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THE COMPONENT QUIVER OF A SELF-INJECTIVE ARTIN ALGEBRA

BY

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Abstract. We prove that the component quiver Σ_A of a connected self-injective artin algebra A of infinite representation type is fully cyclic, that is, every finite set of components of the Auslander–Reiten quiver Γ_A of A lies on a common oriented cycle in Σ_A.

Throughout this note, by an *algebra* is meant a connected associative artin algebra with an identity over a fixed commutative artinian ring R. For an algebra A, we denote by $\operatorname{mod} A$ the category of finitely generated right A-modules and by rad_A the Jacobson radical of $\operatorname{mod} A$, generated by all non-invertible morphisms between indecomposable modules in $\operatorname{mod} A$. Then the infinite Jacobson radical $\operatorname{rad}_A^\infty$ of $\operatorname{mod} A$ is the intersection of all powers rad_A^i, $i \geq 1$, of rad_A. By a result of M. Auslander [2], $\operatorname{rad}_A^\infty = 0$ if and only if A is of finite representation type, that is, there are in $\operatorname{mod} A$ only finitely many indecomposable modules up to isomorphism. Recall also that an algebra A is called *self-injective* if A_A is an injective module, or equivalently, in $\operatorname{mod} A$ projective modules coincide with injective modules.

An important combinatorial and homological invariant of the module category $\operatorname{mod} A$ of an algebra A is its Auslander–Reiten quiver Γ_A whose vertices are the isoclasses of indecomposable modules in $\operatorname{mod} A$ and the arrows correspond to irreducible morphisms between indecomposable modules [4]. In fact, the Auslander–Reiten quiver Γ_A describes the structure of the quotient category $\operatorname{mod} A/\operatorname{rad}_A^\infty$ (see [3]). In general, it is important to study the behaviour of the connected components of Γ_A in the category $\operatorname{mod} A$. Following [18] a component $\mathcal{C}$ of Γ_A is called *generalized standard* if $\operatorname{rad}_A^\infty(X, Y) = 0$ for all modules X and Y in $\mathcal{C}$. Further, the *component quiver* Σ_A of an algebra A is defined in [19] as follows: the vertices of Σ_A are the connected components of Γ_A, and two connected components $\mathcal{C}$ and $\mathcal{D}$ of Γ_A are linked in Σ_A by an arrow $\mathcal{C} \to \mathcal{D}$ if and only if $\operatorname{rad}_A^\infty(X, Y) \neq 0$ for some modules $X \in \mathcal{C}$ and $Y \in \mathcal{D}$. Observe that a connected component

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$\mathcal{C}$ of Γ_A is generalized standard if and only if Σ_A has no loop at $\mathcal{C}$. Moreover, for different connected components $\mathcal{C}, \mathcal{D}$ in Γ_A and $X \in \mathcal{C}$, $Y \in \mathcal{D}$, we have $\operatorname{Hom}_A(X,Y) = \operatorname{rad}_A^\infty(X,Y)$.

A prominent role in the study of module categories is played by paths and cycles of indecomposable modules (see [19]). Recall that a *path* in the module category $\operatorname{mod} A$ of an algebra A is a sequence

$$(*) \qquad X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \to X_{t-1} \xrightarrow{f_t} X_t$$

of non-zero non-isomorphisms between indecomposable modules in $\operatorname{mod} A$, and if $X_0 = X_t$ then $(*)$ is called a *cycle* in $\operatorname{mod} A$. A cycle $(*)$ for which the homomorphisms $f_1, \ldots, f_t$ do not belong to $\operatorname{rad}_A^\infty$ is said to be *finite*. Finally, $\operatorname{mod} A$ is said to be *cycle-finite* if all cycles in $\operatorname{mod} A$ are finite. We note that the module category $\operatorname{mod} A$ of an algebra A of finite representation type is cycle-finite, since then $\operatorname{rad}_A^\infty = 0$.

The structure of the component quiver Σ_A of an algebra A as well as properties of cycles in $\operatorname{mod} A$ carry much information on A and $\operatorname{mod} A$. For example, the tameness of important classes of algebras of small homological dimension (tilted algebras [9], double tilted algebras [14], generalized double tilted algebras [15], quasitilted algebras of canonical type [10], [21], generalized muliticoil algebras [12]) is equivalent to the absence of oriented cycles in their component quivers, or equivalently the absence of infinite cycles in their module categories. Similarly, it has been shown in [20] that a strongly simply connected algebra A over an algebraically closed field is of polynomial growth if and only if the component quiver Σ_A has no oriented cycles, and if and only if $\operatorname{mod} A$ is cycle-finite.

