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ON THE CRITICAL NEUMANN PROBLEM WITH LOWER ORDER PERTURBATIONS

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Abstract. We investigate the solvability of the Neumann problem (1.1) involving a critical Sobolev exponent and lower order perturbations in bounded domains. Solutions are obtained by min max methods based on a topological linking. A nonlinear perturbation of a lower order is allowed to interfere with the spectrum of the operator $-\Delta$ with the Neumann boundary conditions.

1. Introduction. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary $\partial\Omega$. In this paper we investigate the nonlinear Neumann problem

$$(1.1)\qquad \begin{cases} -\Delta u = |u|^{2^*-2}u + g(x,u) & \text{in } \Omega, \\ \dfrac{\partial}{\partial \nu} u(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

where $2^* = 2N/(N-2)$, $N \geq 3$, is a critical Sobolev exponent.

It is assumed that the nonlinearity $g(x,u)$ satisfies the following three basic assumptions:

(g_1) $g : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function and for every $M > 0$,
$$\sup\{|g(x,s)|;\, x \in \Omega,\ |s| \leq M\} < \infty,$$

(g_2) there exist constants $a_1, a_2 > 0$ and $\sigma \in (0,2)$ such that
$$\frac{1}{2}\, g(x,s)s - G(x,s) \geq -a_1 - a_2|s|^\sigma$$
for all $(x,s) \in \Omega \times \mathbb{R}$, where $G(x,t) = \int_0^t g(x,s)\, ds$,

(g_3) $\displaystyle\lim_{|s|\to\infty} \frac{g(x,s)}{|s|^{2^*-1}} = 0$ uniformly in x a.e. in Ω.

Further assumptions will be given in the next sections.

The Neumann problem in bounded domains with $g(x,u) = 0$ has an extensive literature [1]–[3], [5], [16], [19]–[24].

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To motivate our approach we briefly recall the main results for the Neumann problem in the bounded domain Ω,

$$(1.2)\qquad \begin{cases} -\Delta u + \lambda u = |u|^{2^*-2}u & \text{in } \Omega,\\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega,\ u > 0 \text{ on } \Omega. \end{cases}$$

First existence results for problem (1.2) are due to Adimurthi–Mancini [1], Adimurthi–Yadava [6] and X. J. Wang [19]. Solutions to problem (1.2) were obtained as the minimizers of the variational problem

$$(1.3)\qquad \begin{aligned} m_\lambda &= \inf_{u\in H^1(\Omega)-\{0\}} \frac{\int_\Omega(|\nabla u|^2+\lambda u^2)\,dx}{(\int_\Omega |u|^{2^*}\,dx)^{2/2^*}}\\ &= \inf_{u\in H^1(\Omega),\,\int_\Omega |u|^{2^*}dx=1} \int_\Omega (|\nabla u|^2+\lambda u^2)\,dx. \end{aligned}$$

The existence of a minimizer for m_λ is closely related to the best Sobolev constant S. We recall that

$$S = \inf_{u\in D^{1,2}(\mathbb{R}^N)-\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2\,dx}{(\int_{\mathbb{R}^N} |u|^{2^*}\,dx)^{2/2^*}},$$

where $D^{1,2}(\mathbb{R}^N)$ is the Sobolev space defined by

$$D^{1,2}(\mathbb{R}^N) = \{u;\ \nabla u \in L^2(\mathbb{R}^N),\ u \in L^{2^*}(\mathbb{R}^N)\}.$$

The best Sobolev constant is achieved by

$$U(x) = \frac{c_N}{(N(N-2)+|x|^2)^{(N-2)/2}},$$

where $c_N > 0$ is a constant depending on N. The function U, called an *instanton*, satisfies the equation

$$-\Delta U = U^{2^*-1} \quad \text{in } \mathbb{R}^N.$$

We have $\int_{\mathbb{R}^N} |\nabla U|^2\,dx = \int_{\mathbb{R}^N} U^{2^*}\,dx = S^{N/2}$. For future use we introduce the notation

$$U_{\varepsilon,y}(x) = \varepsilon^{-(N-2)/2} U\left(\frac{x-y}{\varepsilon}\right), \qquad y\in\mathbb{R}^N,\ \varepsilon>0.$$

If $y=0$ we write $U_\varepsilon = U_{\varepsilon,0}$.

The main step in establishing the existence of a minimizer for m_λ is to show that

$$(1.4)\qquad m_\lambda < \frac{S}{2^{2/N}}.$$

This can be established by testing m_λ with $U_{\varepsilon,y}$, where $y\in\partial\Omega$ is a point where $\partial\Omega$ has the mean curvature $H(y) > 0$. Solutions of the minimization problem (1.3) are called the *least energy solutions*. These results were extended to the critical Neumann problems involving indefinite weights in the papers [8] and [9].

In [10] the above existence result has been extended to (1.1) with $g(x,u) = \lambda u$ and with λ lying between two consecutive eigenvalues of the operator $-\Delta$ with the Neumann boundary conditions. The purpose of this paper is to obtain the existence of solutions with a more general perturbation $g(x,u)$. In our approach we use a modified topological linking from the paper [12]. In the proofs of our existence results the use of the instanton plays an essential role. In particular, we use some asymptotic properties of $U_{\varepsilon,y}$ with $y \in \partial\Omega$ in terms of the mean curvature of $\partial\Omega$ at y. The influence of the mean curvature on the existence of a solution disappears in the case of the problem involving an indefinite weight in a situation where its global maximum is larger than its maximum on the boundary $\partial\Omega$. It is worth mentioning that the resonance case for the dimensions $N = 3,4$ requires a condition which controls the growth of the perturbation g in the vicinity of at least one boundary point with a positive mean curvature.

The paper is organized as follows. In Section 2 we find the energy level below which the Palais–Smale condition holds for the variational functional associated with problem (1.1). In Sections 3 and 4 we consider nonresonance and resonance cases. Section 5 is devoted to the critical Neumann problem with an indefinite weight.

Throughout this paper we denote strong convergence by "$\to$" and weak convergence by "$\rightharpoonup$". The norms in the Lebesgue spaces $L^p(\Omega)$ are denoted by $\|\cdot\|_p$. By $H^1(\Omega)$ we denote a standard Sobolev space on Ω equipped with the norm

$$\|u\|^2 = \int_\Omega (|\nabla u|^2 + u^2)\,dx.$$

2. Palais–Smale condition. We set

$$J_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega G(x,u)\,dx - \frac{1}{2^*}\int_\Omega |u|^{2^*}\,dx,$$

where $G(x,u) = \int_0^u g(x,s)\,ds$. It is easy to check that J_λ is a C^1-functional on $H^1(\Omega)$. Solutions of problem (1.1) are sought as critical points of J_λ through the topological linking. The important step in this approach is to find the energy level of the functional J_λ below which the Palais–Smale condition holds.

We recall that the functional J_λ satisfies the *Palais–Smale condition at level* c ($(\mathrm{PS})_c$ condition for short) if each sequence $\{u_n\} \subset H^1(\Omega)$ such that $(*)$ $J_\lambda(u_n) \to c$ and $(**)$ $J'_\lambda(u_n) \to 0$ in $H^{-1}(\Omega)$ is relatively compact in $H^1(\Omega)$. Any sequence $\{u_n\} \subset H^1(\Omega)$ satisfying $(*)$ and $(**)$ is called a *Palais–Smale sequence for* J_λ *at level* c (a $(\mathrm{PS})_c$ sequence for short).

PROPOSITION 2.1. *Suppose* (g_1)–(g_3) *hold. Let* $\{u_n\} \subset H^1(\Omega)$ *be a* $(\mathrm{PS})_c$ *sequence for* J_λ. *Then, up to a subsequence,* $u_n \rightharpoonup u$ *in* $H^1(\Omega)$ *and* $J'_\lambda(u) = 0$. *The weak limit* u *is nonzero if* $g(x,0) \neq 0$ *or* $g(x,0) = 0$ *and* $c \in (0, S^{N/2}/2N)$.

