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ON THE UNIFORM CONVEXITY OF THE BESICOVITCH–ORLICZ SPACE OF ALMOST PERIODIC FUNCTIONS WITH ORLICZ NORM

BY

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Abstract. In [5], we characterized the uniform convexity with respect to the Luxemburg norm of the Besicovitch–Orlicz space of almost periodic functions. Here we give an analogous result when this space is endowed with the Orlicz norm.

1. Introduction. The Besicovitch–Orlicz space B^{ϕ}-a.p. of almost periodic functions was introduced and studied in [4]. That paper contains an extensive investigation of the structural and topological properties of this space endowed with the Luxemburg norm.

In [5]–[7], using this norm, we characterized the uniform and strict convexity of this space.

In this paper, we introduce the Orlicz norm in this space and state its different useful reformulations. Finally, we give a characterization of the uniform convexity of B^{ϕ}-a.p. with the Orlicz norm.

Our main result is similar to that obtained in the classical Orlicz space (see [3]), but the method of proof is different.

2. Preliminaries

2.1. *Orlicz functions.* The notation ϕ will be used for an *Orlicz function*, i.e., a function $\phi : \mathbb{R} \to \mathbb{R}^+$ which is even, convex and satisfies $\phi(0) = 0$, $\phi(u) > 0$ iff $u \neq 0$, and $\lim_{u\to 0} \phi(u)/u = 0$, $\lim_{u\to\infty} \phi(u)/u = \infty$.

An Orlicz function ϕ is said to be of Δ_2-*type* if there exist $K > 2$ and $u_0 \geq 0$ such that $\phi(2u) \leq K\phi(u)$ for all $u \geq u_0$. It is *uniformly convex* when, for each $a \in \,]0, 1[$, there exist $\delta(a) \in \,]0, 1[$ and $u_0 \geq 0$ such that

$$\phi\left(\frac{u + au}{2}\right) \leq (1 - \delta(a))\,\frac{\phi(u) + \phi(au)}{2}, \quad \forall u \geq u_0.$$

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In [3], it is shown that if ϕ is uniformly convex for some $u_0 \geq 0$, then, for any $\varepsilon > 0$ and each interval $[a, b] \subset]0, 1[$, there exists $p(\varepsilon) > 0$ for which

$$\phi(\lambda u + (1 - \lambda)v) \leq (1 - p(\varepsilon))\, \frac{\phi(u) + \phi(v)}{2}, \quad \forall \lambda \in [a, b],\ \forall (u, v) \in E, \tag{2.1}$$

where $E = \{(u, v) \in \mathbb{R}^2 : |u - v| \geq \varepsilon \max(|u|, |v|) \geq \varepsilon u_0\}$.

The function $\psi(y) = \sup\{x|y| - \phi(x) : x \geq 0\}$ is called *conjugate* to ϕ. It is an Orlicz function when ϕ is. The pair (ϕ, ψ) satisfies the *Young inequality*

$$xy \leq \phi(x) + \psi(y), \quad x \in \mathbb{R},\ y \in \mathbb{R},$$

with equality iff $x = \psi'(y)$ or $y = \phi'(x)$.

Let us mention that if ϕ is uniformly convex, then its conjugate ψ is of Δ_2-type. In this case we say that ϕ is of ∇_2-*type*.

An Orlicz function admits a derivative ϕ' except possibly on a denumerable set of points. Moreover, $\phi'(0) = 0$, $\phi'(|u|) > 0$ if $u > 0$, and $\lim_{|u|\to\infty} \phi'(|u|) = \infty$, so that ϕ is strictly increasing from zero to infinity (cf. [3], [8]).

