COLLOQUIUM MATHEMATICUM

VOL. 108 2007 NO. 2

PSEUDO-BOCHNER-FLAT LOCALLY CONFORMAL KÄHLER SUBMANIFOLDS

BY

KOJI MATSUO (Ichinoseki)

Dedicated to Professor Shinsuke Yorozu on his sixtieth birthday

Abstract. Let $\widetilde{M}$ be an $(m+r)$-dimensional locally conformal Kähler (l.c.K.) manifold and let M be an m-dimensional l.c.K. submanifold of $\widetilde{M}$ (i.e., a complex submanifold with the induced l.c.K. structure). Assume that both $\widetilde{M}$ and M are pseudo-Bochner-flat. We prove that if $r < m$, then M is totally geodesic (in the Hermitian sense) in $\widetilde{M}$. This is the l.c.K. version of Iwatani's result for Bochner-flat Kähler submanifolds.

1. Introduction. Let $\widetilde{M}$ be an $(m+r)$-dimensional Kähler manifold and let M be an m-dimensional Kähler submanifold of $\widetilde{M}$. If both manifolds are of constant holomorphic sectional curvature, then by Theorem 3 of O'Neill [6], M is totally geodesic in $\widetilde{M}$ if $r < m(m+1)/2$. Kon [4] proved, under the weaker assumption that both manifolds are Bochner-flat, that M is totally geodesic in $\widetilde{M}$ if $r < (m+1)(m+2)/(4m+2)$. In the case of $r = 1$, this is due to Yamaguchi–Sato [8] provided that $m \geq 6$ (see also [9]). On the other hand, by estimating the dimension of the nullity space of normal curvature vectors at a point of M under the same assumption, Iwatani [2] has proved that M is totally geodesic in $\widetilde{M}$ if $r < m$. Since $m > (m+1)(m+2)/(4m+2)$ for $m > 2$, Iwatani's result contains Kon's.

In [5] the author introduced the notion of the pseudo-Bochner curvature tensor on a Hermitian manifold which is constructed out of the curvature tensor of the Hermitian (or Chern) connection and is conformally invariant. In the Kähler case, this tensor coincides with the original Bochner curvature tensor.

We wish to study Hermitian submanifolds making use of Hermitian connections. In this paper, the ambient manifolds are assumed to be locally conformally Kähler (l.c.K.) manifolds. Then their complex submanifolds inherit the l.c.K. structure. By an *l.c.K. submanifold*, we mean a complex submanifold with the induced l.c.K. structure.

2000 *Mathematics Subject Classification*: 53C55, 53C40.

Key words and phrases: Hermitian pseudo-curvature tensor, pseudo-Bochner curvature tensor, l.c.K. submanifolds, PG-equation.

Our purpose is to prove the following theorem corresponding to Iwatani's result mentioned above.

THEOREM 1.1. *Let* $\widetilde{M}$ *be an* $(m+r)$*-dimensional l.c.K. manifold and let* M *be an* m*-dimensional l.c.K. submanifold of* $\widetilde{M}$*. Assume that both* $\widetilde{M}$ *and* M *are pseudo-Bochner-flat. Then* $\{X \in T_xM : \sigma(X,X) = 0\}$ *is a* J*-invariant subspace of* T_xM*. If the complex dimension of this subspace is equal to* ℓ*, then* $r \geq \frac{1}{2}(m-\ell)(m+\ell+1)$*. Therefore, if* $r < m$*, then* M *is totally geodesic* (*in the Hermitian sense*) *in* $\widetilde{M}$*.*

COROLLARY 1.1. *Every pseudo-Bochner-flat l.c.K. hypersurface of a pseudo-Bochner-flat l.c.K. manifold is totally geodesic* (*in the Hermitian sense*)*.*

Throughout this paper, we work in the C^∞-category and deal with connected complex manifolds of (complex) dimension ≥ 2 without boundary only.

Acknowledgements. The author would like to thank Professor Iwatani and the referee for their kind advice and useful comments.

