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SPECTRAL SUBSPACES FOR THE FOURIER ALGEBRA

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Abstract. In this note we define and explore, *à la* Godement, spectral subspaces of Banach space representations of the Fourier–Eymard algebra of a (nonabelian) locally compact group.

Godement, in his basic paper [2] on the Wiener Tauberian theorem and spectral theory of bounded functions on a locally compact abelian group G, defined and studied spectral subspaces of certain Banach space representations of G. In this note we undertake an analogous study for representations of the Fourier algebra of a (nonabelian) locally compact group.

The Fourier algebra $A(G)$ of a locally compact group G was defined and studied by Eymard [1]. All that is needed here about $A(G)$ can be found in that paper. $A(G)$ is a commutative, semisimple, regular Banach algebra with pointwise operations whose Gelfand maximal ideal space is identified with G via point evaluations $\lambda(x)$, $x \in G$. For T in the dual $\mathrm{VN}(G)$ of $A(G)$, the support of T is defined by $\operatorname{supp} T = \{x \in G : u \in A(G), u(x) \neq 0 \Rightarrow u.T \neq 0\}$ where $u.T \in \mathrm{VN}(G)$ is defined by $\langle u.T, v\rangle = \langle T, uv\rangle$, $v \in A(G)$.

Let $\pi : A(G) \to \mathcal{B}(X)$ be an algebra representation of $A(G)$ on a Banach space X which is continuous in the following sense: for each $\xi \in X$ and $\varphi \in X^*$, $T_{\varphi,\xi}$ defined on $A(G)$ by $\langle T_{\varphi,\xi}, u\rangle := \langle \varphi, \pi(u)\xi\rangle$, $u \in A(G)$, is a bounded linear functional on $A(G)$. Fix a continuous representation π of $A(G)$ as above. For a closed subset E of G, define

$$M_E = \{\xi \in X : \operatorname{supp} T_{\varphi,\xi} \subseteq E \text{ for every } \varphi \in X^*\}.$$

PROPOSITION 1. *With notation as above*:

(i) *M_E is a closed linear subspace of X.*
(ii) *M_E is π-invariant*: *$\xi \in M_E \Rightarrow \pi(u)\xi \in M_E$ for all $u \in A(G)$.*

Proof. (i) For $\varphi \in X^*$, $\xi, \eta \in X$ and $\alpha \in \mathbb{C}$, it is easy to check that $T_{\varphi,\xi+\eta} = T_{\varphi,\xi} + T_{\varphi,\eta}$ and $T_{\varphi,\alpha\xi} = \alpha T_{\varphi,\xi}$. These combined with the results of

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Eymard that, for $S, T \in A(G)^*$,

$$\operatorname{supp}(S+T) \subseteq \operatorname{supp} S \cup \operatorname{supp} T, \qquad \operatorname{supp}(\alpha T) \subseteq \operatorname{supp} T,$$

show that M_E is a linear subspace. Further, if $\xi_n \to \xi$ in X, then $\langle T_{\varphi,\xi_n}, u\rangle = \langle \varphi, \pi(u)\xi_n\rangle \to \langle \varphi, \pi(u)\xi\rangle = \langle T_{\varphi,\xi}, u\rangle$, $u \in A(G)$. This means that $T_{\varphi,\xi_n} \to T_{\varphi,\xi}$ in the weak-$*$ topology of $A(G)^*$. By a result [1] of Eymard again, $\operatorname{supp} T_{\varphi,\xi_n} \subseteq E$ for all n implies $\operatorname{supp} T_{\varphi,\xi} \subseteq E$. This proves that M_E is closed.

(ii) A simple computation shows that, for $u \in A(G)$, $\xi \in X$ and $\varphi \in X^*$, $T_{\varphi,\pi(u)\xi} = u.T_{\varphi,\xi}$. Again invoking Eymard [1], we therefore have

$$\operatorname{supp} T_{\varphi,\pi(u)\xi} \subseteq \operatorname{supp} u \cap \operatorname{supp} T_{\varphi,\xi}.$$

Now (ii) is an immediate consequence. ∎

The subspace M_E is called the π-*spectral subspace* of X associated to E. Recall that the representation π is said to be *nondegenerate* if $\pi(u)\xi = 0$ for all $u \in A(G)$ implies $\xi = 0$.

Examples. (i) Suppose $P \in \mathcal{B}(X)$ is a projection, i.e. $P^2 = P$. Fix $x_0 \in G$. Define $\pi = \pi(P, x_0)$ by $\pi(u) = u(x_0)P$, $u \in A(G)$. Then π is a representation of $A(G)$. We have

$$\begin{aligned}\ker P &= \{\xi \in X : \pi(u)\xi = 0 \text{ for all } u \in A(G)\}\\ &= \{\xi \in X : T_{\varphi,\xi} = 0 \text{ for all } \varphi \in X^*\}\end{aligned}$$

and $M_\emptyset = \ker P$, $M_{\{x_0\}} = X$. Observe that π is nondegenerate only when $P = I$, the identity operator on X.

