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THE DETERMINANT OF ORIENTED ROTANTS

BY

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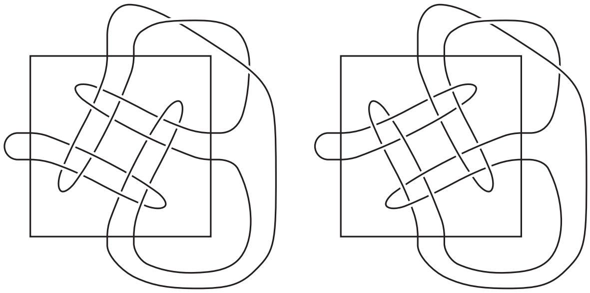
Abstract. We study the determinant of pairs of rotants of Anstee, Przytycki and Rolfsen. We consider various notions of rotant orientations.

0. Introduction. We recall the definition of generalized mutation as given in [APR]. Let D be a diagram of an unoriented link. Assume that the boundary of a regular n-gon intersects D transversally in such a way that the interior of each face of the n-gon contains exactly two points of D. Denote by R the part of D located inside the n-gon. If R has n-fold rotational symmetry, then it is called a *rotor* of order n, or briefly an *n-rotor*, while the complement S of R in D is called a *stator*. A new diagram D' may be constructed in the following manner: we cut out R, flip it over (π-rotate in 3-space about an axis of symmetry of the n-gon) and glue it back to S. This construction does not depend on the choice of the symmetry axis. Denote the flipped rotor by R'. If links L and L' have diagrams D and D' respectively, then we say L and L' are a pair of *rotants*, or that one is a rotant of the other.



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1. Orientation. We will consider various versions of orientations of pairs of rotants. Any orientation of a rotant diagram involves an orientation of the stator and an orientation of the rotor. Of course the two orientations should match on the boundary. In our considerations we will always keep the orientation of the stator fixed, while (possibly) changing the orientation of the flipped rotor.

We will say a rotor is *regularly oriented* if inputs and outputs appear alternately along the boundary as in Figure 1.1. This is called an *orientation-preserving rotor* in [DIPY].

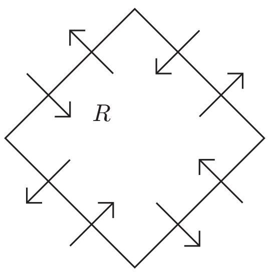


Fig. 1.1

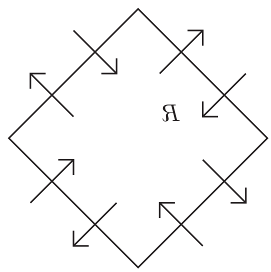


Fig. 1.2

When R is flipped to obtain R' the orientations on the boundary do not match the (unchanged) orientation of the stator (Figure 1.2). To obtain an oriented diagram we reverse *all orientations* in R'. A pair of links obtained in this way are called *regularly oriented* rotants. These are known to have coinciding Conway polynomials (∇) for any order n of rotation ([Tr]). Restricted versions of the above are true for the Jones ($n \leq 5$), Homfly ($n \leq 4$) and Kauffman ($n \leq 3$) polynomials ([APR], [JR]).

In this paper we will consider determinants ($\nabla(-2i)$) of pairs of rotants. Obviously the determinants coincide for regularly oriented rotants (because their Conway polynomials coincide). We will study to what extent this property is preserved when the orientation requirement is relaxed in various ways. In this section we will look at other possible orientations of rotant pairs. In Section 2 we first show how to define the determinant of a link by an evaluation of either the Conway polynomial or the Jones polynomial. We then introduce the Kauffman bracket of an unoriented link and prove that pairs of rotants have the same Kauffman bracket evaluation at a certain point d. In Section 3 we apply our results to rotors which do not contain closed components. Section 4 deals with more complicated rotors. The Appendix gives an example of a pair of nonregularly oriented rotants with different Conway polynomials.