2. Preliminaries. Let M be a Hermitian manifold with complex structure J and compatible Riemannian metric g. The algebra of all C^∞ vector fields on M will be denoted by $\mathfrak{X}M$. The *Kähler form* Ω on M is defined by $\Omega(X,Y) = g(X,JY)$ for all $X,Y \in \mathfrak{X}M$. The *Hermitian connection* (or *Chern connection*) of M is a unique affine connection D on M such that $DJ = 0, Dg = 0$, and the torsion tensor T satisfies $T(JX,Y) = JT(X,Y)$ for all $X,Y \in \mathfrak{X}M$. The Hermitian connection D and the Levi-Civita connection ∇ are related by

$$g(D_XY,Z) = g(\nabla_XY,Z) + \tfrac{3}{2}d\Omega(JX,Y,Z) \tag{2.1}$$

for all $X,Y,Z \in \mathfrak{X}M$. Let H be the Hermitian curvature tensor (the curvature tensor of the Hermitian connection D) on M, i.e.,

$$H(X,Y) = [D_X, D_Y] - D_{[X,Y]}$$

for all $X,Y \in \mathfrak{X}M$. Then H has the following properties.

PROPOSITION 2.1 (cf. [5]). *For all* $X,Y,Z,W \in \mathfrak{X}M$,

$$H(X,Y,Z,W) = -H(Y,X,Z,W) = -H(X,Y,W,Z),$$
$$H(JX,JY,Z,W) = H(X,Y,JZ,JW) = H(X,Y,Z,W),$$
$$\mathfrak{S}\{H(X,Y)Z - T(T(X,Y),Z) - (D_XT)(Y,Z)\} = 0,$$

where $H(X,Y,Z,W) = g(H(Z,W)Y,X)$, *and* $\mathfrak{S}$ *denotes the cyclic sum with respect to* X,Y,Z*.*

For each unit vector X in T_xM, the *Hermitian holomorphic sectional curvature* $\mathcal{H}(X)$ for the holomorphic plane spanned by X and JX is given by

$$\mathcal{H}(X) = H(X, JX, X, JX).$$

In [5] we introduced a unique tensor P, called the *Hermitian pseudo-curvature tensor*, on M defined by

$$\begin{aligned} P(X,Y,Z,W) = \tfrac{1}{8}\{&H(X,Z,Y,W) - H(X,W,Y,Z) \\ &+ H(Y,W,X,Z) - H(Y,Z,X,W) \\ &+ H(X,JZ,Y,JW) - H(X,JW,Y,JZ) \\ &+ H(Y,JW,X,JZ) - H(Y,JZ,X,JW) \\ &+ 2H(X,Y,Z,W) + 2H(Z,W,X,Y)\} \end{aligned}$$

for $X, Y, Z, W \in \mathfrak{X}M$. The tensor P has the same symmetries as the Riemannian curvature tensor on a Kähler manifold, and we proved the following.

THEOREM 2.1 (cf. [5]). *A Hermitian manifold M is of pointwise constant Hermitian holomorphic sectional curvature k if and only if $P = \frac{k}{8}\, g \triangle g$.*

Here, for any two tensors a, b of type $(0,2)$, $a \triangle b$ is defined by

$$a \triangle b = a \owedge b + \overline{a} \owedge \overline{b} + 2\overline{a} \otimes \overline{b} + 2\overline{b} \otimes \overline{a},$$

where $a \owedge b$ denotes the tensor of type $(0,4)$ given by

$$\begin{aligned} (a \owedge sb)(X,Y,Z,W) = a(X,Z)b(Y,W) - a(X,W)b(Y,Z) \\ + a(Y,W)b(X,Z) - a(Y,Z)b(X,W), \end{aligned}$$

and $\overline{a}(X,Y) = a(X,JY)$ for $X, Y \in \mathfrak{X}M$.

Moreover in [5] we introduced a tensor B_H, called the *pseudo-Bochner curvature tensor*, on M defined by

$$B_H = P - \frac{1}{2(m+2)}\, g \triangle P_1 + \frac{p}{8(m+1)(m+2)}\, g \triangle g, \tag{2.2}$$

where $P_1(X,Y) = \frac{1}{2}\operatorname{tr}[Z \to P(X,JY)JZ]$, $p = \operatorname{tr} P_1$, and $m = \dim_{\mathbb{C}} M$. The tensor B_H is conformally invariant, and coincides with the original Bochner curvature tensor in the case where M is a Kähler manifold.

Finally, we recall the notion of an l.c.K. manifold. A Hermitian manifold M is said to be *locally conformal Kähler* (briefly, *l.c.K.*) if there is a closed 1-form ω on M, called the *Lee form*, such that $d\Omega = \omega \wedge \Omega$ (cf. [7]). In particular, if ω is exact, then M is said to be *globally conformal Kähler* (briefly, *g.c.K.*). If $\dim_{\mathbb{C}} M = m \geq 3$, the closedness of ω follows from the condition $d\Omega = \omega \wedge \Omega$. From (2.1), we can easily prove the following.

