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ON PATH COALGEBRAS OF QUIVERS WITH RELATIONS

BY

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Abstract. The notion of the path coalgebra of a quiver with relations introduced in [11] and [12] is studied. In particular, developing this topic in the context of the weak* topology, we give a criterion that allows us to verify whether or not a relation subcoalgebra of a path coalgebra is the path coalgebra of a quiver with relations.

1. Introduction. A very well known result of P. Gabriel (see for instance [2], [3], [10] and the references given there) asserts that any finite-dimensional basic algebra is isomorphic to a quotient of the path algebra of its Gabriel quiver modulo an admissible ideal. The main aim of this note is to study the possibility of an analogous result for coalgebras, through the notion of the path coalgebra $C(Q,\Omega)$ of a quiver Q with relations Ω defined by Simson in [11]. For this purpose, in Section 2, we establish a general framework using the weak* topology on the dual algebra to treat the problem in an elementary context. In Section 3, a result of [7] allows us to obtain a more manageable basis of a relation coalgebra which we use in Section 4 to give a criterion for deciding whether or not a relation subcoalgebra is a path coalgebra of a quiver with relations.

One of the main motivations given in [11] and [12] for presenting a basic coalgebra C in the form $C(Q,\Omega)$ is the fact that, in this case, there is a K-linear equivalence of the category C-comod of finite-dimensional left C-comodules with the category $\mathrm{nilrep}_K^{\mathrm{lf}}(Q,\Omega)$ of nilpotent K-linear representations of finite length of the quiver with relations (Q,Ω) (see [11, p. 135] and [12, Theorem 3.14]). This reduces the study of the category C-comod to the study of linear representations of the bound quiver (Q,Ω).

In order to make the exposition self-contained we collect below some basic definitions and results. Throughout this paper we fix a field K. By a *K-coalgebra* we mean a triple (C,Δ,ϵ) consisting of a K-vector space C and two K-linear maps $\Delta : C \to C \otimes C$ and $\epsilon : C \to K$ such that

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$(\varDelta\otimes I)\varDelta=(I\otimes\varDelta)\varDelta$ and $(I\otimes\epsilon)\varDelta=(\epsilon\otimes I)\varDelta=I$. In what follows we shall refer to the coalgebra $(C,\varDelta,\epsilon)$ as C. A *subcoalgebra* of C is a vector subspace A of C such that $\varDelta(A)\subseteq A\otimes A$ (see [1] and [13] for more details).

Following Gabriel [5], by a *quiver*, Q, we mean a quadruple (Q_0,Q_1,s,e) where Q_0 is the set of vertices (points), Q_1 is the set of arrows, and for each arrow $\alpha\in Q_1$, the vertices $s(\alpha)$ and $e(\alpha)$ are the source (or start point) and the sink (or end point) of α, respectively (see [2], [3] and [10]).

If i and j are vertices, an (oriented) *path* in Q of length m from i to j is a formal composition of arrows

$$p=\alpha_m\cdots\alpha_2\alpha_1$$

where $s(\alpha_1)=i$, $e(\alpha_m)=j$ and $e(\alpha_{k-1})=s(\alpha_k)$ for $k=2,\dots,m$. To any vertex $i\in Q_0$ we attach a trivial path of length 0, say e_i, starting and ending at i such that $\alpha e_i=\alpha$ (resp. $e_j\beta=\beta$) for any arrow α (resp. β) with $s(\alpha)=i$ (resp. $e(\beta)=i$). We identify the set of vertices and the set of trivial paths. A *cycle* is a path which starts and ends at the same vertex.

Let KQ be the K-vector space generated by the set of all paths in Q. Then KQ can be endowed with the structure of a (not necessarily unitary) K-algebra with multiplication induced by concatenation of paths, that is,

$$(\alpha_m\cdots\alpha_2\alpha_1)(\beta_n\cdots\beta_2\beta_1)=\begin{cases}\alpha_m\cdots\alpha_2\alpha_1\beta_n\cdots\beta_2\beta_1 & \text{if } e(\beta_n)=s(\alpha_1),\\ 0 & \text{otherwise;}\end{cases}$$

KQ is the *path algebra* of the quiver Q. The algebra KQ can be graded by

$$KQ=KQ_0\oplus KQ_1\oplus\cdots\oplus KQ_m\oplus\cdots,$$

where Q_m is the set of all paths of length m and Q_0 is a complete set of primitive orthogonal idempotents of KQ. If Q_0 is finite then KQ is unitary and it is clear that KQ has finite dimension if and only if Q is finite and has no cycles.

An ideal $\varOmega\subseteq KQ$ is called an *ideal of relations* or a *relation ideal* if $\varOmega\subseteq KQ_2\oplus KQ_3\oplus\cdots=KQ_{\geq 2}$. An ideal $\varOmega\subseteq KQ$ is *admissible* if it is a relation ideal and there exists a positive integer, m, such that $KQ_m\oplus KQ_{m+1}\oplus\cdots=KQ_{\geq m}\subseteq\varOmega$.

By a *quiver with relations* we mean a pair $(Q,\varOmega)$, where Q is a quiver and $\varOmega$ a relation ideal of KQ. If $\varOmega$ is admissible then $(Q,\varOmega)$ is said to be a *bound quiver* (for more details see [2] and [3]).

The path algebra KQ can be viewed as a graded K-coalgebra with comultiplication induced by the decomposition of paths, that is, if $p=\alpha_m\cdots\alpha_1$ is a path from the vertex i to the vertex j, then

$$\varDelta(p)=e_j\otimes p+p\otimes e_i+\sum_{i=1}^{m-1}\alpha_m\cdots\alpha_{i+1}\otimes\alpha_i\cdots\alpha_1=\sum_{\eta\tau=p}\eta\otimes\tau$$

and for a trivial path, e_i, we have $\Delta(e_i) = e_i \otimes e_i$. The counit of KQ is defined by the formula

$$\epsilon(\alpha) = \begin{cases} 1 & \text{if } \alpha \in Q_0, \\ 0 & \text{if } \alpha \text{ is a path of length } \geq 1. \end{cases}$$

The coalgebra (KQ, Δ, ϵ) is the *path coalgebra* of the quiver Q.

For the convenience of the reader we denote by KQ the path algebra of Q and by CQ the above path coalgebra of Q.

The next theorem can be considered the origin and motivation of this work. We refer to [4] and [14] for the proof.