Proof. First, we show that $\{u_n\}$ is bounded in $H^1(\Omega)$. For large n we have

$$c + 1 + \|u_n\| \geq J(u_n) - \frac{1}{2}\langle J'(u_n), u_n\rangle = \frac{1}{N}\int_\Omega |u_n|^{2^*}\,dx + \int_\Omega \left[\frac{1}{2}\, g(x,u_n)u_n - G(x,u_n)\right] dx.$$

It follows from (g_2) that

$$c + 1 + \|u_n\| \geq \frac{1}{N}\int_\Omega |u_n|^{2^*}\,dx - a_1|\Omega| - a_2\int_\Omega |u_n|^\sigma\,dx. \tag{2.1}$$

In what follows we always denote by C a positive constant independent of n which may change from one inequality to another. Using the Young inequality we obtain

$$\int_\Omega |u_n|^\sigma\,dx \leq \kappa \int_\Omega |u_n|^{2^*}\,dx + C$$

for every $\kappa > 0$, where $C > 0$ is a constant depending on κ and $|\Omega|$. Inserting this inequality with $\kappa = 1/2Na_2$ into (2.1) we obtain

$$\int_\Omega |u_n|^{2^*}\,dx \leq C(\|u_n\| + 1). \tag{2.2}$$

We now use the equality

$$J_\lambda(u_n) - \frac{1}{2^*}\langle J'_\lambda(u_n), u_n\rangle = \frac{1}{N}\int_\Omega |\nabla u_n|^2\,dx + \int_\Omega \left[\frac{1}{2^*}\, g(x,u_n)u_n - G(x,u_n)\right] dx. \tag{2.3}$$

This combined with (g_1) and (g_3) gives the estimate

$$\int_\Omega |\nabla u_n|^2 dx \leq C\Big(\int_\Omega |u_n|^{2^*}\,dx + \|u_n\| + 1\Big).$$

Then it follows from (2.2) that

$$\int_\Omega |\nabla u_n|^2\,dx \leq C(\|u_n\| + 1). \tag{2.4}$$

We now consider the decomposition $H^1(\Omega) = \mathbb{R} \oplus V$, where $V = \{v \in H^1(\Omega);\ \int_\Omega v\,dx = 0\}$. We equip $H^1(\Omega)$ with the equivalent norm

$$\|u\|_V = \Big(\int_\Omega |\nabla v|^2\,dx + t^2\Big)^{1/2}$$

for $u = t + v$, $v \in V$, $t \in \mathbb{R}$. Using this decomposition we can write $u_n = v_n + t_n$, $v_n \in V$, $t_n \in \mathbb{R}$. We claim that $\{t_n\}$ is bounded. Arguing by contradiction we may assume $t_n \to \infty$. The case $t_n \to -\infty$ is similar. We put $w_n = v_n/t_n$. It then follows from (2.4) that

$$\int_\Omega |\nabla w_n|^2\,dx \le C\Big[t_n^{-2} + t_n^{-1}\Big(\int_\Omega (|\nabla w_n|^2\,dx + 1)\Big)^{1/2}\Big].$$

This yields $\int_\Omega |\nabla w_n|^2\,dx \to 0$ and hence $w_n \to 0$ in $L^p(\Omega)$ for every $2 \le p \le 2^*$. (Here we used the fact that the space V equipped with the norm $(\int_\Omega |\nabla v|^2\,dx)^{1/2}$ is continuously embedded into $L^p(\Omega)$ for $2 \le p \le 2^*$.) We now observe that

$$\begin{aligned} t_n^{-2^*}&\Big[J_\lambda(u_n) - \frac{1}{2}\langle J'_\lambda(u_n), u_n\rangle\Big] \\ &= \frac{1}{N}\int_\Omega |w_n+1|^{2^*}dx + t_n^{-2^*}\Big(\frac{1}{2}\int_\Omega g(x,u_n)u_n\,dx - \int_\Omega G(x,u_n)\,dx\Big). \end{aligned}$$

Using (g_3) and letting $n \to \infty$ in this equality we get $N^{-1}\int_\Omega dx = 0$. This is a contradiction.

Since $\{t_n\}$ is bounded, we deduce from (2.4) that $\{|\nabla v_n|\}$ is bounded in $L^2(\Omega)$. Consequently, $\{u_n\}$ is bounded in $H^1(\Omega)$. We may assume that $u_n \rightharpoonup 0$ in $H^1(\Omega)$. Since g is subcritical it is clear that $J'_\lambda(u) = 0$. If we have $g(x,0) \ne 0$, then $u \ne 0$. So it remains to consider the case $g(x,0) = 0$ on Ω. Arguing by contradiction assume $u \equiv 0$ on Ω. Hence

$$\int_\Omega |\nabla u_n|^2\,dx = \int_\Omega |u_n|^{2^*}dx + o(1)$$

and also

$$\int_\Omega |\nabla u_n|^2\,dx \to Nc \quad \text{and} \quad \int_\Omega |u_n|^{2*}\,dx \to Nc.$$

We now apply the following inequality: there exists a constant $A(\Omega) > 0$ such that

$$\frac{S}{2^{2/N}}\Big(\int_\Omega |u|^{2^*}dx\Big)^{2/2^*} \le \int_\Omega |\nabla u|^2\,dx + A(\Omega)\Big(\int_\Omega |u|^{2N/(N-1)}\,dx\Big)^{(N-1)/N}$$

for every $u \in H^1(\Omega)$ (see [25]). We use this inequality with $u = u_n$. Since $2N/(N-1) < 2^*$, letting $n \to \infty$ we deduce that $S(Nc)^{2/2^*}/2^{2/N} \le Nc$. This yields $S^{N/2}/2N \le c$, which is a contradiction. ∎

3. Existence theorem for the nonresonance case near 0. We denote by $0 = \lambda_1 < \lambda_2 < \cdots$ the sequence of the eigenvalues for $-\Delta$ with Neumann boundary conditions. The first eigenvalue is simple and has constant eigenfunctions.

We assume that there exist $k \in \mathbb{N}$, $\delta > 0$, $\sigma > 0$ and $\mu \in (\lambda_k, \lambda_{k+1})$ such that

(g_4) $\frac{1}{2}(\lambda_k + \sigma)s^2 \le G(x,s) \le \frac{1}{2}\mu s^2$ for a.e. $x \in \Omega$ and all $|s| \le \delta$,

(g_5) $G(x,s) \ge \frac{1}{2}(\lambda_k + \sigma)s^2 - \frac{1}{2^*}|s|^{2^*}$ for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$.

For simplicity we assume $0 \in \partial\Omega$. For every $m \in \mathbb{N}$ we define the function $\zeta_m : \Omega \to \mathbb{R}$ by

$$\zeta_m(x) = \begin{cases} 0 & \text{if } x \in \Omega \cap B(0, 1/m), \\ m|x| - 1 & \text{if } x \in A_m = \Omega \cap (B(0,2/m) - B(0,1/m)), \\ 1 & \text{if } x \in \Omega - B(0,2/m). \end{cases}$$

Let $\{e_i\}$ be the orthonormal sequence of eigenfunctions for $-\Delta$ with Neumann boundary conditions. We put $e_i^m = \zeta_m e_i$, $i = 1, 2, \ldots$, and define the spaces H^-, H_m^- and H^+ by

$$H^- = \operatorname{span}\{e_i;\, i = 1, \ldots, k\}, \quad H_m^- = \operatorname{span}\{e_i^m;\, i = 1, \ldots, k\}, \quad H^+ = (H^-)^\perp,$$

so that $H^1(\Omega) = H^- \oplus H^+$.

Lemma 3.1, below, is a modification of Lemma 2 from [12].

LEMMA 3.1. *We have $e_i^m \to e_i$ in $H^1(\Omega)$ as $m \to \infty$ and*

$$\max_{u \in H_m^-, \int_\Omega u^2\,dx = 1} \|\nabla u\|_2^2 \le \lambda_k + c_k m^{2-N},$$

where $c_k > 0$ are constants independent of m.

Proof. Following the proof of Lemma 2 from [12] we have

$$\int_\Omega |\nabla(e_i^m - e_i)|^2\,dx \le C[\|e_i\|_\infty^2 m^{2-N} + \|\nabla e_i\|_\infty \|e_i\|_\infty m^{1-N} + \|\nabla e_i\|_\infty^2 m^{-N}]$$

and similarly

$$\int_\Omega |e_i^m - e_i|^2\,dx = \int_\Omega (\zeta_m - 1)^2 e_i^2\,dx \le C\|e_i\|_\infty^2 m^{-N}.$$

These two estimates give the first part of our assertion. We now use the notation $\partial B = \{u \in H^1(\Omega);\, \int_\Omega u^2\,dx = 1\}$. If $v \in H^- \cap \partial B$, then $v = \sum_{j=1}^k \alpha_j e_j$ with $\sum_{j=1}^k \alpha_j^2 = 1$. If $v \in H^- \cap \partial B$, then $v_m = \zeta_m v = \sum_{j=1}^k \alpha_j \zeta_m e_j = \sum_{j=1}^k \alpha_j e_j^m$. Hence $v_m \in H_m^-$. Let $w \in H_m^- \cap \partial B$. Then $w = \sum_j \beta_j^m e_j^m$ and

$$\begin{aligned} \|w\|_2^2 &= \int_\Omega \sum_{j,k} \beta_j^m \beta_k^m e_j^m e_k^m\,dx \\ &= \int_\Omega \beta_j^m \beta_k^m e_j e_k\,dx + \int_\Omega \sum_{j,k} \beta_j^m \beta_k^m (e_j^m e_k^m - e_j^m e_k + e_j^m e_k - e_j e_k)\,dx. \end{aligned}$$