A *biregular orientation* of a pair of rotants of even order is any orientation of the rotor such that the inputs and outputs are grouped in pairs as in Figure 1.3. This rotor is called *orientation-reversing* in [DIPY]. To obtain

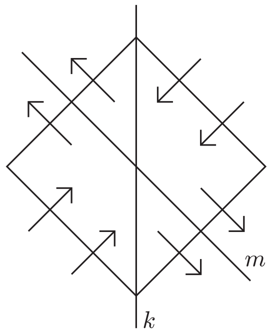


Fig. 1.3

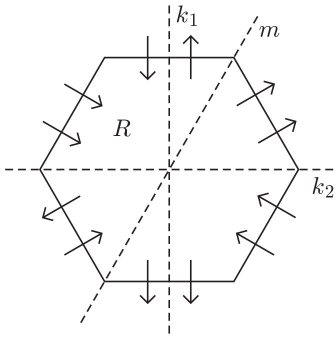


Fig. 1.4

an orientation of the modified diagram we either change *all* orientations in R' or *none* of them. This depends on the choice of the rotation axis. It will be shown that the determinants of a pair of biregularly oriented rotants do coincide. On the other hand, it is known that their Conway polynomials may differ ([DIPY]).

This may be further generalized by allowing any orientation of the initial diagram. It leads to a pair of *nonregularly oriented* rotors (Figure 1.4). Here, the rules for changing or preserving the orientations are more complicated. When R is flipped, the orientation of an arc in R' may or may not agree with the orientation of the stator. We simply choose the orientation for every arc in R' to match the orientation of S. We will show later that it is always possible.

It should be stressed that the orientation of the rotor part of the diagram is not always determined by the orientation on the boundary. This is because a rotor may contain closed components of the relevant link. However, if this is not the case, then for any given orientation of the original diagram D we can consider the *boundary induced* orientation on its rotant D' as described above. We will show that in such a case the determinants do coincide.

We will discuss cases involving components contained in the rotor later.

2. Kauffman bracket. To prove the results described above we will use both the Conway polynomial and the Jones polynomial. In [APR], skein theoretic methods were used to prove the results concerning the Kauffman, Homfly and Jones polynomials for rotants. In [Tr], it was the linear algebraic approach to the Conway polynomial that solved the problem for any order of rotation. Here, we will combine the result concerning the Conway polynomial with skein theoretic methods to get the result for the determinant. This is possible because the determinant can be obtained by suitable substitutions from both the Conway and the Jones polynomial (see (2.6.1) and (2.6.2)).

We begin by describing the Kauffman bracket of an unoriented link diagram, $\langle L\rangle \in \mathbb{Z}[A^{\pm 1}]$, calculated according to the recursions:

$$\langle \times \rangle = A\langle \asymp \rangle + A^{-1}\langle\,)(\, \rangle, \tag{2.1}$$

$$\langle \bigcirc \rangle = 1, \tag{2.2}$$

$$\langle \bigcirc \sqcup L\rangle = (-A^2 - A^{-2})\langle L\rangle. \tag{2.3}$$

Let us call crossings of type $\nearrow\!\!\!\!\nwarrow$ *positive*, while those of type $\nwarrow\!\!\!\!\nearrow$ *negative*.

The *writhe* of an oriented link L, $w(L)$, is defined as the number of positive crossings in a diagram of L minus the number of negative crossings. If the diagram of a link L is oriented, then

$$f_L(A) = (-A)^{-3w(L)}\langle L\rangle(A) \tag{2.4}$$

is an invariant of oriented links. We obtain the *Jones polynomial* of L by substitution

$$V_L(t) = f_L(t^{-1/4}). \tag{2.5}$$

The *determinant* of an oriented link L is a certain evaluation of the link's Conway polynomial or Jones polynomial:

(2.6.1) $D_L = \nabla_L(-2i)$,
(2.6.2) $D_L = V_L(-1)$ (more precisely $\sqrt{t} = -i$).

Other definitions include:

(2.6.3) $D_L = \Delta(-1)$, where Δ is the Alexander polynomial,
(2.6.4) determinant of the symmetrized Seifert form,
(2.6.5) rank of the first homology of the double branched cover,
(2.6.6) determinant of the Goeritz matrix.