THEOREM 1.1. *Let C be a pointed coalgebra* (*i.e.*, *all simple subcoalgebras are one-dimensional*). *Then C is isomorphic to a subcoalgebra of the path coalgebra of its Gabriel quiver. Furthermore*, *C contains the subcoalgebra generated by all vertices and all arrows.* ∎

A subcoalgebra of a path coalgebra is said to be *admissible* if it contains the subcoalgebra generated by all vertices and all arrows, that is, $CQ_0 \oplus CQ_1$ (see [14]). A subcoalgebra C of a path coalgebra CQ is called a *relation subcoalgebra* (see [12]) if C satisfies the following two conditions:

(a) C is admissible.
(b) $C = \bigoplus_{a,b \in Q_0} C \cap CQ(a,b)$, where $CQ(a,b)$ is the subspace generated by all paths starting at a and ending at b.

2. Pairings and weak* topology. This is a technical section devoted to developing some basic facts on topologies induced by pairings of vector spaces, which will be useful in what follows. For further information see [1], [6], [8] and [9].

Let V and W be vector spaces over a field K. A *pairing* (V, W) of V and W is a bilinear map $\langle -, - \rangle : V \times W \to K$.

A pairing $\langle -, - \rangle$ is *non-degenerate* if the following properties hold:

$$\begin{cases} \text{if } \langle v, w \rangle = 0 \text{ for all } v \in V, \text{ then } w = 0, \\ \text{if } \langle v, w \rangle = 0 \text{ for all } w \in W, \text{ then } v = 0. \end{cases}$$

This means that the linear maps $\sigma : V \to W^*$ and $\tau : W \to V^*$ defined by $\sigma(v)(w) = \langle v, w \rangle$ and $\tau(w)(v) = \langle v, w \rangle$ for all $v \in V$ and $w \in W$ are injective.

Throughout this section all pairings will be non-degenerate.

A well known example of a non-degenerate pairing is the dual pairing, (V, V^*), given by the evaluation map $\langle v, f \rangle = f(v)$ for all $v \in V$, $f \in V^*$.

Given a pairing, (V, W), we can relate subspaces of V and W through the dual pairing. Let $v \in V$. The *orthogonal complement* to v is the set

$v^{\perp} = \{f \in V^* \mid f(v) = 0\}$. More generally, for any subset $S \subseteq V$, we may define the orthogonal complement to S to be the space

$$S^{\perp} = \{f \in V^* \mid f(S) = 0\}.$$

Since W can be embedded in V^* by the pairing, we may consider the orthogonal complement to S in W,

$$S^{\perp_W} = S^{\perp} \cap W = \{w \in W \mid \langle S, w\rangle = 0\}.$$

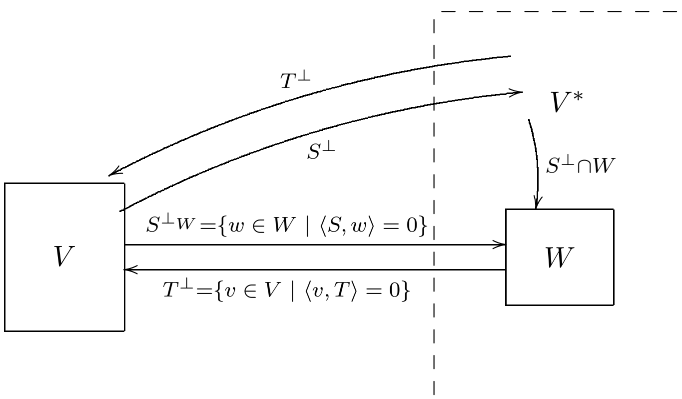
On the other hand, for any subset $T \subseteq V^*$, the *orthogonal complement to T in V* is defined by the formula

$$T^{\perp_V} = \{\ v \in V \mid f(v) = 0 \text{ for all } f \in T\},$$

and if $T \subseteq W$, then we write $T^{\perp_V} = \{v \in V \mid \langle v, w\rangle = 0 \text{ for all } w \in T\}$.

For simplicity we write $\perp$ instead of $\perp_V$.

The following diagram summarizes the above discussion:



The following lemma gives a neighbourhood subbasis and a neighbourhood basis of a topology on V^*. We call it the *weak* topology* on V^* (see [1], [8] and [9]).

LEMMA 2.1. *Let f be a linear map in V^*.*

(a) *The set $\mathcal{U}_f = \{f + v^{\perp} \mid v \in V\}$ is a neighbourhood subbasis at f for a topology on V^*.*

(b) *The sets $\mathcal{B}^f_{x_1,\dots,x_n} = \{g \in V^* \mid g(x_i) = f(x_i)\ \forall i = 1, \dots, n\} \subseteq V^*$, for any $x_1, \dots, x_n \in V$ and $n \in \mathbb{N}^*$, form a neighbourhood basis at f for the topology on V^* defined in* (a).

Proof. (a) This is straightforward.

(b) The finite intersections of elements of a neighbourhood subbasis form a neighbourhood basis and it is easy to check that

$$f + x^{\perp} = \{g \in V^* \mid g(x) = f(x)\} \qquad \text{for any } x \in V. \ \blacksquare$$

If we view W as a subspace of the vector space V^*, the induced topology on W is called the *V-topology*.

In the next proposition we collect some properties of the weak* topology which we shall need.

PROPOSITION 2.2. *Let (V, W) be a pairing of K-vector spaces.*

(a) *The weak* topology is the weakest topology on V^* which makes continuous the elements of V, that is, it is the initial topology for the elements of V.*
(b) *The closed subspaces in the weak* topology are $S^\perp$, where S is a subspace of V.*
(c) *The closure of a subspace T of V^* (in the weak* topology) is $T^{\perp\perp}$.*
(d) *The closed subspaces in the V-topology are $S^{\perp_W}$, where S is a subspace of V.*
(e) *The closure of a subspace T of W (in the V-topology) is $T^{\perp\perp_W}$.*
(f) *Let $\{A_\lambda\}_{\lambda\in\Lambda}$ be a family of subspaces of V. Then*

$$\Big(\sum_{\lambda\in\Lambda} A_\lambda\Big)^{\perp} = \bigcap_{\lambda\in\Lambda} A_\lambda^{\perp} \quad \textit{and} \quad \Big(\sum_{\lambda\in\Lambda} A_\lambda\Big)^{\perp_W} = \bigcap_{\lambda\in\Lambda} A_\lambda^{\perp_W}.$$

(g) *Any finite-dimensional subspace of W is closed.*

Proof. (a) Let $\mathcal{T}$ be the initial topology for the elements of V, and $\mathcal{W}$ the weak* topology on V^*. Let $k \in K$ and $\mathrm{ev}_y \in V$ be the evaluation at $y \in V$. Then

$$(\mathrm{ev}_y)^{-1}(k) = \{f \in V^* \mid f(y) = k\}.$$

But given $g \in (\mathrm{ev}_y)^{-1}(k)$ we obtain $g \in g + y^\perp \subseteq (\mathrm{ev}_y)^{-1}(k)$ so $(\mathrm{ev}_y)^{-1}(k)$ is an open set in the weak* topology and thus $\mathcal{T} \subseteq \mathcal{W}$. Conversely, given $f \in V^*$ and $x \in V$, a neighbourhood of f in the weak* topology is $f + x^\perp = \mathrm{ev}_x^{-1}(f(x))$, which is open in $\mathcal{T}$ and thus $\mathcal{W} \subseteq \mathcal{T}$.

