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L^p-IMPROVING PROPERTIES OF MEASURES OF POSITIVE ENERGY DIMENSION

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Abstract. A measure is called L^p-improving if it acts by convolution as a bounded operator from L^p to L^q for some $q > p$. Positive measures which are L^p-improving are known to have positive Hausdorff dimension. We extend this result to complex L^p-improving measures and show that even their energy dimension is positive. Measures of positive energy dimension are seen to be the Lipschitz measures and are characterized in terms of their improving behaviour on a subset of L^p-functions.

1. Introduction. It is well known that for compact domains we have $L^q \subsetneq L^p$ for $p < q$, with the strict inclusion occurring because of the possibility of singularities in the functions. As a rule, convolution with a summable function, or even a measure, can dampen a singularity, so it is natural to expect that $f * \mu$ might belong to L^q even though f itself might not. One certainly cannot expect this to be true for any measure μ (for example it is not true for a point-mass measure) but there are a class of finite measures which have this property.

A measure is called L^p-improving if it acts by convolution as a bounded operator from L^p to L^q for some $p < q$. Young's inequality implies that any measure whose density function is in L^r, for some $r > 1$, is L^p-improving. More generally, the Hausdorff–Young inequality implies that any measure whose Fourier transform belongs to l^p for some $p < \infty$ is also such an example. In contrast, Riesz product measures ([3], [18]) and the Cantor measure ([14], [5]) are examples of singular measures on the torus whose Fourier transform does not even vanish at infinity, but are L^p-improving.

L^p-improving measures have also been studied in $\mathbb{R}^n$. For example, a classical theorem of Littman [13] implies that a compactly supported, smooth

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measure on an $(n-1)$-dimensional surface in $\mathbb{R}^n$, with appropriate curvature, maps $L^{(n+1)/n}(\mathbb{R}^n)$ to $L^{n+1}(\mathbb{R}^n)$. Motivated by this, Oberlin in [15] proved that the measure on the curve (t,t^2,t^3) maps $L^{3/2}(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$. The L^p-improving behaviour of measures on curves has been extensively studied since; we refer the reader to [4], [6], [20] and the references cited therein. In the more abstract setting of a connected Lie group, Ricci and Stein [17] showed that a smooth measure, compactly supported on a connected submanifold, is L^p-improving if and only if the submanifold generates the group.

It is an open problem, posed by Stein [19], to characterize L^p-improving measures in terms of their "size". Even on the one-dimensional torus, the focus of this paper, the only known characterization is in terms of the size of the level sets of the Fourier transform [10]. It has been speculated that although L^p-improving measures can be singular, they cannot be too "small", in some intuitive sense. For example, L^p-improving measures are always continuous, and positive L^p-improving measures have positive Hausdorff dimension [8]. Because of the possibility for cancellation a measure can be L^p-improving without the same being true for its total variation. Thus questions about the size of non-positive L^p-improving measures tend to be more subtle.

In this note we show that all complex L^p-improving measures have positive Hausdorff dimension and even their energy dimension (which can be smaller than Hausdorff dimension) must be positive. In fact, we obtain an upper bound on the "amount of improvement" in terms of the energy dimension. For measures with decreasing Fourier transform, positive energy dimension is seen to be equivalent to being L^p-improving, however this is not true in general as there are examples of measures with energy dimension one (the maximum possible) which are not L^p-improving.

L^p-improving measures are known to be examples of the so-called Lipschitz measures, measures whose distribution functions are Lipschitz. We show that positive energy dimension is equivalent to being Lipschitz and to acting by convolution as a bounded operator from a restricted class of functions in L^p to L^2 for some $p<2$.

2. L^p-improving measures and energy dimension. For the remainder of the paper by a *measure* we mean a finite, regular, complex, Borel measure on the one-dimensional torus $\mathbb{T}$.

DEFINITION 2.1. A measure μ is called *L^p-improving* if there exists some $1<p<\infty$ and $\varepsilon>0$ such that $\mu * f \in L^p$ whenever $f \in L^{p-\varepsilon}$. In this case the measure μ acts by convolution as a bounded operator from $L^{p-\varepsilon}$ to L^p.

If μ is L^p-improving, then an interpolation argument shows that for every choice of p there exists some $\varepsilon = \varepsilon(p) > 0$ such that $\mu * f \in L^p$ whenever $f \in L^{p-\varepsilon}$. In particular, there is some $p < 2$ such that $\mu * f \in L^2$ whenever $f \in L^p$.