(Note: In [BZ] the determinant is defined as the *absolute value* of (2.6.3)–(2.6.6), but in this paper we will need only (2.6.1) and (2.6.2).) From (2.5) and (2.6.2) we get

$$D_L = f_L(d) \quad \text{for } d = \pm\sqrt{i} = \pm\left(\frac{\sqrt{2}}{2} + i\,\frac{\sqrt{2}}{2}\right). \tag{2.7}$$

From (2.4) and (2.7) we obtain

$$D_L = (-d)^{-3w(L)}\langle L\rangle(d). \tag{2.8}$$

As can be seen from (2.8) we can study the determinat D_L by considering the exponent $-3w(L)$ and the evaluation $\langle L\rangle(d)$ of the bracket polynomial at d quite separately. In this section we will study $\langle L\rangle(d)$. Our aim is to prove the following theorem.

THEOREM 2.9. *If L, L' are a pair of unoriented rotants, then*

$$\langle L\rangle(d) = \langle L'\rangle(d), \qquad \textit{where } d = \pm\sqrt{i}. \tag{2.10}$$

Since the bracket polynomial is defined for unoriented link diagrams, we may temporarily forget the orientation.

To calculate $\langle L\rangle(d)$ we will use the substitution $A = d$ in (2.1)–(2.3). This implies that the right side of (2.3) is 0, since $-d^2 - d^{-2} = -i - i^{-1} = -i + i = 0$.

Using these new recursions we open all the crossings in the stator part of the diagram, leaving the rotor R intact. We obtain trivial stators—ones with no crossings or closed components, which we denote $g_1, \ldots, g_k$. We let $R_i = R \cup g_i$. Then we have

$$\langle L\rangle(d) = \sum_i f_i(d)\langle R_i\rangle(d), \tag{2.11}$$

for certain $f_i \in \mathbb{Z}[A^{\pm 1}]$. If we do the same for L', we get

$$\langle L'\rangle(d) = \sum_i f_i(d)\langle R'_i\rangle(d), \tag{2.12}$$

where $R'_i = R' \cup g_i$. The coefficients f_i are identical in both (2.11) and (2.12), since L and L' have the same stators.

To prove Theorem 2.9 it is sufficient to prove that $\langle R_i\rangle(d) = \langle R'_i\rangle(d)$. We will obtain this directly from Traczyk's theorem [Tr] about the Conway polynomial of oriented rotants. In order to do this we will now consider R_i endowed with regular orientation.

Lemma 2.13. *Trivial* (*crossing-free*) *stators may be given a regular boundary orientation.*

Proof. Suppose the diagram has regular orientation and number the boundary points consecutively. The numbers of all inputs are obviously of the same parity, and similarly for the outputs. Now, consider a stator diagram with no crossings. It consists of several arcs, and it is obvious that all of them connect even points to odd points (otherwise an odd number of inputs/outputs would be trapped in a single area). ■

Using the rotor's n-fold rotational symmetry it can be shown that:

Lemma 2.14. *The rotor R can be given a regular boundary orientation.*

The above two lemmas imply that any pair of rotants with trivial stators may be viewed as a regularly oriented pair. In particular we can consider R_i and R'_i to be a pair of regularly oriented rotants. By Traczyk's theorem they have the same Conway polynomial and (more specifically) the same determinant

$$D_{R_i} = \nabla_{R_i}(-2i) = \nabla_{R'_i}(-2i) = D_{R'_i}. \tag{2.15}$$

This completes the proof of Theorem 2.9 because regularly oriented pairs of rotors have the same writhe and because (2.8) holds.

3. Determinant. Throughout this section we will assume that the rotors have no closed components. We shall prove the following theorem.

THEOREM 3.1. *If L, L' are a pair of nonregularly oriented rotants and their rotors have no closed components, then $D_L = D_{L'}$.*

The determinant of a link L can be calculated from (2.8). In the previous section we showed that rotant pairs with no orientation have the same Kauffman bracket evaluation (2.10). Now we will look at the coefficient $(-d)^{-3w(L)}$ in (2.8). For this we will need our links' orientation again.

Since a rotant L is the union $R \cup S$ of the rotor and stator, both of which have disjoint sets of crossings, the writhe may be written as the sum

$$w(L) = w(R \cup S) = w(R) + w(S). \tag{3.2}$$

Of course $w(S)$ is the same for both L and $L' = R' \cup S$, so we need only investigate $w(R)$ and $w(R')$.