LEMMA 2.1.

$$d\Omega = \omega \wedge \Omega \;\Leftrightarrow\; 2T(X,Y) = \omega(X)Y - \omega(Y)X - \omega(JX)JY + \omega(JY)JX.$$

We shall give a typical example of a pseudo-Bochner-flat l.c.K. manifold.

EXAMPLE 2.1. Let α be any non-zero complex number with $|\alpha| \neq 1$, and let G_α be the cyclic group generated by the transformation $(z^1, \dots, z^m) \mapsto (\alpha z^1, \dots, \alpha z^m)$ of $\mathbb{C}^m - \{0\}$. Then G_α acts freely on $\mathbb{C}^m - \{0\}$ as a properly discontinuous group of complex analytic transformations. Thus the quotient space $H_\alpha^m = (\mathbb{C}^m - \{0\})/G_\alpha$ has the structure of a complex manifold. This manifold H_α^m is called the *Hopf manifold*. As is well known (cf. [3]), H_α^m is diffeomorphic to the product $S^1 \times S^{2m-1}$ of odd-dimensional spheres. In particular H_α^m is compact, and does not admit any Kähler metric. On $\mathbb{C}^m - \{0\}$, we consider a Hermitian metric

$$ds^2 = \frac{2}{\|z\|^2} \sum_{i=1}^{m} dz^i d\bar{z}^i,$$

where $\|z\|^2 = \sum_{i=1}^m z^i \bar{z}^i$. Since this metric is invariant under the action of G_α, it induces a Hermitian metric, called the *Boothby metric*, on H_α^m (cf. [1]). The Hopf manifold H_α^m with the Boothby metric is an l.c.K. manifold whose local Kähler metrics are flat. On such a manifold, the pseudo-Bochner curvature tensor vanishes everywhere.

3. L.c.K. submanifolds. Let $\psi : M \to \widetilde{M}$ be a holomorphic immersion of a complex manifold (M, J) into an l.c.K. manifold $(\widetilde{M}, \widetilde{J}, \widetilde{g})$. Then the Riemannian metric $g = \psi^* \widetilde{g}$ induced on M is Hermitian. Let $\widetilde{\Omega}$ and $\widetilde{\omega}$ be the Kähler and Lee forms on $\widetilde{M}$ respectively. Then $d\widetilde{\Omega} = \widetilde{\omega} \wedge \widetilde{\Omega}$. Putting $\Omega = \psi^* \widetilde{\Omega}$ and $\omega = \psi^* \omega$, it is easy to see that Ω is the Kähler form associated with g and satisfies $d\Omega = \omega \wedge \Omega$. Hence (M, J, g) is an l.c.K. manifold. For all local formulas we may consider ψ as an imbedding and thus identify $x \in M$ with $\psi(x) \in \widetilde{M}$. The tangent space $T_x M$ is identified with a subspace of the tangent space $T_x \widetilde{M}$. The normal space $T_x^\perp M$ is the orthogonal complement of $T_x M$ in $T_x \widetilde{M}$ with respect to $\widetilde{g}$. The tangent bundle of $\widetilde{M}$, restricted to M, is the Whitney sum of the tangent bundle TM and the normal bundle $T^\perp M$;

$$T\widetilde{M}|M = TM \oplus T^\perp M. \tag{3.1}$$

We denote by $\widetilde{D}$ the Hermitian connection of $\widetilde{M}$ with respect to $\widetilde{g}$. Let X and Y be any vector fields on M, and ξ any normal vector field on M. From (3.1) we may then decompose $\widetilde{D}_X Y$ and $\widetilde{D}_X \xi$ respectively as follows:

$$\widetilde{D}_X Y = D_X Y + \sigma(X, Y), \tag{3.2}$$

$$\widetilde{D}_X \xi = -A_\xi X + D_X^\perp \xi, \tag{3.3}$$

where $D_X Y$ (resp. $-A_\xi X$) and $\sigma(X, Y)$ (resp. $D_X^\perp \xi$) are the tangential and normal components respectively of $\widetilde{D}_X Y$ (resp. $\widetilde{D}_X \xi$). We will call (3.2) (resp. (3.3)) the *G-formula* (resp. *W-formula*).