In connection with Stein's question it is natural to ask about the size of the sets on which an L^p-improving measure can be concentrated. A partial answer was provided by Oberlin (private communication; see [8]): a positive measure mapping L^p to L^2 for some $p < 2$ has Hausdorff dimension at least $2/p - 1$, where we recall that the Hausdorff dimension of a measure μ is defined as

$$\dim_{\mathrm{H}} \mu \equiv \inf\{\dim_{\mathrm{H}} E : \mu(E) \neq 0\}.$$

As with dimensions of sets, there are other notions of dimensions of measures which give related ways to quantify the singularity of the measure.

Definition 2.2. We define the *energy dimension* of a measure μ as

$$\dim_{\mathrm{e}} \mu \equiv \sup\Big\{0 < t < 1 : \sum_{n\in\mathbb{Z}\setminus\{0\}} |n|^{t-1}|\widehat{\mu}(n)|^2 < \infty\Big\}.$$

It was shown in [12] that this definition is consistent with the classical definition of energy dimension (valid only for positive measures) given by the formula

$$\dim_{\mathrm{e}} \mu = \sup\Big\{t : \iint \frac{d\mu(x)d\mu(y)}{|x-y|^t} < \infty\Big\}.$$

In [12] the relation $\dim_{\mathrm{H}} \mu \geq \dim_{\mathrm{e}} \mu$ was established, extending the classical result for positive measures ([7, 4.3]). Although the two dimensions can be different, for many measures they coincide. This is the case, for instance, with the classical Cantor measure. On the other hand, all L^1 functions have Hausdorff dimension one, but their energy dimensions can be zero since their Fourier transforms can decay arbitrarily slowly. The function mentioned in Example 2.4 is such an example.

The Hausdorff dimension of a measure and its total variation are always the same. In contrast, $\dim_{\mathrm{e}} \mu \geq \dim_{\mathrm{e}} |\mu|$ and the inequality can occur as was shown in [12].

In this section we obtain a characterization of measures of positive energy dimension which will allow us to show that all L^p-improving measures have not only positive Hausdorff dimension, but even positive energy dimension. For the characterization it is convenient to define another property first.

Definition 2.3. A measure μ is said to belong to $\mathrm{Lip}(\alpha)$ if its distribution function F satisfies a Lipschitz condition of order α. This means there is a constant C such that for all x, h,

$$|F(x+h) - F(x)| \equiv |\mu[x, x+h]| \leq C|h|^\alpha.$$

A measure is said to be *Lipschitz* if it belongs to the class $\mathrm{Lip}(\alpha)$ for some $\alpha > 0$.

L^p-improving measures are known to be examples of Lipschitz measures [8].

THEOREM 2.1. *Let μ be a measure on $\mathbb{T}$. The following are equivalent*:

(i) *The energy dimension of μ is positive.*
(ii) *μ is Lipschitz.*
(iii) *Let D_n denote the Dirichlet kernel of degree n. There exists some $p < 2$ and constant C such that $\|\mu * D_n\|_2 \leq C\|D_n\|_p$ for all n.*

As an immediate consequence we obtain the result stated in the introduction.

COROLLARY 2.2. *Any L^p-improving measure has positive energy and Hausdorff dimensions.*

Proof of Theorem 2.1. (i)⇒(ii). Assume $I_t(\mu) < \infty$ for some $t > 0$. To conclude that $\mu \in \mathrm{Lip}(\alpha)$ it suffices to check that

$$\sum_{|j|=2^n}^{\infty} \left|\frac{\widehat{\mu}(j)}{j}\right| \leq O(2^{-n\alpha})$$

(cf. [1, p. 217]). Applying the Cauchy–Schwarz inequality gives

$$\begin{aligned}\sum_{|j|=2^n}^{\infty} \left|\frac{\widehat{\mu}(j)}{j}\right| &\leq \sum_{k=n}^{\infty} 2^{-k} \sum_{|j|=2^k}^{2^{k+1}-1} |\widehat{\mu}(j)| \leq \sum_{k=n}^{\infty} 2^{-k/2} \Big(\sum_{|j|=2^k}^{2^{k+1}-1} |\widehat{\mu}(j)|^2\Big)^{1/2} \\ &\leq c \sum_{k=n}^{\infty} 2^{-k/2} 2^{k(1-t)/2} \Big(\sum_{|j|=2^k}^{2^{k+1}-1} |j|^{t-1} |\widehat{\mu}(j)|^2\Big)^{1/2} \\ &\leq c \sum_{k=n}^{\infty} 2^{-tk/2} I_t(\mu)^{1/2} = O(2^{-nt/2}).\end{aligned}$$

(ii)⇒(iii). For this we will first prove that if $\mu \in \mathrm{Lip}(\alpha)$, then

$$\sum_{k=-n}^{n} |\widehat{\mu}(k)|^2 \leq O(n^{1-\alpha}).$$

Once this is established (iii) is essentially immediate as the left hand side of the inequality is $\|\mu * D_n\|_2^2$ and the right hand side is $O(\|D_n\|_p^2)$ for $p = 2/(1+\alpha)$.