(ii) Consider the (nondegenerate) regular representation ϱ of $A(G)$ acting on itself by mutiplication: $\varrho(u)(v) = uv$, $u, v \in A(G)$. For a closed subset E of G, it is easy to see that the ϱ-spectral subspace is given by

$$M_E = \{u \in A(G) : \operatorname{supp} u.T \subseteq E \text{ for all } T \in \mathrm{VN}(G)\}.$$

By a result of Eymard [1], $\operatorname{supp} u.T \subseteq \operatorname{supp} u \cap \operatorname{supp} T$, so if $\operatorname{supp} u \subseteq E$, then $u \in M_E$. If u is not supported in E, then there is an $x \notin E$ with $u(x) \neq 0$ and then $x \in \operatorname{supp} u.\lambda(x)$. Thus $u \notin M_E$. Hence

$$M_E = \{u \in A(G) : \operatorname{supp} u \subseteq E\}.$$

Proposition 2. *Suppose that π is a nondegenerate representation of $A(G)$ on a Banach space X. Then*:

(i) $M_\emptyset = \{0\}$ *and* $M_G = X$.
(ii) *If $\{E_i\}$ is a collection of closed subsets of G, then* $M_{\bigcap E_i} = \bigcap M_{E_i}$.
(iii) *If K_1, K_2 are disjoint compact subsets of G, then*

$$M_{K_1 \cup K_2} = M_{K_1} \oplus M_{K_2}.$$

Proof. (i) It is trivially true that $M_G = X$. If $T_{\varphi,\xi} = 0$ for all $\varphi \in X^*$, then $\langle \varphi, \pi(u)\xi\rangle = 0$ for $u \in A(G)$ and all $\varphi \in X^*$. This implies $\pi(u)\xi = 0$ for all $u \in A(G)$, and the assumed nondegeneracy of π now gives $\xi = 0$. This proves $M_\emptyset = \{0\}$. The easy proof of (ii) is omitted.

(iii) Let $\xi \in M_{K_1} \cap M_{K_2}$. Then $\operatorname{supp} T_{\varphi,\xi} \subseteq K_1 \cap K_2 = \emptyset$, so $T_{\varphi,\xi} = 0$ for all $\varphi \in X^*$. This implies that $\pi(u)\xi = 0$ for all $u \in A(G)$ and so $\xi = 0$ since π is nondegenerate. We have thus shown that $M_{K_1} \cap M_{K_2} = \{0\}$.

Next, choose open sets U_i and V_i, $i = 1, 2$, such that $K_i \subseteq U_i \subseteq \overline{U}_i \subseteq V_i$ with $\overline{U}_i$ compact and $V_1 \cap V_2 = \emptyset$. Then there are functions u_1, u_2 in $A(G)$ such that $u_i = 1$ on U_i and $\operatorname{supp} u_i \subseteq V_i$. For $\xi \in X$, write $\xi_i = \pi(u_i)\xi$. Now $T_{\varphi,\xi_i} = T_{\varphi,\pi(u_i)\xi} = u_i.T_{\varphi,\xi}$ and so $\operatorname{supp} T_{\varphi,\xi_i} \subseteq \operatorname{supp} u_i \cap \operatorname{supp} T_{\varphi,\xi}$. Thus if $\xi \in M_{K_1 \cup K_2}$, we have $\operatorname{supp} T_{\varphi,\xi_i} \subseteq V_i \cap (K_1 \cup K_2) = K_i$. This means that $\xi_i \in M_{K_i}$. Moreover, for $u \in A(G)$ and $\varphi \in X^*$,

$$\langle \varphi, \pi(u)(\xi_1 + \xi_2 - \xi)\rangle = \langle \varphi, \pi(uu_1 + uu_2 - u)\xi\rangle = \langle T_{\varphi,\xi}, uu_1 + uu_2 - u\rangle.$$

Now, $uu_1 + uu_2 - u = 0$ on $U_1 \cup U_2$, and if $\xi \in M_{K_1 \cup K_2}$, then $\operatorname{supp} T_{\varphi,\xi} \subseteq K_1 \cup K_2$ and so $\langle T_{\varphi,\xi}, uu_1 + uu_2 - u\rangle = 0$. Thus $\pi(u)(\xi_1 + \xi_2 - \xi) = 0$ for all $u \in A(G)$. Again nondegeneracy of π yields $\xi = \xi_1 + \xi_2 \in M_{K_1} + M_{K_2}$. We have proved that $M_{K_1 \cup K_2} \subseteq M_{K_1} + M_{K_2}$.