(b) Let $S \subseteq V$. If $f \notin S^\perp$ then there exists $x \in S$ such that $f(x) \neq 0$. Thus $(f + x^\perp) \cap S^\perp = \emptyset$ and $f \notin \overline{S^\perp}$. Conversely, let T be a closed subspace; it suffices to prove that $T^{\perp\perp} \subseteq T$. Fix $f \in T^{\perp\perp}$ and $x \in V$; if $x \in T^\perp$ then $f(x) = 0$, hence $0 \in (f + x^\perp) \cap T$. If, on the contrary, $x \notin T^\perp$ then there exists $g \in T$ such that $g(x) \neq 0$, therefore $\frac{f(x)}{g(x)}\, g \in (f + x^\perp) \cap T$. This shows that $f \in \overline{T} = T$.

(c) $T^{\perp\perp}$ is a closed set satisfying $T \subseteq T^{\perp\perp}$, therefore $\overline{T} \subseteq T^{\perp\perp}$. We can now proceed analogously to the proof of (b) to show $T^{\perp\perp} \subseteq \overline{T}$.

(d) The V-topology on W is induced by the weak* topology on V^* so $S^{\perp_W} = S^\perp \cap W$ is closed. If T is closed, then $T = \overline{T}^W = \overline{T} \cap W = T^{\perp\perp} \cap W = T^{\perp\perp_W}$.

(e) The proof is straightforward from (d).

(f) We have

$$\begin{aligned} f \in \bigcap_{\lambda\in\Lambda} A_\lambda^{\perp} &\Leftrightarrow f(A_\lambda) = 0 \ \forall\lambda \in \Lambda, \\ &\Leftrightarrow f\Big(\sum_{\lambda\in\Lambda} A_\lambda\Big) = 0, \\ &\Leftrightarrow f \in \Big(\sum_{\lambda\in\Lambda} A_\lambda\Big)^{\perp}, \\ \bigcap_{\lambda\in\Lambda} A_\lambda^{\perp_W} &= \bigcap_{\lambda\in\Lambda} (A_\lambda^{\perp} \cap W) = \Big(\bigcap_{\lambda\in\Lambda} A_\lambda^{\perp}\Big) \cap W \\ &= \Big(\sum_{\lambda\in\Lambda} A_\lambda\Big)^{\perp} \cap W = \Big(\sum_{\lambda\in\Lambda} A_\lambda\Big)^{\perp_W}. \end{aligned}$$

(g) See [1, Chapter 2]. ∎

Finally, from the point of view of subspaces of V we have

LEMMA 2.3. *Let (V, W) be a pairing of K-vector spaces.*

(a) *Let A be a subspace of V. Then $A^{\perp\perp} = A$.*
(b) *Let A be a finite-dimensional subspace of V. Then $A^{\perp_W\perp} = A$.*
(c) *Let $\{T_i\}_{i\in I}$ be a family of subspaces of V^*. Then*

$$\Big(\sum_{i\in I} T_i\Big)^{\perp} = \bigcap_{i\in I} T_i^{\perp}.$$

Proof. (a) $f(A) = 0$ for each $f \in A^{\perp}$ and so $A \subseteq A^{\perp\perp}$. Conversely, let $v \notin A \subsetneq V$. There exists $f \in V^*$ such that $f(A) = 0$ and $f(v) \neq 0$. By Proposition 2.2, $A^{\perp}$ is closed so $A^{\perp\perp\perp} = A^{\perp}$ and therefore, $\forall g \in V^*$, $g(A) = 0 \Leftrightarrow g(A^{\perp\perp}) = 0$, which implies that $v \notin A^{\perp\perp}$.

(b) See, for instance, [1, Theorem 2.2.1].

(c) We have

$$\begin{aligned} v \in \bigcap_{i\in I} T_i^{\perp} &\Leftrightarrow f(v) = 0 \ \forall f \in T_i \ \forall i \in I, \\ &\Leftrightarrow f(v) = 0 \ \forall f \in \sum_{i\in I} T_i, \\ &\Leftrightarrow v \in \Big(\sum_{i\in I} T_i\Big)^{\perp}. \ \blacksquare \end{aligned}$$

3. Basis of a relation subcoalgebra. The aim of this section is to obtain a more manageable basis for a relation subcoalgebra of a path coalgebra. For more information and technical properties of subcoalgebras see [7].

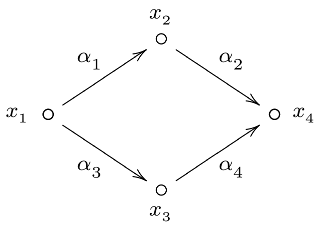
Let $Q = (Q_1, Q_2)$ be a quiver and C a subcoalgebra of CQ. Fix a path $p = \alpha_n\alpha_{n-1}\cdots\alpha_1$ in Q; a *subpath* of p is a path, q, such that either q is a vertex of p or q is a non-trivial path $\alpha_i\alpha_{i+1}\cdots\alpha_j$, where $1 \leq i \leq j \leq n$.

LEMMA 3.1. *Let $C \subseteq CQ$ be a subcoalgebra, and p be a path in C. Then all subpaths of p are in C.*

Proof. See [7]. ∎

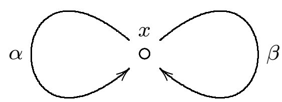
This result could lead the reader to ask if any subcoalgebra could be generated by a set of paths. Unfortunately this is not true as the next examples show.