It is easy to see that if there is an arc in R connecting two points, say p and q, then there is also an arc connecting their images p' and q' under flipping. Of course, the same is true for R'. So we have two arcs which trade ends in the transition $R \leftrightarrow R'$, unless $p = q'$ (and so $q = p'$), in which case there is one arc. The former pair of arcs will be called *symmetric partners* or *m-arcs* (m for moving), while the latter arc is called an *s-arc* (for stable).

EXAMPLE. Figure 3.1 shows three arcs in a 6-rotor. The respective ends of the arcs p_1q_1 and p_2q_2 are symmetric, so they are m-arcs. The arc r_1r_2 is an s-arc.

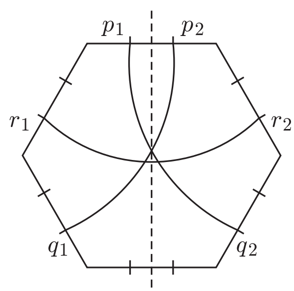


Fig. 3.1

It can be shown that:

LEMMA 3.4. (i) *If n is odd, then the rotor has exactly one s-arc.*
(ii) *If n is even, then the rotor either has two s-arcs, or none at all.* ■

Generally, an m-arc keeps or changes its orientation in R' iff the same is true for the orientation of the arc's symmetric partner, so the orientations of m-arcs are changed in pairs. S-arcs always have their orientation changed.

Now we return to the writhe. In the rotor we will look at six types of arc crossings:

(1) two s-arcs,
(2) two m-arcs which do not change orientation,
(3) two m-arcs which both change orientation,
(4) an s-arc and an m-arc which changes orientation,
(5) two m-arcs, only one of which changes orientation,
(6) an s-arc and an m-arc which does not change orientation.

Since s-arcs always change their orientation, crossings of type (1)–(4) have the same signs in both links of a rotant pair, and so do not change the writhe. In case (5) the sum of the signs of the crossings involved *does* change. However, this change is obviously compensated by the change for symmetric partners of the relevant arcs. Case (6) is similar. This proves

LEMMA 3.5. *If L, L' are a pair of oriented rotants and their rotors have no closed components, then $w(L) = w(L')$.*

Proof of Theorem 3.1. Using formula (2.8), Theorem 2.9 and Lemma 3.4 we obtain

$$D_L = (-d)^{-3w(L)}\langle L\rangle(d) = (-d)^{-3w(L')}\langle L'\rangle(d) = D_{L'}. \ \blacksquare$$

4. Closed components. In this section we will look at the determinants of rotant pairs which have closed components in their rotors. The following example shows that the assumption about closed components in Theorem 3.1 is necessary.

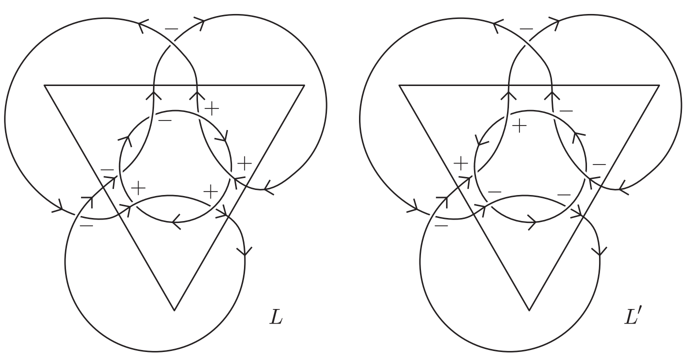


Fig. 4.1

EXAMPLE 4.1. Figure 4.1 shows diagrams of a pair of rotants. If unoriented, they would represent the same link, so $\langle L\rangle = \langle L'\rangle$. But with orientation the rotor contains one pair of m-arcs, one s-arc, and one closed component, which after flipping looks as if it had changed its orientation.