In this note we are concerned with the structure of the module category $\operatorname{mod} A$ and of the component quiver Σ_A of a self-injective algebra A.

The aim of this note is to prove the following theorem on oriented cycles in $\operatorname{mod} A$ and derive some consequences.

THEOREM 1. *Let A be a non-simple connected self-injective algebra and $M_1, \ldots, M_r$ a family of indecomposable modules in* $\operatorname{mod} A$. *Then there is a cycle in* $\operatorname{mod} A$ *passing through all modules $M_1, \ldots, M_r$.*

Proof. Since A is a self-injective algebra, we have the self-equivalence

$$\mathcal{N}_A = \operatorname{D}\operatorname{Hom}_A(-, A_A) : \operatorname{mod} A \to \operatorname{mod} A,$$

called the *Nakayama functor*, where $\operatorname{D} = \operatorname{Hom}_R(-, E)$ with E being a minimal injective cogenerator in $\operatorname{mod} R$ is the standard duality on $\operatorname{mod} A$. Moreover,

$$\mathcal{N}_A^{-1} = \operatorname{Hom}_{A^{\mathrm{op}}}(-, {}_AA)\operatorname{D} : \operatorname{mod} A \to \operatorname{mod} A$$

is the inverse functor of $\mathcal{N}_A$. Further, the Nakayama functor $\mathcal{N}_A$ induces a self-equivalence functor

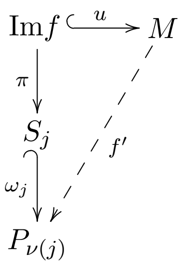
$$\mathcal{N}_A : \operatorname{proj} A \to \operatorname{proj} A$$

for the full subcategory $\operatorname{proj} A$ of $\operatorname{mod} A$ formed by the projective modules (equivalently, injective modules). Moreover, for an indecomposable projective module P in $\operatorname{mod} A$, $\mathcal{N}_A(P)$ is an indecomposable projective module in $\operatorname{mod} A$ such that the simple top, $\operatorname{top}(P) = P/\operatorname{rad} P$, of P is isomorphic to the simple socle, $\operatorname{soc}(\mathcal{N}_A(P))$, of $\mathcal{N}_A(P)$.

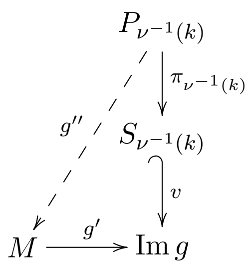
Let $P_1, \dots, P_n$ be a complete set of pairwise non-isomorphic indecomposable projective (equivalently, injective) modules in $\operatorname{mod} A$. Then $S_1 = \operatorname{top}(P_1), \dots, S_n = \operatorname{top}(P_n)$ is a complete set of pairwise non-isomorphic simple modules in $\operatorname{mod} A$ and there is a permutation ν of $\{1, \dots, n\}$, called the *Nakayama permutation*, such that $P_{\nu(i)} \cong \mathcal{N}_A(P_i)$ for any $i \in \{1, \dots, n\}$. Clearly, ν has finite order.

For each $i \in \{1, \dots, n\}$, we have in $\operatorname{mod} A$ the canonical path $P_i \to S_i \to P_{\nu(i)}$, and hence a cycle formed by the modules $P_{\nu^r(i)}$ and $S_{\nu^r(i)}$, $r \in \{1, \dots, m_i\}$, where m_i is the minimal positive integer such that $\nu^{m_i}(i) = i$ (equivalently, the length of the ν-orbit of i in $\{1, \dots, n\}$).

Let M be an indecomposable module in $\operatorname{mod} A$. Assume $\operatorname{Hom}_A(P_j, M) \neq 0$ for some $j \in \{1, \dots, n\}$, and let $f : P_j \to M$ be a non-zero homomorphism in $\operatorname{mod} A$. Then there is a commutative diagram



in $\operatorname{mod} A$ with u, ω_j the canonical monomorphisms and π the canonical epimorphism $\operatorname{Im} f \to \operatorname{top}(\operatorname{Im} f) = S_j$, due to the injectivity of $P_{\nu(j)}$ in $\operatorname{mod} A$. Hence $\operatorname{Hom}_A(M, P_{\nu(j)}) \neq 0$, since $f' \neq 0$. Obviously, if $M \not\cong P_j$ and $M \not\cong P_{\nu(j)}$, then f and f' are non-isomorphisms. We conclude that in all cases there is in $\operatorname{mod} A$ a cycle passing through M and the modules $P_{\nu^s(j)}$, $s \in \{1, \dots, m_j\}$. Similarly, if $\operatorname{Hom}_A(M, P_k) \neq 0$ for some $k \in \{1, \dots, n\}$, we take a non-zero homomorphism $g : M \to P_k$ in $\operatorname{mod} A$. Then there is a commutative diagram