EXAMPLE 3.2. Let Q be the quiver



The subspace generated by $\{e_{x_1}, e_{x_2}, e_{x_3}, e_{x_4}, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_2\alpha_1 + \alpha_4\alpha_3\}$ is a subcoalgebra of CQ which cannot be generated by paths.

EXAMPLE 3.3. Let Q be the quiver



The subcoalgebra $C = K\{e_x, \alpha + \beta\}$ is not generated by paths.

One may observe that, in the preceding examples, the basic elements which are not paths have the common property of being linear combinations of paths with the same source and the same sink. The next proposition asserts that, in general, every subcoalgebra of a path coalgebra has this property.

PROPOSITION 3.4. *Let Q be a quiver and $C \subseteq CQ$ a subcoalgebra. Then there exists a K-linear basis of C such that each basic element is a linear combination of paths with common source and common sink.*

Proof. See [7, Proposition 2.8]. ∎

COROLLARY 3.5. *Any admissible subcoalgebra of a path coalgebra is a relation subcoalgebra.* ∎

Proposition 3.4 is the key tool which allows us to give a more precise description of the basis of a relation subcoalgebra. Throughout, we assume that C is a relation subcoalgebra and $\mathcal{B}$ is a K-linear basis of C as in Proposition 3.4. By definition, C contains the set of all vertices, $V = \{e_i\}_{i\in Q_0}$, and the set of all arrows, $F = \{\alpha\}_{\alpha\in Q_1}$, therefore we rearrange the elements of the basis $\mathcal{B}$ as follows:

$$\mathcal{B} = V \cup F \cup \{G_{ij}^{\tau} \mid \tau \in \mathrm{T}_{ij} \text{ and } i, j \in Q_0\},$$

where, for all $\tau \in \mathrm{T}_{ij}$, the element G_{ij}^{τ} is a K-linear combination of paths with length greater than one which start at i and end at j.

We now assume that $D = \{p_\lambda\}_{\lambda\in\Lambda}$ is the set of all paths of length greater than one in C. Proceeding as before we can write

$$\mathcal{B} = V \cup F \cup D \cup \{R_{ij}^{\upsilon} \mid \upsilon \in \mathrm{U}_{ij} \text{ and } i, j \in Q_0\},$$

where, for all $\upsilon \in \mathrm{U}_{ij}$, the element R_{ij}^{υ} is a K-linear combination of at least two paths of length greater than one which start at i and end at j. Obviously, the paths involved in R_{ij}^{υ} are not in C, for any $\upsilon \in \mathrm{U}_{ij}$ and $i, j \in Q_0$.

For convenience we introduce some notation. We denote by $\mathcal{Q} = Q_0 \cup Q_1 \cup \cdots$ the set of all paths in Q. Let a be an element of KQ. Then we can write $a = \sum_{p\in\mathcal{Q}} a_p p$ for some $a_p \in K$. We define the *path support* of a to be $\mathrm{PSupp}(a) = \{p \in \mathcal{Q} \mid a_p \neq 0\}$. For any set $S \subseteq KQ$, we define $\mathrm{PSupp}(S) = \bigcup_{a\in S} \mathrm{PSupp}(a)$.

DEFINITION 3.6. Let S be a set in KQ. S is called *connected* if $\mathrm{PSupp}(S_1) \cap \mathrm{PSupp}(S_2) \neq \emptyset$ for any subsets $S_1, S_2 \subseteq S$ such that $S_1 \cup S_2 = S$ and $S_1 \cap S_2 = \emptyset$. A subset $S' \subseteq S$ is a *connected component* of S when S' is connected and $\mathrm{PSupp}(S') \cap \mathrm{PSupp}(S \setminus S') = \emptyset$.

Therefore we can break down each set $S_{ij} = \{R_{ij}^{\upsilon}\}_{\upsilon\in\mathrm{U}_{ij}}$ into its connected components and then write the basis $\mathcal{B}$ of C as

$$\mathcal{B} = V \cup F \cup D \cup \bigcup_{\phi\in\Phi} \Upsilon_\phi,$$

where, for any $\phi \in \Phi$, the set Υ_ϕ is a connected set of K-linear combinations of at least two paths such that $\mathrm{PSupp}(\Upsilon_\phi) \subset KQ_{\geq 2}$ and $\mathrm{PSupp}(\Upsilon_{\phi_1}) \cap \mathrm{PSupp}(\Upsilon_{\phi_2}) = \emptyset \Leftrightarrow \phi_1 \neq \phi_2$.

As a final reduction, it will be useful to distinguish those sets Υ_ϕ which are finite. Thus the basis $\mathcal{B}$ of C can be written as

$$\mathcal{B} = V \cup F \cup D \cup \bigcup_{\gamma\in\Gamma} \Pi_\gamma \cup \bigcup_{\beta\in B} \Sigma_\beta,$$

where Π_γ is a finite set for all $\gamma \in \Gamma$ and Σ_β is infinite for all $\beta \in B$.

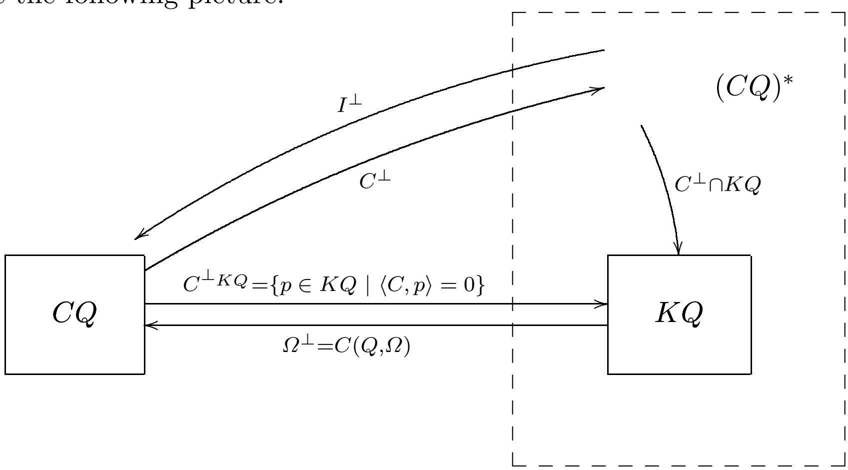
4. Path coalgebras of quivers with relations. In this section we study the notion of the path coalgebra of a quiver with relations introduced by Simson in [11] and [12]. Our main aim is to establish a criterion to decide when it is possible to write an admissible subcoalgebra as the path coalgebra of a quiver with relations.