Proposition 3.1. *D defines the Hermitian connection of M with respect to the induced Hermitian metric $g = \psi^* \widetilde{g}$.*

Proof. Let λ, μ be differentiable functions on M. Then

$$\begin{aligned} \widetilde{D}_{\lambda X}(\mu Y) &= \lambda\{(X\mu)Y + \mu \widetilde{D}_X Y\} \\ &= \{\lambda(X\mu)Y + \lambda\mu D_X Y\} + \lambda\mu\sigma(X,Y). \end{aligned} \tag{3.4}$$

Taking the tangential components of both sides, we find

$$D_{\lambda X}(\mu Y) = \lambda(X\mu)Y + \lambda\mu D_X Y.$$

This equation shows that D is an affine connection of M since additivity is trivial. To show that D is the Hermitian connection with respect to the induced Hermitian metric g on M, it is sufficient to show:

(1) $Dg = 0$,
(2) $DJ = 0$,
(3) the torsion tensor T of D satisfies $T(JX, Y) = JT(X, Y)$ for all $X, Y \in \mathfrak{X}M$.

In order to prove (1), we start from $\widetilde{D}\widetilde{g} = 0$, which implies

$$X\widetilde{g}(Y,Z) = \widetilde{g}(\widetilde{D}_X Y, Z) + \widetilde{g}(Y, \widetilde{D}_X Z) \quad \text{for all } X, Y, Z \in \mathfrak{X}M.$$

We have, however,

$$\widetilde{g}(\widetilde{D}_X Y, Z) = \widetilde{g}(D_X Y + \sigma(X,Y), Z) = g(D_X Y, Z),$$

because $\sigma(X,Y)$ is normal to M. Similarly,

$$\widetilde{g}(Y, \widetilde{D}_X Z) = g(Y, D_X Z).$$

Thus we find

$$Xg(Y,Z) = g(D_X Y, Z) + g(Y, D_X Z),$$

which means $Dg = 0$.

To prove (2), we start from $\widetilde{D}\widetilde{J} = 0$. This implies

$$\widetilde{D}_X \widetilde{J} Y = \widetilde{J}\widetilde{D}_X Y \quad \text{on } M. \tag{3.5}$$

Since $\widetilde{J}$ leaves the decomposition (3.1) invariant, we find, taking the tangential components of both sides,

$$D_X J Y = J D_X Y,$$

which means $DJ = 0$.

To prove (3), let $\widetilde{T}$ be the torsion tensor of $\widetilde{D}$. If we extend X and Y to vector fields $\widetilde{X}$ and $\widetilde{Y}$ on $\widetilde{M}$ (as we may do locally), then the restriction of $[\widetilde{X}, \widetilde{Y}]$ to M is tangent to M and coincides with $[X, Y]$. Of course, we also have

$$\widetilde{D}_{\widetilde{X}} \widetilde{Y} = \widetilde{D}_X Y \quad \text{and} \quad \widetilde{D}_{\widetilde{Y}} \widetilde{X} = \widetilde{D}_Y X \quad \text{on } M.$$

Thus we obtain

$$(3.6)\qquad \begin{aligned}\widetilde{T}(X,Y) &= \widetilde{D}_X Y - \widetilde{D}_Y X - [X,Y]\\ &= D_X Y + \sigma(X,Y) - D_Y X - \sigma(Y,X) - [X,Y]\\ &= T(X,Y) + \sigma(X,Y) - \sigma(Y,X).\end{aligned}$$

From this equation, we get

$$(3.7)\qquad \begin{aligned}0 &= \widetilde{T}(JX,Y) - \widetilde{J}\widetilde{T}(X,Y)\\ &= T(JX,Y) - JT(X,Y)\\ &\quad + \sigma(JX,Y) - \sigma(Y,JX) - \widetilde{J}\sigma(X,Y) + \widetilde{J}\sigma(Y,X).\end{aligned}$$

Taking the tangential components of both sides, we get

$$T(JX,Y) = JT(X,Y). \blacksquare$$

PROPOSITION 3.2. *$\sigma(X,Y)$ is bilinear in X and Y.*

Proof. Taking the normal components of both sides of (3.4), we get $\sigma(\lambda X,\ \mu Y) = \lambda\mu\sigma(X,Y)$. This shows that σ is bilinear in X and Y since additivity is trivial. ∎

LEMMA 3.1. $\sigma(JX,Y) = \sigma(X,JY) = \widetilde{J}\sigma(X,Y)$.