Using the Hölder inequality and the estimates from the first part of the proof we derive

$$1+O(m^{-N}) = \|w\|_2^2 = \int_\Omega \sum_j (\beta_j^m)^2 e_j^2\,dx = \sum_j (\beta_j^m)^2.$$

We put

$$\gamma_j^m = \frac{\beta_j^m}{(\sum_j (\beta_j^m)^2)^{1/2}}.$$

Then we have

$$\begin{aligned}\|w\|^2 &= \Big\|\sum_j \beta_j^m e_j^m\Big\|^2 = \Big\|\sum_j \beta_j^m e_j\Big\|^2 + O(m^{-N+2})\\ &= (1+O(m^{-N}))\Big\|\sum_j \gamma_j^m e_j\Big\|^2 + O(m^{-N+2})\\ &\le \lambda_k(1+O(m^{-N})) + O(m^{-N+2})\end{aligned}$$

and the second assertion follows. ■

In order to apply the Rabinowitz linking theorem [18] we use a family of modified instantons. Let $\eta \in C_{\mathrm{c}}^\infty(B(0,1/m))$ be such that $\eta(x) = 1$ in $B(0,1/2m)$, $0 \le \eta \le 1$ in $B(0,1/m)$ and $\|\nabla\eta\|_\infty \le 4m$. We put $\overline{U}_\varepsilon(x) = \eta(x)U_\varepsilon(x)$. We need the following properties of $\overline{U}_\varepsilon$:

$$\frac{\int_\Omega |\nabla \overline{U}_\varepsilon|^2\,dx}{(\int_\Omega \overline{U}_\varepsilon^{2^*}\,dx)^{2/2^*}} \le \begin{cases} 2^{-2/N}S - A_N H(0)\varepsilon\log(1/\varepsilon) + O(\varepsilon) & \text{if } N=3,\\ 2^{-2/N}S - A_N H(0)\varepsilon + O(\varepsilon^2\log(1/\varepsilon)) & \text{if } N=4,\\ 2^{-2/N}S - A_N H(0)\varepsilon + O(\varepsilon^2) & \text{if } N\ge 5,\end{cases} \tag{3.1}$$

where $H(0)$ denotes the mean curvature of $\partial\Omega$ at 0 and $A_N > 0$ is a constant depending on N. We will also need asymptotic expansions of integrals of $\overline{U}_\varepsilon$. These expansions are taken from [19]. We recall that $0 \in \partial\Omega$. The boundary $\partial\Omega$ near 0 can be represented by

$$x_N = h(x') = \frac{1}{2}\sum_{i=1}^{N-1} \alpha_i^2 x_i^2 + o(|x'|^2) = g(x') + o(|x'|^2) \quad \text{for } x' \in D(0,\delta)$$

for some $\delta > 0$, where $D(0,\delta) = B(0,\delta) \cap (x_N = 0)$ and α_i are the principal curvatures of $\partial\Omega$ at 0. For $N \ge 4$ we have

$$\overline{K}_1(\varepsilon) = \int_\Omega |\nabla \overline{U}_\varepsilon|^2\,dx = \frac{1}{2}K_1 - I(\varepsilon) + o(\varepsilon), \tag{3.2}$$

$$\overline{K}_2(\varepsilon) = \int_\Omega \overline{U}_\varepsilon^{2^*}\,dx = \frac{1}{2}K_2 - \mathit{\Pi}(\varepsilon) + o(\varepsilon), \tag{3.3}$$

where

$$K_1 = (N-2)^2 \int_{\mathbb{R}^N} \frac{|x|^2}{(1+|x|^2)^N}\, dx, \qquad K_2 = \int_{\mathbb{R}^N} \frac{dx}{(1+|x|^2)^N},$$

$$\lim_{\varepsilon\to 0} \varepsilon^{-1} I(\varepsilon) = (N-2)^2 \int_{\mathbb{R}^{N-1}} \frac{|y'|^2 g(y')}{(1+|y'|^2)^N}\, dy',$$

$$\lim_{\varepsilon\to 0} \varepsilon^{-1} \mathit{\Pi}(\varepsilon) = \int_{\mathbb{R}^{N-1}} \frac{g(y')}{(1+|y'|^2)^N}\, dy'.$$

For $N = 3$ we have

$$\overline{K}_1(\varepsilon) \le \frac{1}{2} K_1 - C_\circ \varepsilon |\log \varepsilon| + o(\varepsilon) \tag{3.4}$$

for some constant $C_\circ > 0$ and

$$\overline{K}_2(\varepsilon) \ge \frac{1}{2} K_2 - O(\varepsilon). \tag{3.5}$$

We now define

$$Q^\varepsilon_m = (B(0,R) \cap H^-_m) \oplus [0,R]\{\overline{U}_\varepsilon\}.$$

THEOREM 3.2. *Let $N \ge 3$. Suppose that $G(x,s) \ge 0$ for $(x,s) \in \Omega \times \mathbb{R}$ and that $(g_1), \dots, (g_5)$ hold. Then problem* (1.1) *has a solution.*

Proof. STEP 1. We show that there exist constants $\alpha > 0$ and $\varrho > 0$ such that

$$J(u) \ge \alpha \quad \text{for every } u \in \partial B(0,\varrho) \cap H^+.$$

This follows from assumptions (g_3) and (g_4). Indeed, we have

$$J(u) \ge \frac{1}{2} \int_\Omega |\nabla u|^2\, dx - \frac{\mu}{2} \int_\Omega u^2\, dx - A \int_\Omega |u|^{2^*} dx$$

for some constant $A > 0$. We choose $\varepsilon > 0$ so that $\mu + \varepsilon < \lambda_{k+1}$. From the above inequality we derive

$$\begin{aligned} J(u) &\ge \frac{1}{2} \int_\Omega |\nabla u|^2\, dx + \frac{\varepsilon}{2} \int_\Omega u^2\, dx - \frac{\mu+\varepsilon}{2} \int_\Omega u^2\, dx - A \int_\Omega |u|^{2^*} dx \\ &\ge \frac{1}{2} \int_\Omega |\nabla u|^2\, dx + \frac{\varepsilon}{2} \int_\Omega u^2\, dx - \frac{\mu+\varepsilon}{2\lambda_{k+1}} \int_\Omega |\nabla u|^2\, dx - A \int_\Omega |u|^{2^*} dx \\ &= \left(\frac{1}{2} - \frac{\mu+\varepsilon}{2\lambda_{k+1}}\right) \int_\Omega |\nabla u|^2\, dx + \frac{\varepsilon}{2} \int_\Omega u^2\, dx - A \int_\Omega |u|^{2^*} dx. \end{aligned}$$

Letting

$$C_1 = \min\left(\frac{1}{2} - \frac{\mu+\varepsilon}{2\lambda_{k+1}}, \frac{\varepsilon}{2}\right)$$

and using the Sobolev inequality we derive the estimate

$$J(u) \ge c_1 \|u\|^2 - \overline{A} \|u\|^{2^*}$$

for some constant $\bar{A} > 0$. The claim follows by taking $\varrho > 0$ sufficiently small.

STEP 2. There exists $R > \varrho$ such that

$$\max_{v \in \partial Q_m^\varepsilon} J(v) \le \omega_m \quad \text{with } \omega_m \to 0 \text{ as } m \to \infty. \tag{3.6}$$

It follows from (g_5) that for $u \in H_m^-$ we have

$$\begin{aligned} J(v) &\le \frac{1}{2} \int_\Omega |\nabla v|^2 \, dx - \frac{1}{2} (\lambda_k + \sigma) \int_\Omega v^2 \, dx \\ &\le \frac{1}{2} \int_\Omega |\nabla v|^2 \, dx + \varepsilon \int_\Omega v^2 \, dx - \left(\frac{1}{2} + \frac{\sigma}{4\lambda_k} \right) \int_\Omega |\nabla v|^2 \, dx - \left(\varepsilon + \frac{\sigma}{4} \right) \int_\Omega v^2 \, dx. \end{aligned}$$

Then, if $v \in \partial B(0, R) \cap H_m^-$, we have $J(v) \to -\infty$ as $R \to \infty$. The above inequality also shows that $\lim_{m \to \infty} \max_{v \in H_m^-} J(v) = 0$. Since $G(x, u) \ge 0$ we have

$$J(r\bar{U}_\varepsilon) \le \frac{r^2}{2} \int_\Omega |\nabla \bar{U}_\varepsilon|^2 \, dx - \frac{r^{2^*}}{2^*} \int_\Omega \bar{U}_\varepsilon^{2^*} \, dx.$$

Hence by (3.2) and (3.3) ((3.4) and (3.5) if $N = 3$) we get $J(r\bar{U}_\varepsilon) < 0$ for $r = R$ sufficiently large. We now observe that if $u \in H_m^- \oplus R\{\bar{U}_\varepsilon\}$, then $u = w + R\bar{U}_\varepsilon$ and $\operatorname{supp} w \cap \operatorname{supp} \bar{U}_\varepsilon = \emptyset$. Consequently, $J(u) \le \omega_m$ for $u \in H_m^- \oplus R\{\bar{U}_\varepsilon\}$. Since $\max_{0<r<R} J(r\bar{U}_\varepsilon) < \infty$ we see that if $u \in (\partial B(0, R) \cap H_m^-) \oplus [0, R]\{\bar{U}_\varepsilon\}$, then $J(u) \le 0$ for R sufficiently large. This justifies our claim.