The derivative ϕ' satisfies the following useful inequality:

$$u\phi'(u) \leq \phi(2u) \leq 2u\phi'(2u), \quad \forall u \geq 0.$$

2.2. *The Besicovitch–Orlicz space of almost periodic functions.* Let $M(\mathbb{R})$ be the set of all real Lebesgue measurable functions. The functional

$$\varrho_{B^\phi} : M(\mathbb{R}) \to [0, \infty], \quad \varrho_{B^\phi}(f) = \overline{\lim_{T\to\infty}}\, \frac{1}{2T} \int_{-T}^{T} \phi(|f(t)|)\, dt,$$

is a pseudomodular (cf. [4], [5], [6]). The associated modular space

$$\begin{aligned} B^\phi(\mathbb{R}) &= \{f \in M(\mathbb{R}) : \lim_{\alpha\to 0} \varrho_{B^\phi}(\alpha f) = 0\} \\ &= \{f \in M(\mathbb{R}) : \varrho_{B^\phi}(\lambda f) < \infty,\ \text{for some } \lambda > 0\} \end{aligned}$$

is called the *Besicovitch–Orlicz space*. It is endowed with the pseudonorm (cf. [4]–[6])

$$\|f\|_{B^\phi} = \inf\{k > 0 : \varrho_{B^\phi}(f/k) \leq 1\}, \quad f \in B^\phi(\mathbb{R}),$$

called the *Luxemburg norm*.

Let now $\mathcal{P}$ be the linear set of generalized trigonometric polynomials, i.e.

$$\mathcal{P} = \Big\{P(t) = \sum_{j=1}^{n} \alpha_j \exp(i\lambda_j t) : \lambda_j \in \mathbb{R},\, \alpha_j \in \mathbb{C},\, n \in \mathbb{N}\Big\}.$$

The *Besicovitch–Orlicz space* B^ϕ-a.p. (resp. $\widetilde{B}^\phi$-a.p.) *of almost periodic functions* is the closure of $\mathcal{P}$ in $B^\phi(\mathbb{R})$ with respect to the pseudonorm $\|\cdot\|_{B^\phi}$

(resp. to the modular convergence), more exactly:

$$\begin{aligned}B^{\phi}\text{-a.p.} &= \{f \in B^{\phi}(\mathbb{R}) : \exists (p_n)_{n=1}^{\infty} \text{ in } \mathcal{P} \text{ such that } \lim_{n\to\infty} \|f - p_n\|_{B^{\phi}} = 0\} \\ &= \{f \in B^{\phi}(\mathbb{R}) : \exists (p_n)_{n=1}^{\infty} \text{ in } \mathcal{P} \text{ such that } \\ &\qquad\qquad \forall k > 0,\ \lim_{n\to\infty} \varrho_{B^{\phi}}(k(f - p_n)) = 0\}, \\ \widetilde{B}^{\phi}\text{-a.p.} &= \{f \in B^{\phi}(\mathbb{R}) : \exists (p_n)_{n=1}^{\infty} \text{ in } \mathcal{P} \text{ such that } \\ &\qquad\qquad \exists k > 0,\ \lim_{n\to\infty} \varrho_{B^{\phi}}(k(f - p_n)) = 0\}.\end{aligned}$$

Clearly B^{ϕ}-a.p. $\subset \widetilde{B}^{\phi}$-a.p. and equality holds whenever $\phi \in \Delta_2$ (cf. [4]).

Some structural and topological properties of these spaces are considered in [4]–[6].

From [4], [5], we know that $\phi(|f|) \in B^1$-a.p. when $f \in B^{\phi}$-a.p. Hence, by a classical result (cf. [2]), the upper limit in the expression of $\varrho_{B^{\phi}}(f)$ is a limit, i.e.

$$\varrho_{B^{\phi}}(f) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \phi(|f(t)|)\, dt, \qquad f \in B^{\phi}\text{-a.p.}$$

This fact is very useful in our computations.

Let us denote by {u.a.p.} the classical algebra of Bohr almost periodic functions, or what is the same, the uniform closure of the linear set $\mathcal{P}$. It is known that $\phi(|f|) \in$ {u.a.p.} when $f \in$ {u.a.p.} (cf. [2]).

Also, from [2], we know that if $f \in$ {u.a.p.} and $f \neq 0$, then $M(|f|) > 0$, where

$$M(f) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)\, dt.$$

From now on, B^{ϕ}-a.p. will denote the quotient space obtained by identifying functions whose difference belongs to the subspace $\{f \in B^{\phi}\text{-a.p.} : \|f\|_{B^{\phi}} = 0\}$.