Proof. Taking the normal components of both sides of (3.5) and (3.7), we get $\sigma(X,JY) = \widetilde{J}\sigma(X,Y)$ and $\sigma(JX,Y) = \widetilde{J}\sigma(X,Y)$. ∎

The following is an immediate consequence of Lemma 3.1.

LEMMA 3.2. $\operatorname{tr}\sigma = 0$.

This lemma corresponds to the well known fact that every Kähler submanifold is minimal. We call σ the *Hermitian second fundamental form* of the l.c.K. submanifold M.

PROPOSITION 3.3. *σ is symmetric.*

Proof. From (3.6), a necessary and sufficient condition for σ to be symmetric is that the restriction of the torsion tensor $\widetilde{T}$ to M is tangent to M. Since $\widetilde{M}$ is l.c.K., Lemma 2.1 shows that $\widetilde{T}$ satisfies

$$2\widetilde{T}(X,Y) = \widetilde{\omega}(X)Y - \widetilde{\omega}(Y)X - \widetilde{\omega}(JX)JY + \widetilde{\omega}(JY)JX$$

for all $X,Y \in \mathfrak{X}M$. Hence $\widetilde{T}(X,Y)$ is tangent to M, that is, the Hermitian second fundamental form σ of M is symmetric. ∎

For general affine connections, totally geodesic submanifolds are defined as follows:

DEFINITION 3.1. A submanifold N of a manifold $\widetilde{N}$ is *totally geodesic* if geodesics of N are carried into geodesics of $\widetilde{N}$ by the immersion.

In our l.c.K. case, we have the following.

PROPOSITION 3.4. *An l.c.K. submanifold M of an l.c.K. manifold $\widetilde{M}$ is totally geodesic if and only if $\sigma = 0$ identically.*

Proof. Assume that the Hermitian second fundamental form σ of M in $\widetilde{M}$ vanishes identically. Then, for all $X \in \mathfrak{X}M$, we have

$$\widetilde{D}_X X = D_X X. \tag{3.8}$$

If x_t is a geodesic in M, then $D_{\dot{x}_t}\dot{x}_t \equiv 0$, where $\dot{x}_t$ is the velocity vector field of the curve x_t. Thus $\widetilde{D}_{\dot{x}_t}\dot{x}_t \equiv 0$ by (3.8). This shows that x_t is also a geodesic in $\widetilde{M}$. Hence, M is totally geodesic in $\widetilde{M}$.

Conversely, assume that M is totally geodesic in $\widetilde{M}$. Let $X_x \in T_xM$ be any unit vector at $x \in M$. Choose a geodesic x_t in M such that $x_0 = x$ and $\dot{x}_0 = X_x$. Then $D_{\dot{x}_t}\dot{x}_t = \widetilde{D}_{\dot{x}_t}\dot{x}_t = 0$. Thus $\sigma(X_x, X_x) = 0$. Since σ is symmetric and bilinear, we conclude that $\sigma = 0$ at x. ∎

4. Fundamental equations. Let M be an l.c.K. submanifold of an l.c.K. manifold $\widetilde{M}$. Let $\widetilde{H}$ be the curvature tensor of the Hermitian connection $\widetilde{D}$ of $\widetilde{M}$. Then, for all $X, Y, Z \in \mathfrak{X}M$, we have

$$\widetilde{H}(X,Y)Z = \widetilde{D}_X\widetilde{D}_Y Z - \widetilde{D}_Y\widetilde{D}_X Z - \widetilde{D}_{[X,Y]}Z.$$

Thus, by using the G-formula (3.2) and W-formula (3.3), we obtain

$$\begin{aligned}\widetilde{H}(X,Y)Z = H(X,Y)Z &- A_{\sigma(Y,Z)}X + A_{\sigma(X,Z)}Y \\ &+ \sigma(X, D_Y Z) - \sigma(Y, D_X Z) - \sigma([X,Y],Z) \\ &+ D^{\perp}_X\sigma(Y,Z) - D^{\perp}_Y\sigma(X,Z),\end{aligned}$$

where H is the curvature tensor of the Hermitian connection D of M. Hence, for all $X, Y, Z, W \in \mathfrak{X}M$, we have

$$\begin{aligned}H(X,Y,Z,W) &= \widetilde{H}(X,Y,Z,W) \\ &\quad + \widetilde{g}(\sigma(X,Z),\sigma(Y,W)) - \widetilde{g}(\sigma(X,W),\sigma(Y,Z)).\end{aligned} \tag{4.1}$$

Equation (4.1) will be called the *G-equation*. The following is immediate from Lemma 3.1 and the G-equation (4.1).