Given a measure ω, let $\omega^-(E) = \omega(-E)$ and let $\omega^\sim(E) \equiv \overline{\omega(-E)}$. Consider $\nu_\pm = (\mu \pm \mu^-) * (\mu \pm \mu^-)^\sim$. Replacing $\nu_\pm$ if necessary by $\nu_\pm - \widehat{\nu}_\pm(0)$ we can assume $\widehat{\nu}_\pm(0) = 0$. Let $F_\pm$ denote the distribution function of $\nu_\pm$. Then $\widehat{F}_\pm(n) = \widehat{\nu}_\pm(n)/n \geq 0$ for $n > 0$ and $F_\pm$ are odd/even functions which satisfy a Lipschitz condition of order α since ν belongs to $\mathrm{Lip}(\alpha)$. By [2, 7.20] we have

$$\sum_{k=1}^{n} |\widehat{\nu}_\pm(k)| \leq O(n^{1-\alpha}).$$

The claim holds since $2(|\widehat{\mu}(k)|^2 + |\widehat{\mu}(-k)|^2) = \widehat{\nu}_+(k) + \widehat{\nu}_-(k)$.

(iii)⇒(i). This is a simple consequence of the definition of energy dimension. Since $\|D_n\|_p \sim n^{1/p'}$ we have

$$\begin{aligned}\sum_{n\neq 0} |n|^{t-1}|\widehat{\mu}(n)|^2 &= \sum_{k=0}^{\infty} \sum_{|n|=2^k}^{2^{k+1}-1} |n|^{t-1}|\widehat{\mu}(n)|^2 \\ &\leq c\sum_{k} 2^{k(t-1)}\|\mu * D_{2^{k+1}}\|_2^2 \leq c'\sum_{k} 2^{k(t-1)}2^{k2/p'},\end{aligned}$$

and the latter sum is finite provided $t < 2/p - 1$. ∎

Notice that the proof shows the following relationships.

COROLLARY 2.3. (i) *If* $\dim_{\mathrm{e}} \mu = s$ *then* $\mu \in \mathrm{Lip}(s/2)$.
(ii) *If* $\mu \in \mathrm{Lip}(\alpha)$ *then* $\|\mu * D_n\|_2 \leq C\|D_n\|_p$ *for* $p = 2/(1+\alpha)$.
(iii) *If* $\|\mu * D_n\|_2 \leq C\|D_n\|_p$ *then* $\dim_{\mathrm{e}} \mu \geq 2/p - 1$.
(iv) *If* $\mu \in \mathrm{Lip}(\alpha)$ *then* $\dim_{\mathrm{e}} \mu \geq \alpha$.

COROLLARY 2.4. *Let* μ *be a measure on* $\mathbb{T}$.

(i) *If* $\mu * f \in L^2$ *whenever* $f \in L^p$, *then* $\dim_{\mathrm{H}} \mu \geq \dim_{\mathrm{e}} \mu \geq 2/p - 1 > 0$.
(ii) *If* $\mu : L^p \to L^2$ *for every* $p > 1$, *then* $\dim_{\mathrm{H}} \mu = \dim_{\mathrm{e}} \mu = 1$.

Our first example shows that these results are sharp in some sense.

EXAMPLE 2.1. In [11] (see also the proof of Theorem 3.4) it is shown that there is an integrable function f on the torus comparable to $1/x^t$, for $0 < t < 1$, with Fourier coefficients $\widehat{f}(n) \sim n^{t-1}$. If μ is the measure with density f it is easy to check that $\dim_{\mathrm{e}} \mu = 2 - 2t$ and that $\mu \in \mathrm{Lip}(1-t)$. The Hausdorff–Young inequality shows that μ acts by convolution from L^p to L^2 for $1 - t = 1/p - 1/2$. These observations show that Corollary 2.3(i) and (iii) and Corollary 2.4(i) are sharp.