in $\operatorname{mod} A$ with v the canonical monomorphism from the simple socle $S_{\nu^{-1}(k)}$ of P_k to the non-zero submodule $\operatorname{Im} g$ of P_k, $\pi_{\nu^{-1}(k)} : P_{\nu^{-1}(k)} \to S_{\nu^{-1}(k)}$ the canonical epimorphism, and g' the epimorphism induced by g, due to the projectivity of $P_{\nu^{-1}(k)}$ in $\operatorname{mod} A$. Hence $\operatorname{Hom}_A(P_{\nu^{-1}(k)}, M) \neq 0$, because $g'' \neq 0$. Thus we conclude that there is in $\operatorname{mod} A$ a cycle passing through M and the modules $P_{\nu^t(k)}$, $t \in \{1, \ldots, m_k\}$.

Since A is a connected algebra, we conclude that, for any $l \in \{1, \ldots, n\}$, there is a sequence of indices $j_1 = 1, \ldots, j_{q+1} = l$ in $\{1, \ldots, n\}$ such that

$$\operatorname{Hom}_A(P_{j_i}, P_{j_{i+1}}) \neq 0 \quad \text{or} \quad \operatorname{Hom}_A(P_{j_{i+1}}, P_{j_i}) \neq 0$$

for any $i \in \{1, \ldots, q\}$. Then it follows from the above discussion (by induction on l) that there is in $\operatorname{mod} A$ a cycle passing through P_l and the modules $P_{\nu^p(1)}$, $p \in \{1, \ldots, m_1\}$.

Summing up, we have proved that there is a cycle in $\operatorname{mod} A$ passing through all the projective modules $P_1, \ldots, P_n$. Then for an arbitrary indecomposable module M in $\operatorname{mod} A$ there is a cycle passing through M and the modules $P_1, \ldots, P_n$, since $\operatorname{Hom}_A(P_j, M) \neq 0$ for some $j \in \{1, \ldots, n\}$. Clearly, then, for any family $M_1, \ldots, M_r$ of indecomposable modules in $\operatorname{mod} A$, there is a cycle in $\operatorname{mod} A$ passing through $M_1, \ldots, M_r$ and $P_1, \ldots, P_n$. ■

COROLLARY 2. *Let A be a self-injective algebra. Then A is of finite representation type if and only if* $\operatorname{mod} A$ *is cycle-finite.*

Proof. We know that if A is of finite representation type then $\operatorname{rad}_A^{\infty} = 0$, and hence $\operatorname{mod} A$ is cycle-finite. Conversely, assume that $\operatorname{mod} A$ is cycle-finite and $\operatorname{rad}_A^{\infty} \neq 0$. Then there are indecomposable modules X and Y in $\operatorname{mod} A$ such that $\operatorname{rad}_A^{\infty}(X, Y) \neq 0$. It follows from Theorem 1 that there is in $\operatorname{mod} A$ a cycle containing X and Y. But then there is in $\operatorname{mod} A$ an infinite cycle

$$X \xrightarrow{f} Y \xrightarrow{f_1} Z_1 \xrightarrow{f_2} \cdots \xrightarrow{f_r} Z_r = X$$

with $0 \neq f \in \operatorname{rad}_A^{\infty}(X, Y)$, which contradicts the cycle-finiteness of $\operatorname{mod} A$. Therefore, $\operatorname{mod} A$ cycle-finite forces $\operatorname{rad}_A^{\infty} = 0$, and hence finite representation type of A, by the result of Auslander [2]. ■

THEOREM 3. *Let A be a connected self-injective algebra of infinite representation type and $\mathcal{C}_1, \ldots, \mathcal{C}_r$, $r \geq 1$, a family of connected components of Γ_A. Then there is an oriented cycle in the component quiver Σ_A passing through all components $\mathcal{C}_1, \ldots, \mathcal{C}_r$.*

Proof. We may assume that the components $\mathcal{C}_1, \ldots, \mathcal{C}_r$ are pairwise different. Assume first that $r \geq 2$. For each $i \in \{1, \ldots, r\}$, choose an indecomposable module M_i in $\mathcal{C}_i$. Then $M_1, \ldots, M_r$ is a family of pairwise non-isomorphic indecomposable modules, since the components $\mathcal{C}_1, \ldots, \mathcal{C}_r$

are pairwise different. Applying Theorem 1, we conclude that there is in $\operatorname{mod} A$ a cycle

$$X_1 \to X_2 \to \cdots \to X_t \to X_{t+1} = X_1$$

with $M_1 = X_{j_1}, \ldots, M_r = X_{j_r}$ for some $j_1, \ldots, j_r$ in $\{1, \ldots, t\}$. Taking now the connected components of Γ_A containing the modules $X_1, \ldots, X_t$ we conclude that there is an oriented cycle in Σ_A passing through all these components and hence through $\mathcal{C}_1, \ldots, \mathcal{C}_r$.

