}) \leq \pi\chi(\mathrm{Seq}(\mathcal{F}), s)$.

Now for each $U(t, \phi) \in \mathcal{B}(s)$ and every integer $n > \mathrm{dom}(t)$ take $g^t_\phi(n)$ from the union of all $\phi(u)$ where $u \in U(t, \phi)$ and $\mathrm{dom}(u) = n$ (for example, take the minimum of this union). Then the collection of all g^t_ϕ is a dominating family. Indeed, if f is a branch then we define $\psi_f(u) = \omega \setminus (f(\mathrm{dom}(u)) + 1)$. So if $U(t, \phi) \in \mathcal{B}(s)$ is contained in $U(s, \psi_f)$ then $f \leq^\star g_{\psi_f}$. Thus we get $\pi\chi(\mathrm{Seq}(\mathcal{F}), s) \geq \mathfrak{d}$. ∎

PROPOSITION 3.8.

$$\pi\chi(\mathrm{Seq}(\mathcal{F}), s) \leq \mathrm{cof}(S_{\mathcal{F}}\mathcal{CR}^0)$$

Proof. Assume $\mathcal{B}$ realizes cofinality of $S_{\mathcal{F}}\mathcal{CR}^0$. By Lemma 3.1 for every $M \in \mathcal{B}$ there exists a function ψ_M such that $M \subseteq A_{\psi_M}$. It is enough to check that $\{U(s, \psi_M) : M \in \mathcal{B}\}$ is a base of $\mathrm{Seq}(\mathcal{F})$ at s. So assume $U(s, \phi)$ is given. Then $M_\phi = \omega^\omega \setminus \bigcup\{[U(t, \phi)] : t \in \mathrm{Seq}\}$ is a sequentially nowhere Ramsey set. So there is $M \in \mathcal{B}$ such that $M_\phi \subseteq M$. Hence $M_\phi \subseteq A_{\psi_M}$, and so $\bigcup\{[U(t, \psi_M)] : t \in \mathrm{Seq}\} \subseteq \bigcup\{[U(t, \phi)] : t \in \mathrm{Seq}\}$.

We claim now that there is a sequence t such that $U(t, \psi_M) \subseteq U(s, \phi)$. Assume the contrary and take the first level n_0 of $U(s, \psi_M)$ such that $U_{n_0}(s, \psi_M) \setminus U_{n_0}(s, \phi)$ is non-empty. Consider $t_0 \in U_{n_0}(s, \psi_M) \setminus U_{n_0}(s, \phi)$ and take the first level n_1 of $U(t_0, \psi_M)$ such that $U_{n_1}(t_0, \psi_M) \setminus U_{n_1}(t_0, \phi)$ is non-empty. Consider t_1 in it, and so on. We obtain a sequence $\{n_i\}_{i\in\omega}$ of integers and a collection of sequences $\{t_i\}_{i\in\omega}$ such that $U_{n_{i+1}}(t_i, \psi_M) \setminus U_{n_{i+1}}(t_i, \phi)$ is non-empty. So if we take a branch $f \in [U(s, \psi_M)]$ such that $f{\restriction}(\mathrm{dom}(s) + n_i) = t_i$ then $f \notin \bigcup\{[U(t, \phi)] : t \in \mathrm{Seq}\}$, a contradiction. So there must exist a sequence t which satisfies $U(t, \psi_M) \subseteq U(s, \phi)$. ∎

PROPOSITION 3.9. *If $\mathcal{F}$ is a $P(\omega_1)$-ultrafilter then*

$$\pi\chi(\mathrm{Seq}(\mathcal{F}), s) = \mathrm{cof}(S_{\mathcal{F}}\mathcal{CR}^0) = \chi(\mathcal{F}) \cdot \mathfrak{d}.$$

Proof. It is enough to check that $\mathrm{cof}(S_{\mathcal{F}}\mathcal{CR}^0) \leq \mathfrak{d} \cdot \chi(\mathcal{F})$. So take a dominating collection of branches $\mathcal{D} \subseteq \mathrm{Seq}^\omega$ and a base $\mathcal{B}$ of the ultrafilter $\mathcal{F}$. For any $f \in \mathcal{D}$ and $A \in \mathcal{B}$ we put $\psi_{f,A}(s) = A \setminus (f(s)+1)$ for every sequence s. We shall show that the collection of all $A_{\psi_{f,A}}$ (defined as in Lemma 3.1) is a base of the ideal $S_{\mathcal{F}}\mathcal{CR}^0$. Indeed, assume M is a sequentially nowhere Ramsey set. By Lemma 3.1 there exists a function ϕ which satisfies $M \subseteq A_\phi$. Since $\mathcal{F}$ is a P-ultrafilter, we can choose $B \in \mathcal{B}$ such that $B \subseteq^\star \phi(s)$ for each sequence s. We define $g(s) = \max(B \setminus \phi(s))+1$ for all. By the domination of $\mathcal{D}$

there exists $f \in \mathcal{D}$ such that $g \leq^\star f$. Then $A_\phi \subseteq A_{\psi_{f,B}}$. Indeed, if $h \notin A_{\psi_{f,B}}$ then $h \in [U(s, \psi_{f,B})]$ for some s. So $h(\mathrm{dom}(s)+k) \in B \setminus (f(f{\restriction}_{\mathrm{dom}(s)+k})+1)$ for all $k \in \omega$. But there exists $n \in \omega$ such that $g(t) < f(t)$ for every $t \in U(s, \psi_{f,B})$ such that $\mathrm{dom}(t) > n$. So $h(\mathrm{dom}(s)+k) \in B \setminus (g(f{\restriction}_{\mathrm{dom}(s)+k})+1)$ for every k with $\mathrm{dom}(s)+k > n$. Hence $h \in [U(h{\restriction}_{\mathrm{dom}(s)+n}, \phi)]$, which means $h \notin A_\phi$. ∎

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