As mentioned in the introduction, $\widehat{\mu} \in l^p$ for some $p < \infty$ is a sufficient, but not necessary, condition for a measure μ to be L^p-improving. In comparison, our theorem shows that a necessary condition is for $\widehat{\mu}$ to belong to a weighted l^2 space, with weight of the form $\{n^{t-1}\}$ for some $t > 0$. But this is not a sufficient condition, as our next example illustrates. In fact, even energy dimension one is not enough.

EXAMPLE 2.2. In [8] an example is given of a positive, absolutely continuous measure on the torus which is in every Lipschitz class $\mathrm{Lip}(\alpha)$ for $\alpha < 1$, but is not L^p-improving. By Corollary 2.3(iv) this measure has energy dimension one.

EXAMPLE 2.3. The energy and Hausdorff dimensions of the standard Cantor measure μ are $\log 2/\log 3$, thus Corollary 2.3 implies that if $\mu : L^p \to L^2$ then

$$2/p \le 1 + \log 2/\log 3.$$

This coincides with the necessary condition observed by Oberlin in [16], but leaves a gap with the best known sufficient condition, $p \ge 2/(1+\sqrt{3})$.

Other corollaries follow from the theorem.

COROLLARY 2.5. (i) *If μ is L^p-improving, then* $\dim_e \mu^k \to 1$ *as* $k \to \infty$, *where μ^k denotes μ convolved with itself k times.*

(ii) *Any Borel subgroup on which an L^p-improving measure is concentrated has Hausdorff dimension one.*

Proof. (i) It is known that if μ is L^p-improving, then given any $p > 1$ there exists an integer k such that $\mu^k : L^p \to L^2$ (where μ^k denotes μ convolved with itself k times) ([10]).

(ii) If μ is concentrated on E, then μ^k is concentrated on $E^k \subseteq \mathrm{Grp}\, E$. Thus $\dim_H \mathrm{Grp}\, E \ge \dim_H E^k \ge \dim_e \mu^k$, which tends to one. ∎

REMARK 2.1. It is an open problem if such a subgroup can have Lebesgue measure zero.

COROLLARY 2.6. *Suppose $|\widehat{\mu}(n)|$ decreases monotonically as $|n| \to \infty$. Then μ is L^p-improving if and only if it has positive energy dimension.*

Proof. It was noted in [8] that a Lipschitz measure with monotonic Fourier transform satisfies $\widehat{\mu}(n) = O(n^{-\alpha})$ and thus is L^p-improving. ∎

EXAMPLE 2.4. We remark that even in the monotonic case, positive Hausdorff dimension does not suffice to characterize L^p-improving measures as there are L^1 functions (and therefore measures of Hausdorff dimension one) which are not L^p-improving and yet have monotonically decreasing, convex Fourier coefficients. One example is an integrable function with Fourier coefficients comparable to $1/\log^2 n$. Such a measure has energy dimension zero.

We have already observed that positive energy dimension does not characterize L^p-improving measures. Indeed, there can be no characterization of L^p-improving measures in terms of a weighted l^2 norm of the Fourier transform.

PROPOSITION 2.7. *Let ϕ be any positive sequence tending to zero. There is a measure μ which is not L^p-improving but satisfies*

$$\sum \phi(n)|\widehat{\mu}(n)|^2 < \infty.$$

Proof. Inductively choose positive integers $k_m > k_{m-1} + 2^{m+1}$ such that $\phi(n) < 4^{-(m+1)}$ for all $|n| \geq k_m$. Set

$$\mu = \sum_m \frac{1}{2m^2} K_{2^m}(x) e^{i(k_m+2^m)x},$$

where K_m denotes the Fejér kernel of degree m. Clearly $\mu \in L^1$, $\|\mu\| \leq 1$, $0 \leq \widehat{\mu}(n) \leq 1$ and $\widehat{\mu}(n) = 0$ if $n \notin \bigcup_m [k_m, k_m + 2^{m+1}]$. Moreover, the set $E(1/2m^2)$ contains an arithmetic progression of length 2^m, namely $\{k_m + 2^{m-1}, \ldots, k_m + 2^{m-1}3\}$.

Suppose μ is an L^p-improving measure. It is known ([10]) that there must exist constants a, b such that if A is any arithmetic progression of length N, then

$$|A \cap E(\varepsilon)| \leq a(\log N)^{-b \log \varepsilon}.$$

Since $2^m \gg m^{c \log m}$ this is impossible, therefore μ is not L^p-improving. However,

$$\sum \phi(n)|\widehat{\mu}(n)|^2 \leq \sum_m \sum_{n \in [k_m, k_m+2^{m+1}]} \phi(n) \leq \sum_m (2^{m+1}+1)4^{-(m+1)} < \infty. \qquad \blacksquare$$

3. L^p-improving on characteristic functions of intervals. As the L^p-improving property is quite strong it seems reasonable to introduce a similar, but weaker condition.