DEFINITION 4.1. Let (Q,Ω) be a quiver with relations. The *path coalgebra* of (Q,Ω) is defined to be the subspace of CQ,

$$C(Q,\Omega)=\{a\in KQ \mid \langle a,\Omega\rangle=0\},$$

where $\langle -,-\rangle : CQ\times KQ\to K$ is the bilinear map defined by $\langle v,w\rangle=\delta_{v,w}$ (the Kronecker delta) for any two paths $v,w\in\mathcal{Q}$.

This notion may be reformulated in the notation of Section 2. It is clear that $\langle -,-\rangle$ is a non-degenerate pairing between CQ and KQ, therefore we have the following picture:



First we prove the following result.

LEMMA 4.2. *If Q is any quiver, then the injective morphism $KQ\hookrightarrow (CQ)^*$ defined by the pairing $\langle -,-\rangle$ of* 4.1 *is a morphism of algebras.*

Proof. Recall that in the dual algebra $(CQ)^* := \mathrm{Hom}_K(CQ,K)$ the (convolution) product is defined by

$$(f*g)(p)=\sum_{p=p_2p_1} f(p_2)g(p_1)\quad \text{for any } f,g\in(CQ)^* \text{ and any } p\in\mathcal{Q}.$$

We refer the reader to [13] for more details.

Fix $p\in\mathcal{Q}$ and let $p^*:CQ\to K$ be the linear map defined by $p^*(q)=\delta_{p,q}$ for any $q\in\mathcal{Q}$. It is enough to prove that $(pq)^*=p^**q^*$ for any two paths $p,q\in\mathcal{Q}$. To prove this, let r be a path in Q. Then

$$(p^**q^*)(r)=\sum_{r=r_2r_1}\delta_{p,r_2}\delta_{q,r_1}=\begin{cases}0 & \text{if } r\neq pq\\ 1 & \text{if } r=pq\end{cases}$$
$$=(pq)^*(r),$$

and so $(pq)^*=p^**q^*$. ∎

From now on we will make no distinction between elements of KQ and linear maps $f : CQ \to K$ with finite path support, that is, $f(p) = 0$ for almost all p in $\mathcal{Q}$. On the other hand, it is convenient to note that any element $g \in (CQ)^*$ can be written as a formal sum $g = \sum_{p\in\mathcal{Q}} a_p p$, where $a_p = g(p) \in K$.

Corollary 4.3. *Let Q be a quiver and C a relation subcoalgebra of CQ. Then $C^{\perp_{KQ}}$ is a relation ideal of KQ.*

Proof. Since $C^\perp$ is an ideal of $(CQ)^*$, $C^\perp \cap KQ = C^{\perp_{KQ}}$ is an ideal of KQ by Lemma 4.2. If $c \in KQ_0 \oplus KQ_1$, then $c \in C$ since C is a relation subcoalgebra. Therefore $\langle c, C\rangle \neq 0$, so $c \notin C^{\perp_{KQ}}$, which completes the proof. ∎

The following result, proved in [12], justifies the preceding definition of the path coalgebra of a quiver with relations.

Proposition 4.4. *Let Q be a quiver and Ω a relation ideal of KQ. Then $C(Q,\Omega) = \Omega^\perp$ is a relation subcoalgebra of CQ.* ∎

It is well known that, over an algebraically closed field, a finite-dimensional algebra, A, is isomorphic to KQ_A/Ω, where Q_A is the Gabriel quiver of A and Ω is an admissible ideal of KQ. In [11], it is suggested, as an open problem, to relate the relation subcoalgebras of a path coalgebra CQ and the relation ideals of the path algebra KQ, through the above-mentioned notion of the path coalgebra of a quiver with relations. That is, for any relation subcoalgebra $C \le CQ$, is there a relation ideal $\Omega \le KQ$ such that $C = C(Q,\Omega)$? In other words, in the notation of Section 2, for any relation subcoalgebra $C \le CQ$, is there a relation ideal Ω of KQ such that $\Omega^\perp = C$?

Note that if C has finite dimension, then, by Lemma 2.3, $(C^{\perp_{KQ}})^\perp = C$ and the result follows. This yields a reduction of the problem:

Problem 4.5. *Verify the relation $\Omega^\perp = C$ for the ideal $\Omega = C^{\perp_{KQ}}$.*

Lemma 4.6. *Let Q be a quiver and C a vector subspace of CQ. Then the following conditions are equivalent.*

(a) *There exists a subspace Ω of KQ such that $\Omega^\perp = C$.*
(b) *$C^{\perp_{KQ}}$ is dense in $C^\perp$ in the weak* topology of $(CQ)^*$.*
(c) *$(C^{\perp_{KQ}})^\perp = C$.*

Proof. (a)⇒(b). Since $C = \Omega^\perp$, it follows that $C^\perp = \Omega^{\perp\perp}$ is the closure of Ω in the weak* topology by Proposition 2.2. Thus $\Omega \subset C^\perp \cap KQ = C^{\perp_{KQ}} \subset C^\perp$ and, by Proposition 2.3, $C = C^{\perp\perp} \subset (C^{\perp_{KQ}})^\perp \subset \Omega^\perp = C$. Therefore $C = (C^{\perp_{KQ}})^\perp$ and thus $C^\perp = (C^{\perp_{KQ}})^{\perp\perp} = \overline{C^{\perp_{KQ}}}$.

(b)⇒(c). Since $C^\perp = (C^{\perp_{KQ}})^{\perp\perp}$, we have $C^{\perp\perp} = (C^{\perp_{KQ}})^{\perp\perp\perp}$ and, by Proposition 2.3, $C = (C^{\perp_{KQ}})^\perp$.

(c)⇒(a). It is trivial. ∎

We now assume that C is a relation subcoalgebra of CQ. If we consider the basis of C,

$$\mathcal{B} = V \cup F \cup D \cup \bigcup_{\gamma \in \Gamma} \Pi_\gamma \cup \bigcup_{\beta \in B} \Sigma_\beta,$$

built in Section 3, then we have

$$C = KV \oplus KF \oplus KD \oplus \Big(\bigoplus_{\gamma \in \Gamma} K\Pi_\gamma\Big) \oplus \Big(\bigoplus_{\beta \in B} K\Sigma_\beta\Big) \tag{4.7}$$

as K-vector space. Since the subsets into which we have partitioned $\mathcal{B}$ have disjoint path supports, it is easily seen that $\Omega^\perp = C$ if and only if each direct summand C_i of (4.7) is the orthogonal complement $\Omega_i^\perp$ of a subspace Ω_i and, in this case, $\Omega = \bigcap \Omega_i$.