STEP 3. We put

$$\Gamma = \{h \in C(\bar{Q}_m^\varepsilon, H^1(\Omega));\ h(v) = v \text{ for every } v \in \partial Q_m^\varepsilon\}$$

and

$$c = \inf_{h \in \Gamma} \max_{v \in Q_m^\varepsilon} J(h(v)).$$

This energy level of J generates the $(\mathrm{PS})_c$ sequence. To complete the proof we must show that

$$c < \frac{1}{2N} S^{N/2}. \tag{3.7}$$

Since $\mathrm{id} \in \Gamma$, we have $c \le \max_{v \in Q_m^\varepsilon} J(v)$. Therefore it is sufficient to show that

$$\sup_{v \in Q_m^\varepsilon} J(v) < \frac{1}{2N} S^{N/2} \tag{3.8}$$

for $\varepsilon > 0$ sufficiently small. We argue by contradiction. Assume

$$\sup_{v \in Q_m^\varepsilon} J(v) \ge \frac{1}{2N} S^{N/2} \tag{3.9}$$

for every $\varepsilon > 0$. Since the set $\{v \in Q_m^\varepsilon;\ J(v) \geq 0\}$ is compact for every $\varepsilon > 0$, there exist $w_\varepsilon \in H_m^-$ and $t_\varepsilon \geq 0$ such that

$$J(v_\varepsilon) = \max_{v\in Q_m^\varepsilon} J(v) \geq \frac{1}{2N} S^{N/2}, \qquad v_\varepsilon = w_\varepsilon + t_\varepsilon \overline{U}_\varepsilon. \tag{3.10}$$

This means that

$$\frac{1}{2}\int_\Omega |\nabla v_\varepsilon|^2\,dx - \int_\Omega G(x,v_\varepsilon)\,dx - \frac{1}{2^*}\int_\Omega |v_\varepsilon|^{2^*}\,dx \geq \frac{1}{2N} S^{N/2}. \tag{3.11}$$

It follows from Lemma 3.1 and (g_5) that

$$\begin{aligned} J(w_\varepsilon) &\leq \frac{\lambda_k + c_k m^{2-N}}{2}\int_\Omega w_\varepsilon^2\,dx - \int_\Omega G(x,w_\varepsilon)\,dx - \frac{1}{2^*}\int_\Omega |w_\varepsilon|^{2^*}\,dx \\ &\leq \frac{c_k m^{2-N} - \sigma}{2}\int_\Omega w_\varepsilon^2\,dx. \end{aligned}$$

We now choose m so large that

$$c_k m^{2-N} \leq \sigma.$$

With this choice of m we get $J(w_\varepsilon) < 0$. Consequently, since $G \geq 0$, we derive from (3.1) that

$$\begin{aligned} J(v_\varepsilon) &= J(w_\varepsilon) + J(t_\varepsilon \overline{U}_\varepsilon) \leq J(t_\varepsilon \overline{U}_\varepsilon) \\ &\leq \max_{t\geq 0} J(t\overline{U}_\varepsilon) = \frac{1}{N}\left(\frac{\int_\Omega |\nabla \overline{U}_\varepsilon|^2\,dx}{(\int_\Omega \overline{U}_\varepsilon^{2^*}\,dx)^{(N-2)/N}}\right)^{N/2} < \frac{1}{2N} S^{N/2} \end{aligned}$$

for $\varepsilon > 0$ small enough. This contradicts inequality (3.11) and so the proof is complete. ∎

4. Resonance near the origin. In the case of the resonance near the origin we replace assumptions (g_4) and (g_5) by

(g_6) there exist constants $\delta > 0$ and $\mu \in (\lambda_k, \lambda_{k+1})$ such that

$$\frac{1}{2}\lambda_k s^2 \leq G(x,s) \leq \frac{1}{2}\mu s^2$$

for a.e. $x \in \Omega$ and every $|s| \leq \delta$,

(g_7) there exists $\sigma \in (0, 1/2^*)$ such that

$$G(x,s) \geq \frac{1}{2}\lambda_k s^2 - \left(\frac{1}{2^*} - \sigma\right)|s|^{2^*}$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$.

We need asymptotic estimates for $\|\nabla \overline{U}_\varepsilon\|_2^2$ and $\|\overline{U}_\varepsilon\|_{2^*}^{2^*}$ emphasizing the dependence on m.

LEMMA 4.1. *For $N \geq 5$ we have*

$$\overline{K}_1(\varepsilon) = \int_\Omega |\nabla(\eta U_\varepsilon)|^2\,dx \leq \frac{K_1}{2} - I(\varepsilon) + o(\varepsilon) + C\varepsilon^{N-2}m^N, \tag{4.1}$$

$$\overline{K}_2(\varepsilon) = \int_\Omega |\eta U_\varepsilon|^{2^*}\,dx = \frac{K_2}{2} - \mathit{\Pi}(\varepsilon) + o(\varepsilon) + \varepsilon^N m^{2N}. \tag{4.2}$$

Proof. To show (4.1) we write

$$\begin{aligned}\overline{K}_1(\varepsilon) &= \int_\Omega |\nabla\eta U_\varepsilon + \eta\nabla U_\varepsilon|^2\,dx \\ &\leq \int_\Omega \eta^2|\nabla U_\varepsilon|^2\,dx + \int_\Omega (\eta^2 + |\nabla\eta|^2)U_\varepsilon^2\,dx + \int_\Omega |\nabla\eta|^2|\nabla U_\varepsilon|^2\,dx.\end{aligned} \tag{4.3}$$

We now estimate terms on the right-hand side of this inequality. It follows from (3.2) (see also [19]) that

$$\int_\Omega \eta^2|\nabla U_\varepsilon|^2\,dx \leq \int_\Omega |\nabla U_\varepsilon|^2\,dx = \frac{K_1}{2} - I(\varepsilon) + o(\varepsilon)$$

and

$$\int_\Omega \eta^2 U_\varepsilon^2\,dx \leq \int_\Omega U_\varepsilon^2\,dx = O(\varepsilon^2). \tag{4.4}$$

We also have

$$\begin{aligned}\int_\Omega |\nabla\eta|^2|\nabla U_\varepsilon|^2\,dx &\leq Cm^2 \int_{\Omega\cap(1/2m\leq|x|\leq 1/m)} \frac{|x|^2\varepsilon^{N-2}}{(\varepsilon^2+|x|^2)^N}\,dx \\ &\leq Cm^2\varepsilon^{N-2}\int_{1/2m}^{1/m} r^{1-N}\,dr = C\varepsilon^{N-2}m^N\end{aligned} \tag{4.5}$$

and

$$\int_\Omega |\nabla\eta|^2 U_\varepsilon^2\,dx = \int_{\Omega\cap(1/2m\leq|x|\leq 1/m)} \frac{m^2\varepsilon^{N-2}}{(\varepsilon^2+|x|^2)^{N-2}}\,dx \leq C\varepsilon^{N-2}m^{N-2}. \tag{4.6}$$

Combining (4.3)–(4.6) we get (4.1). In a similar way we derive (4.2). ∎

THEOREM 4.2. *Let $N \geq 5$. Suppose* (g_1), (g_2), (g_3), (g_6) *and* (g_7) *hold. Then problem* (1.1) *admits a solution.*