Since the stator controls only the orientations of the rotor arcs, and *not* those of the closed components, the latter retain their orientations after flipping. Below we calculate the determinants of the two links:

$$w(L) = 4 - 4 = 0 \quad \Rightarrow \quad D_L = (-d)^0 \langle L \rangle (d) = -30i,$$
$$w(L') = 2 - 6 = -4 \;\Rightarrow\; D_{L'} = (-d)^{-12} \langle\, L' \,\rangle (d) = 30i.$$

We see that $D_{L'} = -D_L \neq D_L$.

As was shown in Example 4.1, pairs of oriented rotants do not always have identical determinants. Looking at (2.8) and (2.10) we see that the problem must be in the writhe of the rotors with closed components. With $d = \pm\sqrt{i}$ we have

$$(-d)^4 = -1, \tag{4.2}$$
$$(-d)^8 = 1, \tag{4.3}$$

so we look at $w(R)$ and $w(R')$ mod 8.

It can be shown that the determinants of pairs of rotants of *even* order coincide, while this is not necessarily so for rotants of *odd* order. This is a consequence of Lemma 3.3. All is not lost, though. It turns out that if the determinants do not coincide, then they only differ in sign. If $w(\bigcup O_i, l_1)$ is the sum of all crossings between the rotor's single s-arc l_1 and closed components O_i, then

$$\frac{D_{L'}}{D_L} = \begin{cases} -1 & \text{if } w(\bigcup O_i, l_1) \equiv 4 \pmod 8, \\ +1 & \text{if } w(\bigcup O_i, l_1) \equiv 0 \pmod 8. \end{cases} \tag{4.4}$$

This gives us

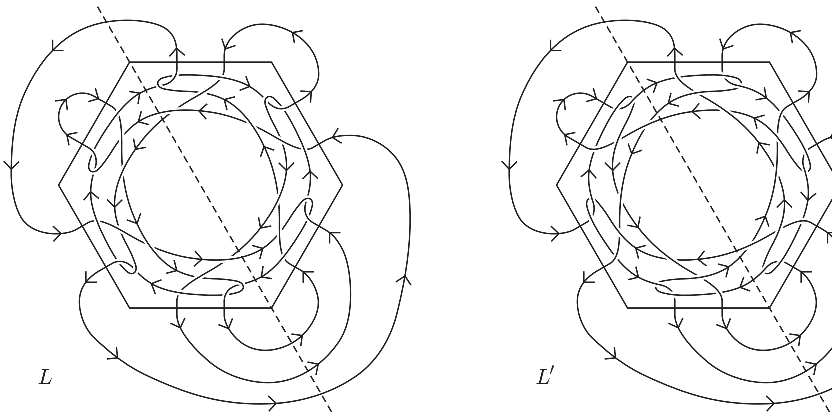
THEOREM 4.5. *If L, L' are a pair of oriented rotants of order n then*:

(i) *if* $2 \mid n$ *then* $D_L = D_{L'}$,
(ii) *if* $2 \nmid n$ *and the rotors have no closed components then* $D_L = D_{L'}$,
(iii) *if* $2 \nmid n$ *and* $w(\bigcup O_i, l_1) \equiv 0 \pmod 8$ *then* $D_L = D_{L'}$,
(iv) *if* $2 \nmid n$ *and* $w(\bigcup O_i, l_1) \equiv 4 \pmod 8$ *then* $D_L = -D_{L'}$.

5. Appendix. In this paper we relied on Traczyk's theorem [Tr], which states that pairs of regularly oriented rotants have the same Conway polynomial. This brings up a natural question: can Traczyk's theorem be generalized to cover *nonregularly oriented* rotants? The following example gives a negative answer.

EXAMPLE 5.1. The following figures show a pair of nonregularly oriented 6-rotants. The first five nonzero coefficients (mod 256) of the Conway polynomial for L are as follows:

$$-3z - 2z^3 + 13z^5 + 3z^7 - 24z^9,$$



while the coefficients for L' are

$$-3z - 14z^3 - 14z^5 - 31z^7 - 71z^9.$$

This proves that nonregularly oriented rotant pairs may have *different* Conway polynomials.

Recently Dąbkowski, Ishiwata, Przytycki and Yasuhara discovered a pair of *biregularly* oriented rotants with different Conway polynomials (see [DIPY]), which further proves that Traczyk's theorem cannot be improved.

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