There are two trivial cases:

Case 1. It is immediate that $KV = K(\mathcal{Q} \setminus V)^\perp$, $KF = K(\mathcal{Q} \setminus F)^\perp$ and $KD = K(\mathcal{Q} \setminus D)^\perp$.

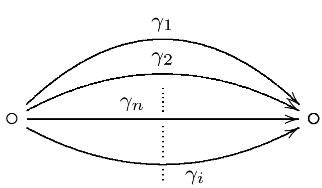
Case 2. For each $\gamma \in \Gamma$, $K\Pi_\gamma$ is a finite-dimensional subspace and so, by Lemma 2.3, $K\Pi_\gamma = ((K\Pi_\gamma)^{\perp_{KQ}})^\perp$. As a consequence we get

Corollary 4.8. *With the above notation*, $C = \Omega^\perp$ *if and only if* $\Sigma_\beta = (\Sigma_\beta)^{\perp_{KQ}\perp}$ *for each* $\beta \in B$. ∎

In particular, this implies the following proposition proved in [12].

Proposition 4.9. *Let Q be a quiver without cycles such that the set of paths in Q from i to j is finite, for all $i, j \in Q_0$. Then the map $C \mapsto C^{\perp_{KQ}}$ defines a bijection between the set of all relation subcoalgebras of CQ and the set of all relation ideals of KQ. The inverse map is defined by $\Omega \mapsto \Omega^\perp$, for any relation ideal Ω of KQ.* ∎

Therefore, we can reduce Problem 4.5 to the situation of a quiver Q with the following structure:

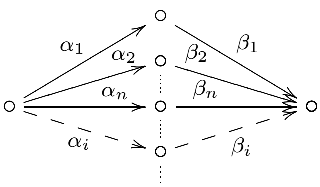
(4.10) $Q \equiv$ 

length(γ_i)> 1, $i \in I$, I infinite

and C a relation subcoalgebra generated, as vector space, by $V \cup F \cup D \cup \Sigma$, where Σ is an infinite connected set with $\mathrm{PSupp}(\Sigma) = \{\gamma_i\}_{i \in I}$. We may assume that $\gamma_i \notin C$ for all $i \in I$. Then the question is: when does the equality $\Sigma = \Sigma^{\perp_{KQ}\perp}$ hold?

Let us first show that, at least, there is an example of a relation subcoalgebra $C \subseteq CQ$ such that C is not of the form $C = \Omega^{\perp}$, where Ω is a relation ideal of KQ.

EXAMPLE 4.11. Let Q be the quiver

(4.12) 

$\gamma_i = \beta_i\alpha_i$ for all $i \in \mathbb{N}$

and let H be the relation subcoalgebra of CQ as in (4.10) with $\Sigma = \{\gamma_i - \gamma_{i+1}\}_{i\in\mathbb{N}}$. Assume that $x = \sum_{i\geq 1} a_i\gamma_i$ belongs to $H^{\perp}$ and $a_i = 0$ for $i \geq n$ we have for some $n \in \mathbb{N}$. Then $\langle \gamma_i - \gamma_{i+1}, x\rangle = a_i - a_{i+1} = 0$ for all $i \in \mathbb{N}$, so $a_i = a_{i+1}$ for all $i \in \mathbb{N}$. But $a_n = 0$ and it follows that $x = 0$. Hence $H^{\perp_{KQ}} = 0$.

By a similar argument $H^{\perp} = \langle f\rangle$, where $f(\gamma_i) = 1$ for all $i \in \mathbb{N}$. That is, $f \equiv \sum_{i\geq 1}\gamma_i$. Obviously, $H^{\perp_{KQ}}$ is not dense in $H^{\perp}$.

Here we present a positive example:

EXAMPLE 4.13. Let Q be the quiver of (4.12), and C the relation subcoalgebra generated by $\Sigma = \{\gamma_{2n-1} + \gamma_{2n} + \gamma_{2n+1}\}_{n\geq 1}$. A straightforward calculation shows that $\Omega^{\perp} = C$, where $\Omega = \langle\{\gamma_1 - \gamma_2, \{\gamma_{2n} - \gamma_{2n+1} + \gamma_{2n+2}\}_{n\geq 1}\}\rangle$.

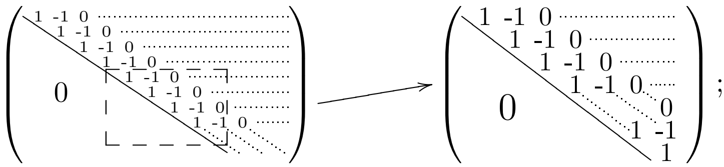
We now analyse these examples to provide a criterion which allows us to check when a relation subcoalgebra of CQ is the path coalgebra $C(Q,\Omega)$ of a quiver with relations.

First, it is convenient to see Examples 4.11 and 4.13 from a more graphic point of view. We write the elements of Σ in matrix form. Thus we have the associated infinite matrices

$$\begin{pmatrix} 1 & -1 & 0 & \cdots & & & & & \\ & 1 & -1 & 0 & \cdots & & & & \\ & & 1 & -1 & 0 & \cdots & & & \\ & & & 1 & -1 & 0 & \cdots & & \\ & 0 & & & 1 & -1 & 0 & \cdots & \\ & & & & & 1 & -1 & 0 & \cdots \\ & & & & & & 1 & -1 & 0 \cdots \\ & & & & & & & 1 & -1 \; 0 \cdots \\ & & & & & & & & \ddots \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 1 & 0 & \cdots & & & & \\ & & 1 & 1 & 1 & 0 & \cdots & & \\ & & & & 1 & 1 & 1 & 0 & \cdots \\ & & & & & & 1 & 1 & 1 \; 0 \cdots \\ & & & & & & & & 1 \; 1 \; 1 \cdots \\ & & 0 & & & & & & 1 \; 1 \; 1 \cdots \\ & & & & & & & & 1 \cdots \\ & & & & & & & & \ddots \end{pmatrix}$$

Example 4.11 Example 4.13

We can observe that Example 4.11 has an infinite diagonal of non-zero elements. Let $h \in H^{\perp_{KQ}}$. Then h must have finite path support, and so, if we want to know h, we only have to solve a finite linear system of equations with associated matrix



but zero is the unique solution.