Proof. We argue as in the proof of Theorem 3.2. The main point is to establish inequality (3.7). Arguing by contradiction we assume (3.9). We now stress the dependence on m which comes from the definition of η. Therefore for large m and all $\varepsilon > 0$ there exist $v_\varepsilon^m \in Q_m^\varepsilon$, $v_\varepsilon^m = w_\varepsilon^m + t_\varepsilon^m\overline{U}_\varepsilon^m$, such that

$$\frac{1}{2}\int_\Omega |\nabla v_\varepsilon^m|^2\,dx - \int_\Omega G(x, v_\varepsilon^m)\,dx - \frac{1}{2^*}\int_\Omega |v_\varepsilon^m|^{2^*}\,dx \geq \frac{1}{2N}\,S^{N/2}.$$

As in [12] we show that $\{t_\varepsilon^m\}$ and $\{w_\varepsilon^m\}$ satisfy $t_\varepsilon^m \ge C_1$ and $\|w_\varepsilon^m\| \le C_2$ for some constants $C_1 > 0$ and $C_2 > 0$ independent of ε and m provided $\varepsilon = m^{-(N+2)}$. With this choice of ε we have $\varepsilon^{N-2}m^{2N} \to 0$ as $m \to \infty$ (see Lemma 4.1). Also, with this choice we have

$$\frac{\varepsilon^{N-2}m^{2N}}{m^{-(N+2)}} = m^{-N^2+3N+6} \to 0 \quad \text{as } m \to \infty$$

and

$$\frac{m^{(-N^2+2N)/2}}{m^{-(N+2)}} = m^{(-N^2+4N+4)/2} \to 0 \quad \text{as } m \to \infty$$

if $N \ge 5$. From Lemma 3.1 and assumption (g_7) (see also the proof of Lemma 8 in [12]) we derive the estimate

$$J(w_\varepsilon^m) \le Cm^{-N(N-2)/2} \quad \text{for large } m.$$

We now observe that the quantities $\varepsilon^{N-2}m^{2N}$ and $m^{(-N^2+2N)/2}$ behave as $o(\varepsilon)$ and can be incorporated into the $o(\varepsilon)$ appearing in the asymptotic estimates for $\overline{K}_1(\varepsilon)$ and $\overline{K}_2(\varepsilon)$ (see (3.2) and (3.3)). Hence

$$\begin{aligned} J(v_\varepsilon^m) = J(w_\varepsilon^m) + J(t_\varepsilon^m U_\varepsilon^m) &\le Cm^{(-N^2+2N)/2} + \frac{(t_\varepsilon^m)^2}{2}\,\overline{K}_1(\varepsilon) - \frac{(t_\varepsilon^m)^{2^*}}{2^*}\overline{K}_2(\varepsilon) \\ &\le \max_{t\ge 0}\left(\frac{t^2}{2}\,\overline{K}_1(\varepsilon) - \frac{t^{2^*}}{2^*}\,\overline{K}_2(\varepsilon)\right) < \frac{S^{N/2}}{2N} \end{aligned}$$

for sufficiently small ε. This contradiction completes the proof. ■

To extend Theorem 4.2 to the cases $N = 3, 4$ we additionally assume that

(A) if $N = 3$, then

$$\lim_{s\to\infty} \frac{G(x,s)}{s^{17/3}} = \infty$$

uniformly in $x \in B(0,\varrho_\circ) \cap \Omega$, and if $N = 4$, then

$$\lim_{s\to\infty} \frac{G(x,s)}{s^{11/3}} = \infty$$

uniformly in $x \in B(0,\varrho_\circ) \cap \Omega$, for some $\varrho_\circ > 0$.

Notice that if $N = 3$, then $17/3 < 2^*|_{N=3} = 6$, and if $N = 4$, then $11/3 < 2^*|_{N=4} = 4$. Therefore under this assumption G has a subcritical growth at infinity.

THEOREM 4.3. *Let* $N = 3, 4$ *and suppose that the assumptions of Theorem* 4.2 *hold. If*, *additionally*, *assumption* (A) *is satisfied*, *then problem* (1.1) *has a solution.*

Proof. CASE $N = 3$. In this case, using the argument from the proof of Lemma 4.1, we have

$$\overline{K}_1(\varepsilon) \leq \frac{1}{2}\, K_1 - C_\circ \varepsilon |\log \varepsilon| + o(\varepsilon) + C\varepsilon m^3, \qquad \overline{K}_2(\varepsilon) \geq \frac{1}{2}\, \overline{K}_2 - O(\varepsilon) + \varepsilon^3 m^6.$$

We choose $\varepsilon = m^{-8}$. Then $\varepsilon m^3 \to 0$ and $\varepsilon^3 m^6 \to 0$ as $m \to \infty$ and we can show that $t_\varepsilon^m \geq C > 0$. We now modify the argument from [12] (see Lemmas 5 and 6 there). Assume that $\varepsilon \leq \varrho_\circ$. Let h be the function giving a local representation of $\partial\Omega$ around 0. By taking ε smaller if necessary we derive from assumption (A) that

$$(\varepsilon^2 - |x'|^2)^{1/2} - h(x') \geq C\varepsilon$$

for $x' \in D(0, \varepsilon/2)$, where $C > 0$ is a constant independent of ε. We then have

$$\begin{aligned}
&\int_\Omega G(x, t_\varepsilon^m \overline{U}_\varepsilon^n)\, dx \\
&\quad \geq \int_{D(0,\varepsilon/2)} dx' \int_{h(x')}^{(\varepsilon^2 - |x'|^2)^{1/2}} \left(\frac{\varepsilon^{1/2}}{(\varepsilon^2 + |x|^2)^{1/2}} \right)^{17/3} \phi\left(c\, \frac{\varepsilon^{1/2}}{(\varepsilon^2 + |x|^2)^{1/2}} \right) dx_3 \\
&\quad \geq C \int_{D(0,\varepsilon/2)} \varepsilon (\varepsilon^{-1/2})^{17/3} \phi(c\varepsilon^{-1/2})\, dx' = C \int_0^{\varepsilon/2} \varepsilon (\varepsilon^{-1/2})^{17/3} r\, dr\, \phi(c\varepsilon^{-1/2}) \\
&\quad = C\varepsilon^{1/6} \phi(c\varepsilon^{-1/2}),
\end{aligned}$$

where $D(0, \varepsilon/2) = B(0, \varepsilon/2) \cap (x_3 = 0)$ and ϕ is an increasing function such that $\phi(s) \to \infty$ as $s \to \infty$. Letting $\varepsilon = m^{-8}$ we get

$$\int_\Omega G(x, t_\varepsilon^m \overline{U}_\varepsilon^m)\, dx \geq C m^{-4/3} \phi(cm^4).$$

We put

$$\widetilde{K}_1(\varepsilon) = \frac{1}{2}\, K_1 - C_\circ \varepsilon |\log \varepsilon| + o(\varepsilon), \qquad \widetilde{K}_2(\varepsilon) = \frac{1}{2}\, K_2 - O(\varepsilon)$$

and

$$A(\varepsilon) = \max_{t \geq 0} \left(\frac{t^2}{2}\, \widetilde{K}_1(\varepsilon) - \frac{t^{2^*}}{2^*}\, \widetilde{K}_2(\varepsilon) \right).$$

If $\varepsilon = m^{-8}$, then $A(\varepsilon) < S^{3/2}/6$ for large m. As in [12] we can show that

$$J(w_\varepsilon^m) \leq C m^{-3/2} \quad \text{for large } m.$$

Hence

$$\begin{aligned}
J(v_\varepsilon^m) &= J(w_\varepsilon^m) + J(t_\varepsilon^m \overline{U}_\varepsilon^m) \leq C m^{-3/2} + C m^{-5} \\
&\qquad - C m^{-4/3} \phi(cm^4) + A(m^{-1/8}) \\
&= A(m^{-1/8}) + m^{-4/3} (C m^{-3/2 + 4/3} + C m^{-5 + 4/3} - C\phi(cm^4)).
\end{aligned}$$

Since $\phi(cm^4)\to\infty$, the last expression becomes negative for large m. Hence $J(v_\varepsilon^m)<S^{3/2}/6$ for large m and we have arrived at a contradiction.