DEFINITION 3.1. We will say a measure μ is *L^p-improving on characteristic functions of intervals* (*L^p-improving on intervals*, for short) if there is some $1 < p < \infty$, $\varepsilon > 0$ and constant C such that

$$\|\mu * \chi_I\|_p \leq C\|\chi_I\|_{p-\varepsilon} \tag{1}$$

for all intervals I.

For positive measures it is straightforward to show that a measure is L^p-improving on characteristic functions of intervals if and only if it is Lipschitz, and therefore if and only if it has positive energy dimension. The main contribution of this section (Theorem 3.4) is to show that this result continues to hold for general measures.

Obviously L^p-improving measures are L^p-improving on intervals, but since we have already seen that there are measures with positive energy dimension that are not L^p-improving, the converse is not true.

As with L^p-improving measures, if the inequality (1) holds for some p, then it holds for each p, with a different choice of ε. However, the interpolation theorem does not apply and we instead give a direct proof.

PROPOSITION 3.1. *If μ is L^p-improving on characteristic functions of intervals, then for every $1 < p < \infty$ there exists an $\varepsilon = \varepsilon(p) > 0$ and a constant C_p such that*

$$\|\mu * \chi_I\|_p \le C_p \|\chi_I\|_{p-\varepsilon}.$$

Proof. Assume $\|\mu * \chi_I\|_q \le C_q \|\chi_I\|_{q-\varepsilon}$ and fix $1 < p < \infty$, $p \ne q$. Choose $t \in (0,1)$ and define r and s by

$$\frac{1}{p} = \frac{t}{q} + \frac{1-t}{r}, \qquad \frac{1}{s} = \frac{t}{q-\varepsilon} + \frac{1-t}{r}.$$

The generalized Hölder inequality gives

$$\begin{aligned}\|\mu * \chi_I\|_p &\le \|\mu * \chi_I\|_q^t \|\mu * \chi_I\|_r^{1-t} \le C \|\chi_I\|_{q-\varepsilon}^t \|\chi_I\|_r^{1-t} \|\mu\|^{1-t} \\ &\le C' |I|^{t/(q-\varepsilon)+(1-t)/r} = C' \|\chi_I\|_s.\end{aligned}$$

As $t \ne 0$, $s < p$. ∎

The following implication is almost a consequence of the definition.

PROPOSITION 3.2. *If $\mu \in \mathrm{Lip}(\alpha)$ for some $\alpha > 0$, then μ is L^p-improving on characteristic functions of intervals.*

Proof. Choose $p > 1/\alpha$. As $\mu \in \mathrm{Lip}(\alpha)$, $|\mu * \chi_I(x)| = |\mu(x - I)| \le C|I|^\alpha$. Thus if $q = 1/\alpha < p$,

$$\|\mu * \chi_I\|_p \le \sup_x |\mu(x - I)| \le C|I|^\alpha = C\|\chi_I\|_q. \ ∎$$

Our next objective is to prove that the two properties are equivalent.

PROPOSITION 3.3. *If μ is L^p-improving on characteristic functions of intervals, then there exists $p < 2$ and a constant C such that*

$$\|\mu * f\|_2 \le C \|f\|_p$$

for all monotonic, integrable functions f.

Proof. Choose $q < 2$ such that $\|\mu * \chi_I\|_2 \le C\|\chi_I\|_q$ for all intervals I. Suppose first that f is a positive, decreasing step function, say

$$f = \sum_{k=1}^{N} a_k \chi_{I_k},$$

where a_k are decreasing and I_k are disjoint, adjacent intervals. Let $b_k = a_k - a_{k+1}$ ($b_N = a_N$). Then $f = \sum_{k=1}^N b_k \chi_{U_k}$, where U_k are the intervals $U_k = \bigcup_{j=1}^k I_j$.