THEOREM 4.1. *Let M be an l.c.K. submanifold of an l.c.K. manifold $\widetilde{M}$. Then, for all $X \in \mathfrak{X}M$,*

$$H(X,JX,X,JX) = \widetilde{H}(X,JX,X,JX) - 2\widetilde{g}(\sigma(X,X),\sigma(X,X)).$$

We see from Theorem 4.1 that the Hermitian holomorphic sectional curvature of M does not exceed that of the ambient space $\widetilde{M}$. In particular, we have

THEOREM 4.2. *Let $\widetilde{M}$ be an l.c.K. manifold with non-positive Hermitian holomorphic sectional curvature. Then every l.c.K. submanifold M of $\widetilde{M}$ also has non-positive Hermitian holomorphic sectional curvature.*

Between the Hermitian pseudo-curvature tensors P and $\widetilde{P}$, there is a relation similar to the G-equation (4.1):

$$P(X,Y,Z,W) = \widetilde{P}(X,Y,Z,W) + \widetilde{g}(\sigma(X,Z),\sigma(Y,W)) - \widetilde{g}(\sigma(X,W),\sigma(Y,Z)). \tag{4.2}$$

Equation (4.2) will be called the *PG-equation*.

REMARK 4.1. In our l.c.K. case, there also exist equations corresponding to the Codazzi or Ricci equation. But we do not deal with those equations in this paper.

5. Proof of Theorem 1.1. Let M be an m-dimensional l.c.K. submanifold of an $(m+r)$-dimensional l.c.K. manifold $\widetilde{M}$. From (4.2) and Lemma 3.1, we have

$$P(X,Y,X,Y) = \widetilde{P}(X,Y,X,Y) + \widetilde{g}(\sigma(X,X),\sigma(Y,Y)) - \|\sigma(X,Y)\|^2, \tag{5.1}$$

$$P(X,JY,X,JY) = \widetilde{P}(X,JY,X,JY) - \widetilde{g}(\sigma(X,X),\sigma(Y,Y)) - \|\sigma(X,Y)\|^2 \tag{5.2}$$

for all $X,Y \in T_xM$. Assume that both M and $\widetilde{M}$ are pseudo-Bochner-flat. Then by (2.2) we easily get

$$P(X,Y,X,Y) = P(X,JY,X,JY), \tag{5.3}$$

$$\widetilde{P}(X,Y,X,Y) = \widetilde{P}(X,JY,X,JY) \tag{5.4}$$

for any orthonormal vectors $X,Y \in T_xM$ with $g(X,JY) = 0$. Using (5.1)–(5.4), we obtain

$$\widetilde{g}(\sigma(X,X),\sigma(Y,Y)) = 0 \tag{5.5}$$

for orthonormal vectors $X,Y \in T_xM$ with $g(X,JY) = 0$. Replacing X and Y in (5.5) by $\frac{1}{\sqrt{2}}(X+Y)$ and $\frac{1}{\sqrt{2}}(X-Y)$ respectively, we get

$$4\|\sigma(X,Y)\|^2 = \|\sigma(X,X)\|^2 + \|\sigma(Y,Y)\|^2. \tag{5.6}$$

Since σ is symmetric and bilinear, (5.5) also means

$$\widetilde{g}(\sigma(X,X),\sigma(Y,Z)) = 0, \tag{5.7}$$

$$\widetilde{g}(\sigma(X,Y),\sigma(Z,W)) = 0 \tag{5.8}$$

for orthonormal vectors $X,Y,Z,W,JX,JY,JZ,JW \in T_xM$.