CASE $N=4$. In this case we have

$$\overline{K}_1(\varepsilon)=\frac{1}{2}K_1-I(\varepsilon)+o(\varepsilon)+c\varepsilon^2m^4,\qquad \overline{K}_2(\varepsilon)=\frac{1}{2}K_2-\mathit{\Pi}(\varepsilon)+o(\varepsilon)+\varepsilon^4m^8.$$

We commence with the estimate

$$\begin{aligned}\int_\Omega G(x,t_\varepsilon^m\overline{U}_\varepsilon^m)\,dx &\geq C\int_{D(0,\varepsilon/2)}dx'\int_{h(x')}^{(\varepsilon^2-|x'|^2)^{1/2}}\left(\frac{\varepsilon}{(\varepsilon^2+|x|^2)}\right)^{11/3}\phi\left(c\,\frac{\varepsilon}{(\varepsilon^2+|x|^2)}\right)dx_4\\ &\geq C\int_{D(0,\varepsilon/2)}\varepsilon(\varepsilon^{-1})^{11/3}\phi(c\varepsilon^{-1})\,dx'=C\varepsilon^{1/3}\phi(c\varepsilon^{-1}).\end{aligned}$$

We choose $\varepsilon=m^{-6}$. Then $\varepsilon^2m^4\to0$ and $\varepsilon^4m^8\to0$ as $m\to\infty$. Let

$$\widetilde{K}_1(\varepsilon)=\frac{1}{2}K_1-I(\varepsilon)+o(\varepsilon),\qquad \widetilde{K}_2(\varepsilon)=\frac{1}{2}K_2-\mathit{\Pi}(\varepsilon)+o(\varepsilon)$$

and put

$$A(\varepsilon)=\max_{t\geq0}\left(\frac{t^2}{2}\widetilde{K}_1(\varepsilon)-\frac{t^4}{4}\widetilde{K}_2(\varepsilon)\right).$$

If $\varepsilon=m^{-6}$, then $A(\varepsilon)<S^2/8$ for m large. We also have

$$J(w_\varepsilon^m)\leq Cm^{-4}\quad\text{ for } m \text{ large.}$$

Thus

$$\begin{aligned}J(w_\varepsilon^m+t_\varepsilon^m\overline{U}_\varepsilon^m)&\leq A(m^{-4})+Cm^{-4}-Cm^{-2}\phi(cm^6)\\ &=A(m^{-6})+Cm^{-2}(m^{-2}-\phi(cm^6))<S^2/8,\end{aligned}$$

which is impossible. ■

5. Extension to a problem with weight. In this section we will examine the effect of a weight in the critical nonlinearity on the existence of a solution. We consider the problem

$$\text{(5.1)}\qquad \begin{cases}-\Delta u=Q(x)|u|^{2^*-2}u+g(x,u) & \text{in } \Omega,\\ \dfrac{\partial u}{\partial\nu}=0 & \text{on } \partial\Omega.\end{cases}$$

We assume that $Q(x)$ is continuous and positive on $\overline{\Omega}$. Solutions to problem (5.1) will be obtained as critical points of the functional

$$I(u)=\frac{1}{2}\int_\Omega|\nabla u|^2\,dx-\frac{1}{2^*}\int_\Omega Q(x)|u|^{2^*}\,dx-\int_\Omega G(x,u)\,dx.$$

Obviously, I is a C^1-functional on $H^1(\Omega)$. Let $Q_{\mathrm{m}} = \max_{x\in\partial\Omega} Q(x)$ and $Q_{\mathrm{M}} = \max_{x\in\overline{\Omega}} Q(x)$.

PROPOSITION 5.1. *Suppose that* (g_1)–(g_3) *hold. Let* $\{u_m\} \subset H^1(\Omega)$ *be a* $(\mathrm{PS})_c$ *sequence for* I. *Then, up to a subsequence,* $u_m \rightharpoonup u$ *in* $H^1(\Omega)$ *and* $I'(u) = 0$. *The weak limit* u *is nonzero if* $g(x,0) \neq 0$ *or* $g(x,0) = 0$ *and* $c \in (0, s_\infty)$ *with*

$$s_\infty = \min\left(\frac{S^{N/2}}{2NQ_{\mathrm{m}}^{(N-2)/2}}, \frac{S^{N/2}}{NQ_{\mathrm{M}}^{(N-2)/2}}\right).$$

Proof. As in Proposition 2.1 we show that $\{u_m\}$ is bounded in $H^1(\Omega)$. Hence, up to a subsequence, $u_m \rightharpoonup u$ in $H^1(\Omega)$. Consider the case $g(x,0) = 0$ and $c \in (0, s_\infty)$. Arguing by contradiction assume $u = 0$. By P.-L. Lions' concentration-compactness principle [15] there exist at most countable collections of points $\{x_j\} \subset \overline{\Omega}$, $j \in J$, and positive numbers $\{\mu_j\}, \{\nu_j\}$, $j \in J$, such that

$$|u_m|^{2^*} \rightharpoonup d\nu = \sum_{j\in J} \nu_j \delta_{x_j}, \qquad |\nabla u|^2\,dx \rightharpoonup d\mu = \sum_{j\in J} \mu_j \delta_{x_j}$$

in the sense of measures. Moreover,

$$\frac{S}{2^{2/N}}\,\nu_i^{2/2^*} \le \mu_i \quad \text{if } x_i \in \partial\Omega \quad \text{and} \quad S\nu_i^{2/2^*} \le \mu_i \quad \text{if } x_i \in \Omega.$$

We also have $\mu_i = Q(x_i)\nu_i$. We write

$$\begin{aligned} c + o(1) &= I(u_m) - \frac{1}{2}\langle I'(u_m), u_m\rangle \\ &= \frac{1}{N}\int_\Omega Q(x)|u_m|^{2^*}\,dx + \int_\Omega \left(\frac{1}{2}\,g(x,u_m)u_m - G(x,u_m)\right) dx. \end{aligned} \tag{5.2}$$

Since g is subcritical, letting $m \to \infty$ in (5.2), we get

$$c = \frac{1}{N}\sum_{j\in J} Q(x_j)\nu_j.$$

If $x_i \in \partial\Omega$ and $\nu_i > 0$, then $\nu_i \ge S^{N/2}/2Q(x_i)^{N/2}$. If $x_i \in \Omega$ and $\nu_i > 0$, then $\nu_i \ge S^{N/2}/Q(x_i)^{N/2}$. Assuming that one of the ν_i's is not 0 we derive that

$$c \ge \begin{cases} \dfrac{1}{2N}\,\dfrac{S^{N/2}}{Q(x_i)^{(N-2)/2}} \ge \dfrac{1}{2N}\,\dfrac{S^{N/2}}{Q_{\mathrm{m}}^{(N-2)/2}} & \text{if } x_i \in \partial\Omega, \\[2ex] \dfrac{1}{N}\,\dfrac{S^{N/2}}{Q(x_i)^{(N-2)/2}} \ge \dfrac{1}{N}\,\dfrac{S^{N/2}}{Q_{\mathrm{M}}^{(N-2)/2}} & \text{if } x_i \in \Omega. \end{cases}$$

In both cases we have a contradiction. Hence $\nu_i = \mu_i = 0$ for all $i \in J$. This means that $u_m \to 0$ in $H^1(\Omega)$. This yields $I(u_m) \to 0$, which is again a contradiction. ∎

Case $Q_{\mathrm{M}} > 2^{2/(N-2)} Q_{\mathrm{m}}$. In this case we have

$$s_\infty = \frac{S^{N/2}}{N Q_{\mathrm{M}}^{(N-2)/2}}.$$

First we consider the nonresonance case. Without loss of generality we may assume that $0 \in \Omega$ and $Q(0) = Q_{\mathrm{M}}$. We replace assumption (g_5) by

(g_5') there exist constants $\sigma > 0$ and $0 < \alpha \le Q_* = \min_{x \in \bar{\Omega}} Q(x)$ such that

$$G(x,s) \ge \frac{1}{2}(\lambda_k + \sigma)s^2 - \frac{\alpha}{2^*}|s|^{2^*}$$

for all $(x,s) \in \Omega \times \mathbb{R}$.