Let g^* denote the non-increasing rearrangement of g and consider the Lorentz norm

$$\|g\|_{q,1}^* \equiv \int (t^{1/q} g^*(t)) \, \frac{dt}{t}.$$

Notice that $f^* = \sum_{k=1}^N b_k (\chi_{U_k})^*$ (of course, here $(\chi_{U_k})^* = \chi_{U_k}$) and

$$\|f\|_{q,1}^* = \sum_{k=1}^N |b_k| \, \|\chi_{U_k}\|_{q,1}^*.$$

As μ is improving on intervals,

$$\|\mu * f\|_2 \le \sum_{k=1}^N |b_k| \, \|\mu * \chi_{U_k}\|_2 \le C \sum_{k=1}^N |b_k| \, \|\chi_{U_k}\|_q.$$

But $\|\chi_{U_k}\|_q \le \|\chi_{U_k}\|_{q,1}^*$, hence

$$\|\mu * f\|_2 \le C \sum_{k=1}^N |b_k| \, \|\chi_{U_k}\|_{q,1}^*.$$

As $\|f\|_{q,1}^* \le C_p \|f\|_p$ for any $p > q$, we have the desired result for functions of this form.

By taking limits we can extend the result to any positive, decreasing function. We can handle the general monotonic case by considering the positive and negative parts of the function. ∎

Remark 3.1. Note that if the function is decreasing symmetrically away from the origin the same result holds.

Theorem 3.4. *A measure μ is L^p-improving on characteristic functions of intervals if and only if μ has positive energy dimension.*

Proof. We have already seen that positive energy dimension implies Lipschitz and this implies L^p-improving on intervals.

So assume μ is L^p-improving on intervals. It was shown in [11] that for any $t \in (0,1)$ there are functions F_t, ϕ and E_t satisfying

$$F_t(x) = \frac{\phi(x)}{|x|^t} + E_t(x),$$

with $\widehat{F_t}(n) \sim |n|^{t-1}$, E_t bounded, and ϕ positive, bounded and bounded away from zero on $\mathbb{T}$. Moreover, ϕ can be chosen to be decreasing symmetrically away from the origin.

The previous proposition implies that

$$\left\| \mu * \frac{\phi(x)}{|x|^t} \right\|_2 \le C \left\| \frac{\phi(x)}{|x|^t} \right\|_p$$

for some $p < 2$. Because ϕ is bounded, the right hand side is finite provided t is chosen such that $tp < 1$. Since E_t is also bounded it follows that $\|\mu * F_t\|_2 < \infty$.

But

$$\|\mu * F_t\|_2^2 = \sum |\widehat{\mu}(n)|^2 |\widehat{F}_t(n)|^2 \sim \sum |n|^{2(t-1)} |\widehat{\mu}(n)|^2 + |\widehat{\mu}(0)|^2$$

and the latter is comparable to $I_s(\mu)$, where $s = 2t-1$. This implies $\dim_{\mathrm{e}} \mu \geq 2/p - 1 > 0$. ∎

Remark 3.2. The measure of Example 2.2 is L^p-improving on intervals, being of energy dimension one, but not L^p-improving.

The known characterization of L^p-improving measures is in terms of the "size" of the sets $E(\varepsilon) \equiv \{n : |\widehat{\mu}(n)| > \varepsilon\}$ (see [10]). An analogous (but simpler) characterization can be given for measures which are L^p-improving on intervals.

Proposition 3.5. *The energy dimension of μ is positive if and only if there is some $\alpha < 1$ such that for each $\varepsilon > 0$ and integer N,*

$$|E(\varepsilon) \cap [-N, N]| \leq O(N^{\alpha} \varepsilon^{-2}).$$

Proof. Without loss of generality we can assume $\|\mu\| \leq 1$. First, suppose $I_t(\mu) < \infty$. Then

$$\varepsilon^2 |E(\varepsilon) \cap [-N, N]| N^{t-1} \leq \sum_{0<|n|\leq N} |n|^{t-1} |\widehat{\mu}(n)|^2 + |\widehat{\mu}(0)|^2 \leq c I_t(\mu).$$

To prove the converse, we will first verify that $\mu * \mu$ has positive energy dimension. Consider

$$\begin{aligned}
\sum |n|^{t-1} |\widehat{\mu}(n)|^4 &= \sum_{k=0}^{\infty} \sum_{n \in E(2^{-k}) \setminus E(2^{-(k-1)})} |n|^{t-1} |\widehat{\mu}(n)|^4 \\
&\leq C \sum_k 2^{-4(k-1)} \sum_j \Big(\sum_{n \in E(2^{-k});\, 2^j \leq |n| < 2^{j+1}} |n|^{t-1} \Big) \\
&\leq C' \sum_k 2^{-4k} \sum_j 2^{j(t-1)} |E(2^{-k}) \cap [-2^{j+1}, 2^{j+1}]| \\
&\leq C'' \sum_k 2^{-4k} \sum_j 2^{j(t-1)} 2^{2k} 2^{j\alpha}.
\end{aligned}$$

This sum is finite provided $t < 1 - \alpha$, hence $\dim_{\mathrm{e}} \mu * \mu \geq 1 - \alpha$.