Now we define the function $f_1 : S_x(1) \to \mathbb{R}$ on the unit sphere $S_x(1)$ of T_xM by

$$f_1(X) = \|\sigma(X,X)\|^2$$

and choose an orthonormal frame $\{E_1,\dots,E_m,JE_1,\dots,JE_m\}$ such that E_i is the point at which f_1 restricted to $S_x(1) \cap \{E_1,\dots,E_{i-1},JE_1,$

$\dots,\ JE_{i-1}\}^{\perp}$ attains its maximum, provided that E_0 is the zero-vector. Thus, by the choice of the frame, the function $F(\theta) = f_1(\cos\theta\, E_i + \sin\theta\, E_j)$ for $i < j$ satisfies $F'(0) = 0$, so that

$$\widetilde{g}(\sigma(E_i, E_i), \sigma(E_i, E_j)) = 0 \quad \text{for } i < j. \tag{5.9}$$

For orthonormal vectors $\{E_i, E_j, X, JE_i, JE_j, JX\}$ $(i < j)$ we consider the function $f_2 : \mathbb{R} \to \mathbb{R}$ defined by

$$f_2(\theta) = 4\|\sigma(X, \cos\theta\, E_i + \sin\theta\, E_j)\|^2.$$

Then by (5.6) we get

$$f_2(\theta) = \|\sigma(X, X)\|^2 + F(\theta).$$

Since $f_2'(0) = F'(0) = 0$, we obtain

$$\widetilde{g}(\sigma(X, E_i), \sigma(X, E_j)) = 0. \tag{5.10}$$

On the other hand, since the Hermitian pseudo-curvature tensor P has the Riemannian curvature symmetries, for orthonormal vectors $\{E_i, E_j\}$ $(i < j)$, we have

$$\begin{aligned} P(\cos\theta\, E_i + \sin\theta\, E_j, \sin\theta\, E_i - \cos\theta\, E_j, \cos\theta\, E_i + \sin\theta\, E_j, \sin\theta\, E_i - \cos\theta\, E_j) \\ = \cos^4\theta\, P(E_i, E_j, E_i, E_j) - \cos^2\theta \sin^2\theta\, P(E_i, E_j, E_j, E_i) \\ - \sin^2\theta \cos^2\theta\, P(E_j, E_i, E_i, E_j) + \sin^4\theta\, P(E_j, E_i, E_j, E_i) \\ = (\cos^2\theta + \sin^2\theta)^2 P(E_i, E_j, E_i, E_j) = P(E_i, E_j, E_i, E_j). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \widetilde{P}(\cos\theta\, E_i + \sin\theta\, E_j, \sin\theta\, E_i - \cos\theta\, E_j, \cos\theta\, E_i + \sin\theta\, E_j, \sin\theta\, E_i - \cos\theta\, E_j) \\ = \widetilde{P}(E_i, E_j, E_i, E_j). \end{aligned}$$

Thus, by (5.1) and (5.5), we get

$$\|\sigma(\cos\theta\, E_i + \sin\theta\, E_j, \sin\theta\, E_i - \cos\theta\, E_j)\|^2 = \|\sigma(E_i, E_j)\|^2.$$

We define the function $f_3 : \mathbb{R} \to \mathbb{R}$ by the left hand side of the above equation. Then f_3 is a constant function, so that $f_3'(0) = 0$. Therefore

$$\widetilde{g}(\sigma(E_i, E_j), \sigma(E_j, E_j)) = 0. \tag{5.11}$$

Hence, by (5.5) and (5.7)–(5.11), we see that the $m(m+1)$ normal vectors

$$\begin{array}{l} \sigma(E_1, E_1), \sigma(E_1, JE_1), \sigma(E_1, E_2), \sigma(E_1, JE_2), \dots, \sigma(E_1, E_m), \sigma(E_1, JE_m), \\ \qquad\qquad \sigma(E_2, E_2), \sigma(E_2, JE_2), \dots, \sigma(E_2, E_m), \sigma(E_2, JE_m), \\ \qquad\qquad\qquad\qquad \ddots \qquad \vdots \qquad \vdots \\ \qquad\qquad\qquad\qquad\qquad \sigma(E_m, E_m), \sigma(E_m, JE_m) \end{array}$$

are mutually orthogonal.