Conversely, suppose $\xi_i \in M_{K_i}$ and $\xi = \xi_1 + \xi_2$. Then, for $\varphi \in X^*$, $T_{\varphi,\xi} = T_{\varphi,\xi_1} + T_{\varphi,\xi_2}$ and $\operatorname{supp} T_{\varphi,\xi} \subseteq \operatorname{supp} T_{\varphi,\xi_1} \cup \operatorname{supp} T_{\varphi,\xi_2} \subseteq K_1 \cup K_2$ by the result of Eymard mentioned earlier. Thus $\xi \in M_{K_1 \cup K_2}$ and the proof is complete. ∎

Here is the main result on spectral subspaces.

THEOREM 3. *Let π be a nondegenerate representation of $A(G)$ on a Banach space X. Suppose π has only the trivial spectral subspaces $\{0\}$ and X. Then there is an $x_0 \in G$ such that $\pi(u) = u(x_0)I$ for all $u \in A(G)$.*

Proof. By Proposition 2, there is a smallest nonempty closed set E in G with the property $M_E = X$. We first prove that E is a singleton.

Let $x_0 \in E$. Suppose that there is an $y_0 \in E$, $y_0 \neq x_0$. Choose a $v_0 \in A(G)$ with $v_0 = 1$ near x_0 and $v_0 = 0$ near y_0. For $u \in A(G)$, write $u = v + w$, where $v = u - uv_0$ and $w = uv_0$. Observe that $x_0 \notin \overline{V}$, where $V := \{x \in G : v(x) \neq 0\}$. Hence there is a $w_0 \in A(G)$ such that $w_0 = 1$ in a neighbourhood W of x_0 and $\operatorname{supp} w_0 \cap V = \emptyset$. Note that $vw_0 \equiv 0$. For $\xi \in X$ and $\varphi \in X^*$,

$$\operatorname{supp} T_{\varphi,\pi(v)\xi} \subseteq W^{\mathrm{c}}.$$

For, if $x \in W$, then $w_0(x) = 1$ and $w_0.T_{\varphi,\pi(v)\xi} = T_{\varphi,\pi(vw_0)\xi} = 0$ since $vw_0 = 0$ and so $x \notin \operatorname{supp} T_{\varphi,\pi(v)\xi}$. Thus, $M_{W^{\mathrm{c}}} = X$ if $\pi(v)\xi \neq 0$. But $x_0 \notin W^{\mathrm{c}}$ and so E is not a subset W^{c}. The choice of E now forces that $\pi(v)\xi = 0$ for $\xi \in X$. Thus $\pi(v) = 0$. In the same way, we can show that $\pi(w) = 0$. Hence $\pi(u) = 0$ for every $u \in A(G)$, leading to a contradiction because π is different from zero.

We have thus proved that $E = \{x_0\}$, so $M_{\{x_0\}} = X$. This means that $\operatorname{supp} T_{\varphi,\xi} \subseteq \{x_0\}$ for all $\xi \in X$ and $\varphi \in X^*$. Appealing to Eymard [1] once more we conclude that $T_{\varphi,\xi} = c_{\varphi,\xi}\lambda(x_0)$ for some scalar $c_{\varphi,\xi}$. Now choose a $u_0 \in A(G)$ with $u_0(x_0) = 1$. Then

$$c_{\varphi,\xi} = \langle T_{\varphi,\xi}, u_0\rangle = \langle \varphi, \pi(u_0)\xi\rangle$$

and so, for $u \in A(G)$,

$$\langle \varphi, \pi(u)\xi\rangle = \langle T_{\varphi,\xi}, u\rangle = \langle \varphi, \pi(u_0)\xi\rangle u(x_0) = \langle \varphi, u(x_0)\pi(u_0)\xi\rangle.$$

Since this is true for all $\varphi \in X^*$, we have

$$\pi(u)\xi = u(x_0)\pi(u_0)\xi, \qquad \xi \in X.$$

Hence $\pi(u) = u(x_0)\pi(u_0)$. But since π is an algebra representation, $\pi(u_0)^2 = \pi(u_0)$, i.e. $\pi(u_0)$ is a projection. The nondegeneracy of π forces $\pi(u_0)$ to be the identity operator on X and the proof is complete. ∎

REMARK. The content of the theorem is that if π is a nondegenerate representation having only the trivial spectral subspaces, then π is essentially a "character", i.e. $\pi(u)$ is just the multiplication by the value of the character $\lambda(x_0)$ of $A(G)$ at u. Thus if π is not a "character", then nontrivial spectral subspaces exist.

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