It is an elementary exercise to check that if $\sum |n|^{-t} |a_n|^2 < \infty$ for some $0 < t < 1$, then $\sum |n|^{-s} |a_n| < \infty$ for some other $0 < s < 1$. Thus μ has positive energy dimension if $\mu * \mu$ has positive energy dimension. ∎

The same argument given in Proposition 3.1 for characteristic functions of intervals shows that if there are some p and $\varepsilon > 0$ such that

$$\|\mu * D_n\|_p \le c_p \|D_n\|_{p-\varepsilon}$$

for all Dirichlet kernels D_n, then a similar inequality holds for all indices p and $\varepsilon = \varepsilon(p)$. Thus our results may be summarized as follows.

COROLLARY 3.6. *The following are equivalent for a measure μ:*

(i) $\dim_e \mu > 0$.
(ii) μ *is Lipschitz.*
(iii) μ *is L^p-improving on characteristic functions of intervals.*
(iv) *There is some p, $\varepsilon > 0$ and constant c such that $\|\mu * D_n\|_p \le c\|D_n\|_{p-\varepsilon}$ for all Dirichlet kernels D_n.*
(v) *There is some $\alpha < 1$ such that for each $\varepsilon > 0$ and integer N,*

$$|E(\varepsilon) \cap [-N, N]| \le O(N^\alpha \varepsilon^{-2}).$$

REMARK 3.3. Note that Example 2.4 shows that Hausdorff dimension one is not enough to ensure L^p-improving on intervals.

4. Examples

4.1. *Cantor measures.* By a *Cantor set*, we mean a compact, totally disconnected, perfect subset of $[0, 1]$, which has a construction similar to that of the standard middle-third Cantor set, however, at step k in the construction, rather than keeping closed intervals of length 3^{-k}, we keep closed intervals whose lengths are r_k times the length of the parent interval, where $r_k \in (0, 1/2)$. The numbers r_k are known as the *ratios of dissection*. The *Cantor measure* associated with a Cantor set is the probability measure uniformly distributed on the Cantor set. These measures are all singular; indeed, their Hausdorff dimension is equal to

$$\liminf_k \left(\frac{k \log 2}{|\log r_1 \cdots r_k|} \right).$$

It is known that Cantor measures are L^p-improving if their ratios of dissection are bounded away from zero [5], but this is not necessary as there are Cantor measures with ratios not bounded away from zero, whose Fourier transform decays sufficiently rapidly to be L^p-improving [9].

In contrast, for Cantor measures, L^p-improving on intervals is characterized by ratios bounded away from zero “on average”.

COROLLARY 4.1. *A Cantor measure μ is L^p-improving on characteristic functions of intervals if and only* $\dim_H \mu > 0$.

Proof. Using the method of [21, pp. 296–297] one can easily check that μ is Lipschitz if $\limsup (r_1 \cdots r_k)^{-1/k} < \infty$, equivalently, $\dim_{\mathrm{H}} \mu > 0$. ∎

With this observation we can give another example of a measure which is L^p-improving on intervals, but not L^p-improving.

EXAMPLE 4.1. Let μ be a Cantor measure with ratios of dissection r_k at step k satisfying $r_k = 1/z_k$ for some $z_k \in \mathbb{N}$, where $\inf r_k = 0$ and $\limsup (r_1 \cdots r_k)^{-1/k} < \infty$. As seen in [9], such a Cantor measure satisfies $\limsup |\widehat{\mu}| = \|\mu\|$ and this cannot hold for an L^p-improving measure [8]. But according to the previous corollary, μ is L^p-improving on intervals.

4.2. *Other examples.* In Corollary 2.5 we saw that an L^p-improving measure must satisfy $\dim_{\mathrm{e}} \mu^k \to 1$. This need not be the case with measures that are L^p-improving on characteristic functions of intervals.