In this way we obtain a class of relation subcoalgebras which are not path coalgebras of quivers with relations:

Definition 4.14. Let Q be a quiver as in (4.10), and C be a relation subcoalgebra generated by a connected set Σ with $\mathrm{PSupp}(\Sigma) = \{\gamma_i\}_{i\in I}$ and $\gamma_i \notin C$ for all $i \in I$. We say that C has the *infinite diagonal property* (IDP for short) if there exists a subset $\Sigma' \subseteq \Sigma$ with $\mathrm{PSupp}(\Sigma') = \{\gamma_n\}_{n\in\mathbb{N}}$ such that by means of elementary transformations, Σ' can be reduced to $\{\gamma_n + \sum_{j>n} a_j^n \gamma_j\}_{n\in\mathbb{N}}$, where $a_j^n \in K$ for all $j, n \in \mathbb{N}$.

Proposition 4.15. *Let Q be a quiver as in* (4.10) *and C be a relation subcoalgebra generated by a connected set Σ with* $\mathrm{PSupp}(\Sigma) = \{\gamma_i\}_{i\in I}$. *Suppose that $\gamma_i \notin C$ for each $i \in I$. If C has IDP, then there is no relation ideal $\Omega \subseteq KQ$ such that $C = C(Q, \Omega)$.*

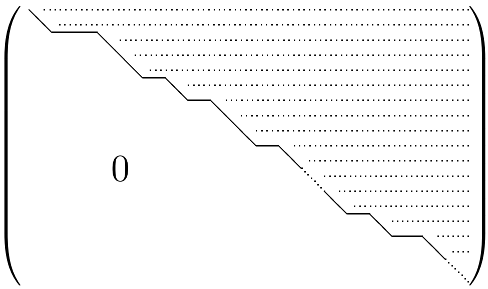
Proof. Let $\Sigma' = \{\gamma_n + \sum_{j>n} a_j^n \gamma_j\}_{n\in\mathbb{N}} \subseteq \Sigma$. Assume that the assertion is not true, i.e., there is a relation ideal $\Omega \subseteq KQ$ such that $C = C(Q, \Omega)$. By Lemma 4.6, $C^{\perp_{KQ}}$ is dense in $C^{\perp}$. Since $\gamma_1 \notin C$, there exists a linear map $g \in C^{\perp}$ such that $g(\gamma_1) \neq 0$. By the density of $C^{\perp_{KQ}}$ in $C^{\perp}$, there exists a linear map h with finite path support such that $h(\gamma_1) = g(\gamma_1)$. Defining $x_i := h(\gamma_i)$ for any $i \in \mathbb{N}$, we see that $h(\Sigma') = 0$ is the infinite system of linear equations $\{x_n + \sum_{j>n} a_j x_j = 0\}_{n\in\mathbb{N}}$. Since h has finite path support, there exists an integer m such that $x_k = 0$ for $k \geq m$. Hence $x_1, \ldots, x_m$ satisfy the finite system of linear equations

$$\begin{aligned} x_1 + a_2^1 x_2 + \cdots + a_m^1 x_m &= 0, \\ x_2 + \cdots + a_m^2 x_m &= 0, \\ &\vdots \\ x_m &= 0, \end{aligned}$$

which has the unique solution $x_m = x_{m-1} = \cdots = x_1 = h(\gamma_1) = 0$, and we get a contradiction. ∎

We claim that Example 4.13 does not have IDP. This means that for any infinite countable subset $\Sigma' \subseteq \Sigma$, the associated matrix can be reduced to a matrix of a "staircase" form

(4.16)



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that is, for any positive integer n, the first n rows have at least n variables and there is an integer $m > n$ such that the first m rows have more than m variables. We can prove that for any linear map $f \in C^{\perp}$ and any finite set $\{\gamma_1, \dots, \gamma_n\}$ of paths of Q, we obtain a linear map $g \in C^{\perp}$ such that $f(\gamma_i) = g(\gamma_i)$ for all $i = 1, \dots, n$. That is, $C^{\perp_{KQ}}$ is dense on $C^{\perp}$.

Proposition 4.17. *Under the assumptions of Proposition* 4.15, *if C fails IDP*, *then there exists a relation ideal Ω such that $C = C(Q, \Omega)$.*

Proof. It suffices to show that $\Sigma^{\perp_{KQ}}$ is dense in $\Sigma^{\perp}$, that is, given $f \in \Sigma^{\perp}$ and $\gamma_1, \dots, \gamma_n \in \mathrm{PSupp}(\Sigma)$ there exists $h \in \Sigma^{\perp}$, with finite path support, such that $h(\gamma_i) = f(\gamma_i)$ for all $i = 1, \dots, n$. We give the proof for $n = 1$; the general case is analogous and left to the reader.

We know that $h(\Sigma) = 0$ produces an infinite system of linear equations with variables $\{h(\gamma_i) = x_i\}_{i \in I}$. We rewrite the system in the following way:

Step 1. Fix an equation, say E_1, such that the coefficient of x_1 is not zero. We may assume that it is the only one with this property. Suppose that

$$E_1 \equiv x_1 + a_2^1 x_2 + \cdots + a_{r_1}^1 x_{r_1} + \cdots + a_m^1 x_m = 0,$$

where $a_2^1, \dots, a_m^1$ are non-zero and $x_1, \dots, x_{r_1 - 1}$ do not appear in any other equation of the system.

Step 2. We now take x_{r_1}. There is at least one equation, say E_2, different from E_1, in which the coefficient of x_{r_1} is not zero. We eliminate it from the remaining equations different from E_1. Choose variables $x_{r_1+1}, \dots, x_{r_2-1}$ which only appear in E_1 or E_2; then the system starts with

$$\begin{aligned} x_1 + a_2^1 x_2 + \cdots + a_{r_1}^1 x_{r_1} + \cdots + a_m^1 x_m &= 0, \\ x_{r_1} + \cdots + a_m^2 x_m + \cdots + a_l^2 x_l &= 0. \end{aligned}$$

Step 3. We do the same with x_{r_2} to obtain

$$\begin{aligned} x_1 + \cdots + a_{r_1}^1 x_{r_1} + \cdots + a_{r_2}^1 x_{r_2} + \cdots + a_m^1 x_m &= 0, \\ x_{r_1} + \cdots + a_{r_2}^2 x_{r_2} + \cdots\cdots + a_l^2 x_l &= 0, \\ x_{r_2} + a_{r_2+1}^3 x_{r_2+1} + \cdots + a_h^3 x_h &= 0. \end{aligned}$$

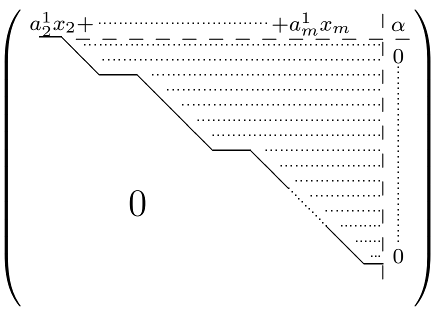
Step 4. We continue in this fashion. When we finish with the variables of E_1, we proceed with the variables of E_2 and so on. The reader should observe that the variables $x_1, \ldots, x_{r_1}, x_{r_1+1}, \ldots, x_{r_i}$ only appear in the equations $E_1, E_2, \ldots, E_{i+1}$, for all $i \in \mathbb{N}$.