Assume now that $r = 1$. Since A is of infinite representation type, we have $\operatorname{rad}_A^\infty(X, Y) \neq 0$ for some indecomposable modules X and Y in $\operatorname{mod} A$. Then, by Theorem 1, for an arbitrary module M in $\mathcal{C} = \mathcal{C}_1$, we have a cycle in $\operatorname{mod} A$ of the form

$$M \to \cdots \to X \xrightarrow{f} Y \to \cdots \to M$$

for some $0 \neq f \in \operatorname{rad}_A^\infty(X, Y)$. Hence there is an oriented cycle in Σ_A passing through $\mathcal{C}$ and through the connected components of Γ_A containing the modules X and Y. ∎

A component $\mathcal{C}$ of an Auslander–Reiten quiver Γ_A is said to be a *sink* (respectively, *source*) of Σ_A if $\mathcal{C}$ is not a source (respectively, sink) of an arrow of Σ_A.

COROLLARY 4. *Let A be a connected self-injective algebra of infinite representation type. Then no connected component of Γ_A is a sink or a source in Σ_A.*

Proof. Let $\mathcal{C}$ be a connected component of Γ_A and assume that $\mathcal{C}$ is a sink or a source of Σ_A. It follows from Theorem 3 that $\mathcal{C}$ is a unique component of Γ_A and is generalized standard. Hence $\operatorname{rad}_A^\infty(X, Y) = 0$ for all indecomposable modules X, Y in $\operatorname{mod} A$, and so $\operatorname{rad}_A^\infty = 0$. This contradicts our assumption that A is of infinite representation type. ∎

A component $\mathcal{C}$ of an Auslander–Reiten quiver Γ_A is said to be a *weak source* (respectively, a *weak sink*) if there is no arrow $\mathcal{C}' \to \mathcal{C}$ in Σ_A with $\mathcal{C}' \neq \mathcal{C}$ (respectively, there is no arrow $\mathcal{C} \to \mathcal{C}''$ with $\mathcal{C} \neq \mathcal{C}''$). We note that in [13] a weak source (respectively, weak sink) of Σ_A is called the starting (respectively, ending) component.

COROLLARY 5. *Let A be a connected self-injective algebra and $\mathcal{C}$ a connected component of Γ_A. Assume that $\mathcal{C}$ is either a weak source or a weak sink of Σ_A. Then $\mathcal{C} = \Gamma_A$.*

Proof. Suppose, to the contrary, that $\mathcal{C} \neq \Gamma_A$. Since A is connected, we conclude that A is of infinite representation type and there is a connected component $\mathcal{D}$ of Γ_A different from $\mathcal{C}$. Then, applying Theorem 3, we deduce that there is an oriented cycle in Σ_A passing through $\mathcal{C}$ and $\mathcal{D}$, and this contradicts the assumption on $\mathcal{C}$. ∎

We mention that it is still not clear (see [11, Problem 1]) if a connected artin algebra A with Γ_A connected is necessarily of finite representation type.

From Drozd's tame and wild theorem [8] the class of finite-dimensional algebras over an algebraically closed field K may be divided into two disjoint classes. One class consists of the tame algebras for which the indecomposable modules occur, in each dimension d, in a finite number of discrete and a finite number of one-parameter families. The second class is formed by the wild algebras whose representation theory contains the representation theories of all finite-dimensional algebras over K (for more details on tame and wild algebras we refer to [17, Chapter XIX]).

COROLLARY 6. *Let A be a connected tame self-injective algebra of infinite representation type over an algebraically closed field K, and $\mathcal{C}$ be a component of Γ_A. Then $\mathcal{C}$ is neither a weak source nor a weak sink of Σ_A.*

Proof. Since A is of infinite representation type, it follows from the validity of the second Brauer–Thrall conjecture [5], [6] that there are infinitely many pairwise non-isomorphic indecomposable A-modules of a fixed dimension d. Further, since A is tame, we know by a theorem of W. Crawley-Boevey [7] that all but finitely many indecomposable A-modules of dimension d lie in stable tubes of rank one. Therefore, Γ_A admits infinitely many stable tubes of rank one. In particular, we have $\mathcal{C} \neq \Gamma_A$. Then it follows from Corollary 5 that $\mathcal{C}$ is neither a weak source nor a weak sink. ∎

For basic background on the representation theory of algebras we refer to the monographs [1], [4], [16], [17].

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