EXAMPLE 4.2. Given any $s \in (0,1)$ we construct a measure μ with $\dim_{\mathrm{e}} \mu^k = s$ for all $k = 1, 2, \ldots$. Set $r = s/(1-s)$. Take

$$\mu = \sum_{n=1}^{\infty} \frac{K_{m_n}(d_n x)}{n^2},$$

where K_{m_n} is the Fejér kernel of degree m_n and the integers $d_n \ge 2^n$ and m_n are inductively defined so that $m_n \sim d_n^{1/r}$ and $d_{n+1} > d_n m_n$. This ensures that $\operatorname{supp} \widehat{K_{m_n}(d_n x)} \cap \operatorname{supp} \widehat{K_{m_j}(d_j x)} = \{0\}$ if $n \neq j$. Let $t \ge s$. Then

$$I_t(\mu^k) \sim \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right)^{2k} + 2 \sum_{n=1}^{\infty} \frac{1}{n^{4k}} \sum_{j=1}^{\infty} |\widehat{K_{m_n}(d_n x)}(j)|^{2k} j^{t-1}.$$

Since for any fixed k,

$$\begin{aligned} d_n^{t-1} m_n^t &\sim \sum_{l=1}^{m_n/2} (l d_n)^{t-1} \frac{1}{2^{2k}} \\ &\le \sum_{j=1}^{\infty} |\widehat{K_{m_n}(d_n x)}(j)|^{2k} j^{t-1} \le \sum_{l=1}^{m_n} (l d_n)^{t-1} \sim d_n^{t-1} m_n^t, \end{aligned}$$

it follows that

$$I_t(\mu^k) \sim C^{2k} + \sum_{n=1}^{\infty} \frac{1}{n^{4k}} d_n^{t-1} m_n^t \sim C^{2k} + \sum_{n} \frac{1}{n^{4k}} d_n^{(r(t-1)+t)/r}.$$

If $t = s$, then $r(t-1)+t = 0$ and thus $I_t(\mu^k) < \infty$. Otherwise $r(t-1)+t > 0$ (as $t \ge s$) and since $d_n \ge 2^n$, $I_t(\mu^k) = \infty$. Hence $\dim_{\mathrm{e}} \mu^k = s$ for all k.

In [8, 1.3] an example is given of a measure which is L^p-improving, but whose total variation measure is not. A modification in [12] produced an example of a measure of energy dimension one, whose total variation measure had energy dimension zero. As a final example, we construct an L^p-improving measure of energy dimension one, whose total variation measure has energy dimension zero, and so is not even L^p-improving on characteristic functions of intervals.

EXAMPLE 4.3. *An L^p-improving measure μ with* $\dim_e \mu = 1$ *and* $\dim_e |\mu| = 0$. We construct singular measures μ_m and polynomials f_m with $\operatorname{supp} \widehat{f_m} \subseteq [-N_m, N_m]$, as in [12, Section 3]. These measures have the property that $\widehat{\mu}_m(0) = 1$ and $|\widehat{\mu}_m(n)| \leq 1/N_m$ for $n \neq 0$. Let λ denote Lebesgue measure on $\mathbb{T}$ and let $\|\cdot\|_{p,2}$ denote the norm of a convolution operator from L^p to L^2. Notice

$$\|\mu_m^k - \lambda\|_{p,2} = \|\mu_m^{k-1} * (\mu_m - \lambda)\|_{p,2} \leq \sup\{|\widehat{\mu}_m(j)|^{k-1} : j \neq 0\}\|\mu_m - \lambda\|_{p,2},$$

thus we can pick integers $n(m)$ such that $\|\mu_m^{n(m)} - \lambda\|_{p,2} \leq |\operatorname{supp} \widehat{f}_m|^{-1}$. This ensures that $\|f_m(\mu_m^{n(m)} - \lambda)\|_{p,2} \leq 1$ and hence $\mu = \sum 2^{-m}(f_m(\mu_m^{n(m)} - \lambda))$ maps L^p to L^2. The singularity of the measures μ_m implies that $I_t(|\mu|) \geq I_t(2^{-m} f_m \lambda)$ and therefore, as in [12], $\dim_e |\mu| = 0$. Since the Fourier transform of μ is even smaller than that of the measure μ constructed in [12], its dimension is at least as big and hence is also one.

The total variation of μ is another example of a measure with Hausdorff dimension one, but not L^p-improving on intervals.

REMARK 4.1. More generally, it can be shown that if μ is any non-zero measure, then there is a probability measure ν mutually absolutely continuous with respect to μ with $\dim_e \nu = 0$. This can be proven in a similar fashion to [8, 1.4]. The main difference is to show that the set of multipliers which are bounded as operators from the characteristic functions of intervals in L^p to L^2 is a Banach space with the norm of the operator ϕ equal to

$$\sup\left\{\frac{\|\phi * \chi_I\|_2}{\|\chi_I\|_p} : I \text{ interval}\right\}.$$

We note, in contrast, that mutually absolutely continuous measures have the same Hausdorff dimension.

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