Assume that the complex dimension of $\{X \in T_xM : \sigma(X,X) = 0\}$ is equal to ℓ. Then, by the choice of the frame, we see that this space is spanned by $\{E_{m-\ell+1}, \dots, E_m, JE_{m-\ell+1}, \dots, JE_m\}$, that is, $\sigma(E_i, E_i) \neq 0$ for $i \leq m-\ell$ and $\sigma(E_i, E_i) = 0$ for $i > m-\ell$. Moreover, by (5.6), we also see that $\sigma(E_i, E_j) \neq 0$ for $i \leq m-\ell$ and any j. Hence we obtain $(m-\ell)(m+\ell+1)$ linearly independent normal vectors, i.e., $r \geq \frac{1}{2}(m-\ell)(m+\ell+1)$. This means that if $r < m$, then $\ell > m-1$. Thus, since $\ell \leq m$, we conclude that $\ell = m$, i.e., $\sigma = 0$ at $x \in M$. This completes the proof of the theorem. ∎

REMARK 5.1. Let $\iota : \mathbb{C}^m \to \mathbb{C}^{m+r}$ be the natural injection, i.e., $(z^1, \dots, z^m) \mapsto (z^1, \dots, z^m, 0, \dots, 0)$. It induces a holomorphic imbedding $\psi : H_\alpha^m \to H_\alpha^{m+r}$. Moreover the metric on H_α^m induced by the Boothby metric of H^{m+r} coincides with the Boothby metric of H_α^m. By Theorem 1.1, H_α^m is a totally geodesic submanifold (in the Hermitian sense) of H_α^{m+r} if $r < m$. On the other hand, it is known (cf. [1]) that H_α^m is a totally umbilical submanifold (in the usual Riemannian sense) of H_α^{m+r}. Indeed, on an l.c.K. submanifold M of $\widetilde{M}$, by (2.1) we have

$$\widetilde{D}_X Y = \widetilde{\nabla}_X Y - \tfrac{1}{2}\omega(X)Y - \tfrac{1}{2}\omega(JX)JY + \tfrac{1}{2}g(X,Y)\widetilde{B},$$
$$D_X Y = \nabla_X Y - \tfrac{1}{2}\omega(X)Y - \tfrac{1}{2}\omega(JX)JY + \tfrac{1}{2}g(X,Y)B$$

for all $X, Y \in \mathfrak{X}M$, where $\widetilde{B} = \widetilde{\omega}^\#$ and $B = \omega^\#$ are the Lee vector fields of $\widetilde{M}$ and M, respectively. From these equations, we get

$$\sigma(X,Y) = h(X,Y) + \tfrac{1}{2}g(X,Y)\widetilde{B}^\perp,$$

where h denotes the (Riemannian) second fundamental form and $\widetilde{B}^\perp$ the normal component of $\widetilde{B}$, i.e., $\widetilde{B}^\perp = \widetilde{B} - B$. Therefore $\sigma = 0$ means that for any normal vector field ξ on M, we have

$$g(A_\xi^\nabla(X), Y) = \widetilde{g}(h(X,Y), \xi) = -\tfrac{1}{2}\widetilde{\omega}(\xi)g(X,Y),$$

that is, M is totally umbilical in $\widetilde{M}$.

REFERENCES

[1] S. Dragomir and L. Ornea, *Locally Conformal Kähler Geometry*, Progr. Math. 155, Birkhäuser, 1998.

[2] T. Iwatani, *Kähler submanifolds with vanishing Bochner curvature tensor*, Mem. Fac. Sci. Kyushu Univ. Ser. A 30 (1976), 319–321.

[3] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry, Vol. II*, Interscience, New York, 1969.

[4] M. Kon, *Kaehler immersions with vanishing Bochner curvature tensors*, Kōdai Math. Sem. Rep. 27 (1976), 329–333.

[5] K. Matsuo, *Pseudo-Bochner curvature tensor on Hermitian manifolds*, Colloq. Math. 80 (1999), 201–209.

[6] B. O'Neill, *Isotropic and Kähler immersions*, Canad. J. Math. 17 (1965), 907–917.

[7] I. Vaisman, *Generalized Hopf manifolds*, Geom. Dedicata 13 (1982), 231–255.
[8] S. Yamaguchi and S. Sato, *On complex hypersurfaces with vanishing Bochner curvature tensor in Kählerian manifolds*, Tensor (N.S.) 22 (1971), 77–81.
[9] K. Yano, *Manifolds and submanifolds with vanishing Weyl or Bochner curvature tensor*, in: Proc. Sympos. Pure Math. 27, Amer. Math. Soc., 1975, 253–262.

Department of Mathematics
Ichinoseki National College of Technology
Ichinoseki 021-8511, Japan
E-mail: matsuo@ichinoseki.ac.jp

Received 28 September 2005;
revised 20 November 2006 (4672)