Theorem 5.2. *Let $Q_{\mathrm{M}} > 2^{2/(N-2)} Q_{\mathrm{m}}$. Suppose that assumptions (g_1)–(g_4) and (g_5') hold. Moreover assume that*

$$|Q(x) - Q(0)| = \begin{cases} O(|x|^{(1-\alpha)(N-2)}) & \text{for some } \alpha \in (0, \frac{N-4}{2(N-2)}) \\ & \text{if } N \ge 5, \\ O(|x|^2) & \text{if } N = 4, \\ O(|x|) & \text{if } N = 3. \end{cases} \tag{5.3}$$

If $N = 3$, additionally assume that

$$\lim_{s \to \infty} \frac{G(x,s)}{s^4} = \infty$$

uniformly in $B(0, \varrho_\circ) \subset \Omega$ for some $\varrho_\circ > 0$. Then problem (5.1) *has a solution.*

Proof. It is clear that the assumptions of the linking theorem [18] are satisfied. We choose m so large that

$$c_k m^{2-N} < \sigma. \tag{5.4}$$

We must show that

$$\sup_{v \in Q_m^\varepsilon} I(v) < \frac{S^{N/2}}{N Q_{\mathrm{M}}^{(N-2)/2}}. \tag{5.5}$$

Arguing by contradiction assume that for every $\varepsilon > 0$ we have

$$\sup_{v \in Q_m^\varepsilon} I(v) \ge \frac{S^{N/2}}{N Q_{\mathrm{M}}^{(N-2)/2}}.$$

As the set $\{v \in Q_m^\varepsilon\,;\, I(v) \ge 0\}$ is compact, the above supremum is attained. Therefore for every $\varepsilon > 0$ there exist $w_\varepsilon \in H_m^-$ and $t_\varepsilon > 0$ such that

$$I(v_\varepsilon) = \max_{v \in Q_m^\varepsilon} I(v) \ge \frac{S^{N/2}}{N Q_{\mathrm{M}}^{(N-2)/2}}, \qquad v_\varepsilon = w_\varepsilon + t_\varepsilon \overline{U}_\varepsilon,$$

that is,

$$\frac{1}{2}\int_\Omega |\nabla v_\varepsilon|^2\,dx - \int_\Omega G(x,v_\varepsilon)\,dx - \frac{1}{2^*}\int_\Omega Q(x)|v_\varepsilon|^{2^*}\,dx \ge \frac{S^{N/2}}{NQ_{\mathrm{M}}^{(N-2)/2}}. \tag{5.6}$$

Since $\{t_\varepsilon\}$ and $\{w_\varepsilon\}$ are bounded we may assume that $t_\varepsilon \to t_\circ \ge 0$ and $w_\varepsilon \to w_\circ \in H_m^-$. By (5.4), (g_5') and Lemma 3.1, we have $I(w_\varepsilon) \le 0$. We now show that $t_\circ = Q_{\mathrm{M}}^{-(N-2)/4}$. Since G is subcritical we have

$$\lim_{\varepsilon\to 0}\int_\Omega G(x,t_\varepsilon \overline{U}_\varepsilon)\,dx = 0.$$

Since $\|\nabla \overline{U}_\varepsilon\|_2^2 = S^{N/2} + O(\varepsilon^{N-2})$ and $\|\overline{U}_\varepsilon\|_{2^*}^{2^*} = S^{N/2} + O(\varepsilon^N)$, and since $\lim_{\varepsilon\to0}\int_\Omega Q(x)|\overline{U}_\varepsilon|^{2^*}\,dx = S^{N/2}Q_{\mathrm{M}}$, we get

$$I(t_\varepsilon \overline{U}_\varepsilon) \le S^{N/2}\left(\frac{t_\circ^2}{2} - \frac{Q_{\mathrm{M}} t_\circ^{2^*}}{2^*}\right) + o(1) = S^{N/2}\Phi(t_\circ) + o(1).$$

We now observe that $\Phi(t)$ attains its maximum at $\overline{t} = Q_{\mathrm{M}}^{-(N-2)/4}$ and $\Phi(\overline{t}) = N^{-1}Q_{\mathrm{M}}^{-(N-2)/2}$. So if $t_\circ \ne \overline{t}$ we get a contradiction. We only consider the case $N \ge 5$. To proceed further we need an analogue of Lemma 4 from [12]. We claim that

$$\frac{t_\varepsilon^2}{2}\int_\Omega |\nabla \overline{U}_\varepsilon|^2\,dx - \frac{t_\varepsilon^{2^*}}{2^*}\int_\Omega Q(x)|\overline{U}_\varepsilon|^{2^*}\,dx \le \frac{S^{N/2}}{NQ_{\mathrm{M}}^{(N-2)/2}} + O(\varepsilon^l),$$

where $l = (1-\alpha)(N-2)$. The following inequalities are easy to verify:

$$\frac{t_\varepsilon^2}{2}\|\nabla \overline{U}_\varepsilon\|_2^2 \le \frac{S^{N/2}}{2Q_{\mathrm{M}}^{(N-2)/2}} + \frac{t_\varepsilon^2 - Q_{\mathrm{M}}^{-(N-2)/2}}{2} S^{N/2} + O(\varepsilon^{N-2})$$

and

$$\begin{aligned}
&\frac{t_\varepsilon^{2^*}}{2^*}\int_\Omega Q(x)|\overline{U}_\varepsilon|^{2^*}\,dx \\
&\quad\ge \frac{Q_{\mathrm{M}} S^{N/2} t_\varepsilon^{2^*}}{2^*} + \frac{t_\varepsilon^{2^*}}{2^*}\int_\Omega (Q(x)-Q(0))|\overline{U}_\varepsilon|^{2^*}\,dx + O(\varepsilon^N) \\
&\quad= \frac{Q_{\mathrm{M}} S^{N/2}}{2^*} Q_{\mathrm{M}}^{-N/2} + \frac{Q_{\mathrm{M}} S^{N/2}}{2^*}(t_\varepsilon^{2^*} - Q_{\mathrm{M}}^{-N/2}) + O(\varepsilon^N) + O(\varepsilon^l).
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{t_\varepsilon^2}{2}\int_\Omega |\nabla \overline{U}_\varepsilon|^2\,dx - \frac{t_\varepsilon^{2^*}}{2^*}\int_\Omega Q|\overline{U}_\varepsilon|^{2^*}\,dx \\
&\quad\le \frac{S^{N/2}}{NQ_{\mathrm{M}}^{(N-2)/2}} + S^{N/2}\frac{t_\varepsilon^2 - Q_{\mathrm{M}}^{-(N-2)/2}}{2} - \frac{Q_{\mathrm{M}} S^{N/2}}{2^*}(t_\varepsilon^{2^*} - Q_{\mathrm{M}}^{-N/2}) + O(\varepsilon^l).
\end{aligned}$$

Since

$$\max_{x\ge 0}\left[\frac{x^2 - Q_{\mathrm{M}}^{-(N-2)/2}}{2} - \frac{Q_{\mathrm{M}}}{2^*}\,(x^{2^*} - Q_{\mathrm{M}}^{-N/2})\right] = 0,$$

the claim follows. To estimate $\int_\Omega G(x, t_\varepsilon \overline{U}_\varepsilon)\,dx$ we use Lemma 5 from [12] (formula (24)). We then have

$$\begin{aligned} I(v_\varepsilon) &= I(w_\varepsilon) + I(t_\varepsilon \overline{U}_\varepsilon) \\ &\le \frac{S^{N/2}}{NQ_{\mathrm{M}}^{(N-2)/2}} + O(\varepsilon^{(1-\alpha)(N-2)}) - c\varepsilon^{N-2}\varepsilon^{-(N-4)/2} \\ &= \frac{S^{N/2}}{NQ_{\mathrm{M}}^{(N-2)/2}} + \varepsilon^{(1-\alpha)(N-2)}(C - c\varepsilon^{\alpha(N-2)-(N-4)/2}) < \frac{S^{N/2}}{NQ_{\mathrm{M}}^{(N-2)/2}} \end{aligned}$$

for $\varepsilon > 0$ small enough and we have arrived at a contradiction. ■

REMARK 5.3. The flatness condition (5.3) can be replaced by the condition $|Q(x) - Q(0)| = O(|x|)$ (locally Lipschitz around 0) if we add the assumption

$$\lim_{s\to\infty} \frac{G(x,s)}{s^{2(N-1)/(N-2)}} = \infty$$

uniformly in $x \in B(0, \varrho_\circ)$ (if $N = 3$, we get assumption (8) from [12]).

Indeed, we only need to observe that Lemmas 5 and 6 from [12] give

$$\begin{aligned} \int_\Omega G(x, t_\varepsilon \overline{U}_\varepsilon)\,dx &\ge C\int_0^\varepsilon \left(\frac{\varepsilon^{\frac{N-2}{2}}}{(\varepsilon^2 + r^2)^{\frac{N-2}{2}}}\right)^{\frac{2(N-1)}{N-2}} \tau\left(c\frac{\varepsilon^{\frac{N-2}{2}}}{(\varepsilon^2 + r^2)^{\frac{N-2}{2}}}\right) r^{N-1}\,dr \\ &\ge C(\varepsilon^{-\frac{N-2}{2}})^{\frac{2(N-1)}{N-2}}\tau(c\varepsilon^{-\frac{N-2}{2}})r^N|_0^\varepsilon = C\varepsilon^{-(N-1)}\varepsilon^N\tau(c\varepsilon^{-\frac{N-2}{2}}) \\ &= C\varepsilon\tau(c\varepsilon^{-\frac{N-2}{2}}), \end{aligned}$$

with $\tau(s) \to \infty$ as $s \to \infty$.

We now turn our attention to the resonance case. Assumption (g_7) is replaced by

(g_7') there exists $\alpha \in (0, Q_*)$ such that

$$G(x,s) \ge \frac{\lambda_k s^2}{2} - \frac{\alpha}{2^*}\,|s|^{2^*}$$

for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$.