There are two cases to consider:

Case 1. This process stops after a finite number of steps. Then we consider $x_\alpha = 0$, for all variables outside the finite subsystem which we have obtained. Since any equation has at least two variables, the subsystem has more variables than equations and maximal range. It follows that there is a solution for $x_1 = -f(\gamma_1)$.

Case 2. This process is infinite. Then we stop after finding a variable x_{r_k}, where r_k is the minimal integer such that $r_k > n$ and $r_{k+1} - r_k > 1$ (this is possible because C fails IDP). Roughly speaking, this means that we stop the process on the first "step" (horizontal segment in (4.16)) after processing the variables of E_1.

We consider $x_i = 0$, for all $i \neq 1, \ldots, r_k + 1$, and therefore it suffices to prove that the finite system of $k+1$ equations and r_k variables

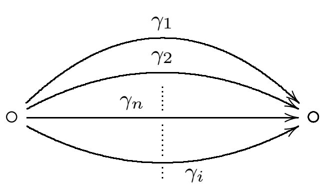


has a solution, where $\alpha = -f(\gamma_1)$. But this is clearly true, because $r_k \geq k+1$ and the matrix of coefficients has maximal range. ∎

Let Q be a quiver as in (4.10) and C be a relation subcoalgebra as in the assumptions of Proposition 4.15. Suppose that there exists a subset $\Sigma' \subseteq \Sigma$ such that $\Sigma' = \{\gamma_n + \sum_{j>n} a_j^n \gamma_j\}_{n \in \mathbb{N}}$, where $a_j^n \in K$ for all $j, n \in \mathbb{N}$, and $\gamma_i = \alpha_{n_i}^i \alpha_{n_i-1}^i \cdots \alpha_2 \alpha_1$ for all $i \in \mathbb{N}$. We may consider the subquiver $Q' = (Q'_0, Q'_1)$, where $Q'_0 = \{e(\alpha_j^i), s(\alpha_j^i)\}_{j=1,\ldots,n_i}^{i \in \mathbb{N}}$ and $Q'_1 = \{\alpha_j^i\}_{j=1,\ldots,n_i}^{i \in \mathbb{N}}$. Then C contains the relation subcoalgebra of CQ' generated by Σ'.

Therefore we turn to the case of a quiver Q with the following structure:

(4.18) $Q \equiv$



length$(\gamma_i) > 1$, $i \in \mathbb{N}$

and C a relation subcoalgebra of CQ generated by an infinite countable connected set $\Sigma = \{\gamma_n + \sum_{j>n} a_j^n \gamma_j\}_{n\in\mathbb{N}}$, where $a_j^n \in K$ for all $j, n \in \mathbb{N}$. We may suppose that $\gamma_i \notin C$ for all $i \in \mathbb{N}$.

Under these conditions, we denote by $\mathbb{H}_Q^n$ the class of relation subcoalgebras of CQ such that $\dim_K(\langle \mathrm{PSupp}(\Sigma)\rangle/\langle\Sigma\rangle) = n$ and by $\mathbb{H}_Q$ the class of relation subcoalgebras of CQ such that $\dim_K(\langle \mathrm{PSupp}(\Sigma)\rangle/\langle\Sigma\rangle) = \infty$. Finally, we set

$$\mathbb{H}_Q^\infty = \mathbb{H}_Q \cup \bigcup_{n\in\mathbb{N}} \mathbb{H}_Q^n.$$

THEOREM 4.19. *Let Q be any quiver and C be a relation subcoalgebra of CQ. There exists a relation ideal Ω of KQ such that $C = C(Q,\Omega)$ if and only if there is no subquiver Γ of Q such that C contains a subcoalgebra in $\mathbb{H}_\Gamma^\infty$.*

Proof. This follows from Propositions 4.15 and 4.17, and the arguments mentioned above. ■

REMARK. The reader could ask if a relation subcoalgebra, C, of CQ_C, which contains a subcoalgebra in $\mathbb{H}_\Gamma^\infty$ can be written as $C(Q',\Omega')$, where Q' is a quiver which is not the Gabriel quiver of C.

We know that there exists an injective map $f : C \to CQ$ such that $f|_{C_1} = \mathrm{id}$. If there is a quiver Q' and an inclusion $C \hookrightarrow CQ'$, then the following diagram commutes:

$$\begin{tikzcd} C_1 \arrow[r, leftrightarrow, "\mathrm{id}"] \arrow[dr, hook, "i"'] & CQ_1 \arrow[d, hook, dashed, "i"] \\ & CQ_1' \end{tikzcd}$$

We need the following lemma to finish our remark.

LEMMA 4.20. *Let $f : C \to D$ be a morphism of coalgebras.*

(a) *If e is a group-like element of C, then $f(e)$ is a group-like element of D.*
(b) *If f is injective and x is a non-trivial (e,d)-primitive element of C, then $f(x)$ is a non-trivial $(f(e),f(d))$-primitive element of D.* ■

Thus, since CQ_1 and CQ_1' are generated by the sets of all vertices and arrows of Q and Q', respectively, using Lemma 4.20 we conclude that Q is a subquiver of Q'; so it contains some coalgebra in $\mathbb{H}_\Gamma^\infty$.

As a consequence, we get a negative answer to the following open problem considered by Simson in [11] and [12]: *Is any basic coalgebra, over an algebraically closed field, isomorphic to the path coalgebra of a quiver with relations?*

It is easy to see that our counterexamples are of wild comodule type in the sense of [11]. Recently Simson asked a modified realization question:

Is any basic coalgebra of tame comodule type isomorphic to the path coalgebra of a quiver with relations?

Unfortunately, we are not able to answer this question at the moment.

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