THEOREM 5.4. *Let $Q_{\mathrm{M}} > 2^{2/(N-2)}Q_{\mathrm{m}}$ and let $Q(x_\circ) = Q_{\mathrm{M}}$ with $x_\circ \in \Omega$ and*

$$|Q(x) - Q(x_\circ)| = O(|x - x_\circ|^l) \tag{5.7}$$

for x close to $x_\circ$ and $l > N(N-2)/(N+2)$. Furthermore, assume that

(g_1)–(g_3), (g_6) and (g_7') hold and that

$$\lim_{s\to\infty}\frac{G(x,s)}{s^{8N/(N^2-2)}}=\infty \quad \textit{uniformly in } x\in B(0,\varrho_\circ)\subset\Omega \tag{5.8}$$

for some $\varrho_\circ>0$. Then problem (5.1) *has a solution.*

Proof. For simplicity we assume that $x_\circ=0$. We proceed as in Theorem 5.2. By Lemma 6 in [12] we have

$$\|\nabla\overline{U}_\varepsilon^m\|_2^2=S^{N/2}+O((\varepsilon m)^{N-2}),\qquad \|\overline{U}_\varepsilon^m\|_{2^*}^{2^*}=S^{N/2}+O((\varepsilon m)^N).$$

Repeating the argument from the proof of Theorem 5.2 we show that $t_\varepsilon=Q_{\mathrm{M}}^{-(N-2)/4}$. We now notice that assumption (5.7) yields

$$\begin{aligned}\int_\Omega Q(x)(\overline{U}_\varepsilon^m)^{2^*}\,dx&=\int_\Omega Q_{\mathrm{M}}(\overline{U}_\varepsilon^m)^{2^*}\,dx+\int_\Omega(Q(x)-Q_{\mathrm{M}})(\overline{U}_\varepsilon^m)^{2^*}\,dx\\&=Q_{\mathrm{M}}S^{N/2}+O(\varepsilon^l)+O((\varepsilon m)^N).\end{aligned}$$

Hence

$$\begin{aligned}&\frac{t_\varepsilon^{2^*}}{2^*}\int_\Omega Q(x)(\overline{U}_\varepsilon^m)^{2^*}\,dx\\&\quad=\frac{Q_{\mathrm{M}}S^{N/2}}{2^*}Q_{\mathrm{M}}^{-N/2}+Q_{\mathrm{M}}\frac{S^{N/2}}{2^*}(t_\varepsilon^{2^*}-Q_{\mathrm{M}}^{-N/2})+O((\varepsilon m)^N)+O(\varepsilon^l).\end{aligned}$$

Similarly we have

$$\frac{t_\varepsilon^2}{2}\int_\Omega|\nabla\overline{U}_\varepsilon|^2\,dx=\frac{S^{N/2}}{2Q_{\mathrm{M}}^{(N-2)/2}}+S^{N/2}\frac{t_\varepsilon^2-Q_{\mathrm{M}}^{-(N-2)/2}}{2}+O((\varepsilon m)^{N-2}).$$

The last two relations yield

$$\begin{aligned}&\frac{t_\varepsilon^2}{2}\int_\Omega|\nabla\overline{U}_\varepsilon|^2\,dx-\frac{t_\varepsilon^{2^*}}{2^*}\int_\Omega Q(\overline{U}_\varepsilon^m)^{2^*}\,dx\\&\qquad\le\frac{S^{N/2}}{NQ_{\mathrm{M}}^{(N-2)/2}}+S^{N/2}\frac{t_\varepsilon^2-Q_{\mathrm{M}}^{-(N-2)/2}}{2}\\&\qquad\quad-\frac{Q_{\mathrm{M}}S^{N/2}}{2^*}(t_\varepsilon^{2^*}-Q_{\mathrm{M}}^{-N/2})+O(\varepsilon^l)+O((\varepsilon m)^{N-2}).\end{aligned}\tag{5.9}$$

We now observe that the function

$$f(x)=x^2-Q_{\mathrm{M}}^{-(N-2)/2}-\frac{Q_{\mathrm{M}}(N-2)}{N}(x^{2^*}-Q_{\mathrm{M}}^{-N/2})$$

has $\max_{x\ge0}f(x)=0$. Therefore we derive from (5.9) that

$$\begin{aligned}&\frac{t_\varepsilon^2}{2}\int_\Omega|\nabla\overline{U}_\varepsilon^m|^2\,dx-\frac{t_\varepsilon^{2^*}}{2^*}\int_\Omega Q(\overline{U}_\varepsilon^m)^{2^*}\,dx\\&\qquad\le\frac{S^{N/2}}{NQ_{\mathrm{M}}^{(N-2)/2}}+O(\varepsilon^l)+O((\varepsilon m)^{N-2}).\end{aligned}\tag{5.10}$$

Assumption (g_7') allows us to establish the analogue of Lemma 8 from [12], that is, the estimate

$$I(w_\varepsilon^m) \le Cm^{-N(N-2)/2} \tag{5.11}$$

for large m. It is easy to show that assumption (5.8) implies the estimate (see Lemma 6 in [12])

$$\int_\Omega G(x, \overline{U}_\varepsilon^m)\,dx \ge \varepsilon^{N(N-2)/(N+2)}\phi(\varepsilon^{-1}), \tag{5.12}$$

where $\phi(\varepsilon^{-1}) \to \infty$ as $\varepsilon \to 0$. Combining (5.9)–(5.12) we have

$$\begin{aligned} I(v_\varepsilon^m) &= I(w_\varepsilon^m) + I(t_\varepsilon^m \overline{U}_\varepsilon^m) \\ &\le c_k m^{-N(N-2)/2} + \frac{S^{N/2}}{NQ_{\mathrm{M}}^{(N-2)/2}} + O(\varepsilon^l) \\ &\quad + O((\varepsilon m)^{N-2}) - c\varepsilon^{N(N-2)/(N+2)}\phi(\varepsilon^{-1}). \end{aligned}$$

We now put $\varepsilon = m^{-(N+2)/2}$. With this choice of ε the above relation becomes

$$\begin{aligned} I(v_\varepsilon^m) &\le c_k m^{-N(N-2)/2} + \frac{S^{N/2}}{NQ_{\mathrm{M}}^{(N-2)/2}} + O(\varepsilon^{-l(N+2)/2}) \\ &\quad + O(m^{-(N^2-4)/2+N-2}) - cm^{-N(N-2)/2}\phi(cm^{(N+2)/2}) \\ &= \frac{S^{N/2}}{NQ_{\mathrm{M}}^{(N-2)/2}} + m^{-N(N-2)/2}[c_k + O(m^{-l(N+2)/2+N(N-2)/2}) \\ &\quad + O(m^{(-N^2+2N)/2+N(N-2)/2}) - c\phi(m^{(N+2)/2})]. \end{aligned}$$

Since $-l(N+2)/2 + N(N-2)/2 < 0$, we see that

$$I(v\varepsilon^m) < \frac{1}{N}\,\frac{S^{N/2}}{Q_{\mathrm{M}}^{(N-2)/2}}$$

for large m. This contradiction completes the proof. ■

Case $Q_{\mathrm{M}} \le 2^{2/(N-2)}Q_{\mathrm{m}}$. In this case we have

$$s_\infty = \frac{S^{N/2}}{2NQ_{\mathrm{m}}^{(N-2)/2}}.$$

We now formulate two theorems dealing with nonresonance and resonance cases.

Theorem 5.5. *Let $Q_{\mathrm{M}} \le 2^{2/(N-2)}Q_{\mathrm{m}}$ and $Q(x_\circ) = Q_{\mathrm{m}}$ with $x_\circ \in \partial\Omega$ and $H(x_\circ) > 0$. Suppose that (g_1)–(g_4) and (g_5') hold and that*

$$|Q(x) - Q(x_\circ)| = o(|x - x_\circ|) \tag{5.13}$$

for x close to $x_\circ$. Then problem (5.1) *has a solution.*

THEOREM 5.6. *Let* $N \geq 5$, $Q_{\mathrm{M}} \leq 2^{2/(N-2)} Q_{\mathrm{m}}$ *and* $Q_{\mathrm{m}} = Q(x_\circ)$ *with* $x_\circ \in \partial\Omega$ *and* $H(x_\circ) > 0$. *Suppose that assumptions* (g_1)–(g_3), (g_6) *and* (g_7') *hold. Moreover assume that* (5.13) *is satisfied. Then problem* (5.1) *admits a solution.*

The proofs of these two theorems are similar to those of Theorems 3.2, 4.2 and are omitted.

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