To every $f \in B^{\phi}$-a.p., we may associate a formal Fourier series. More precisely, define the *Bohr transform* of $f \in B^{\phi}$-a.p. by $a(\lambda, f) = M(f \exp(i\lambda t))$ for $\lambda \in \mathbb{R}$. There is at most a denumerable set $\{\lambda_1, \lambda_2, \dots\}$ of scalars for which $a(\lambda, f) \neq 0$ (these are called the *Fourier–Bohr exponents*). The associated coefficients $\{a(\lambda_i, f)\}_{i\geq 1}$ are the *Fourier–Bohr coefficients.*

Questions concerning the convergence of the formal Fourier series

$$S(f)(x) = \sum_{n\geq 1} a(\lambda_n, f) \exp(i\lambda_n x)$$

are nontrivial and only partial results are available. The Bochner approximation result will be of importance here:

If $f \in B^{\phi}$-a.p. and

$$S_n(f)(x) = \sum_{k=1}^{n} a(\lambda_k, f) \exp(i\lambda_k x)$$

are the partial sums of its Fourier series, then there exists a sequence of *Bochner–Fejér polynomials*

$$\sigma_m(f)(x) = \sum_{k=1}^{m} \mu_{m_k} a(\lambda_k, f) \exp(i\lambda_k x),$$

with the convergence factors μ_{m_k} depending only on the sequence $\{\lambda_k\}$, satisfying $0 < \mu_{m_k} \le 1$, such that (cf. [4]):

(1) $\|\sigma_m(f)\|_{B^\phi} \le \|f\|_{B^\phi}$, $m = 1, 2, \ldots$ (and $\varrho_{B^\phi}(\sigma_m(f)) \le \varrho_{B^\phi}(f)$).
(2) $\|\sigma_m(f) - f\|_{B^\phi} \to 0$ as $m \to \infty$ ($\forall \alpha > 0$, $\varrho_{B^\phi}(\alpha(\sigma_m(f) - f)) \to 0$ as $m \to \infty$).

To end this section, we define the *Orlicz pseudonorm* in the B^{ϕ}-a.p. space by setting, as usual,

$$|\!|\!| f |\!|\!|_{B^\phi} = \sup\{M(|fg|) : g \in B^{\psi}\text{-a.p.}, \varrho_{B^\psi}(g) \le 1\},$$

where ψ denotes the conjugate function to ϕ.

3. Convergence results in the B^{ϕ}-a.p. space. A sequence $\{f_k\}_{k\ge1}$ in $B^{\phi}(\mathbb{R})$ is said to be *modular convergent* to some $f \in B^{\phi}(\mathbb{R})$ if $\lim_{k\to\infty} \varrho_{B^\phi}(f_k - f) = 0$.

Let $P(\mathbb{R})$ be the family of subsets of $\mathbb{R}$ and $\Sigma(\mathbb{R})$ the Σ-algebra of Lebesgue measurable sets. We define the set function

$$\overline{\mu}(A) = \overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{-T}^{T} \chi_A(t)\, dt = \overline{\lim_{T\to\infty}} \frac{1}{2T} \mu(A \cap [-T, T]).$$

Clearly, $\overline{\mu}$ is null on sets with μ-finite measure and $\overline{\mu}$ is not σ-additive. As usual, a sequence of Σ-measurable functions $\{f_k\}_{k\ge1}$ will be called $\overline{\mu}$-*convergent* to f if, for all $\varepsilon > 0$,

$$\lim_{k\to\infty} \overline{\mu}\{t \in \mathbb{R} : |f_k(t) - f(t)| \ge \varepsilon\} = 0.$$

Let now $\{A_i\}_{i\ge1}$ with $A_i \in \Sigma$ for all $i \in \mathbb{N}$ be such that $A_i \cap A_j = \emptyset$ if $i \ne j$ and $\bigcup_{i\ge1} A_i \subset [0, \alpha]$, $\alpha < 1$. Put $f = \sum_{i\ge1} a_i \chi_{A_i}$ with $\sum_{i\ge1} \phi(a_i)\mu(A_i) < \infty$ and let $\widetilde{f}$ be the periodic extension of f to the whole $\mathbb{R}$ (with period 1). Then there exist $P_m \in \mathcal{P}$, $m \ge 1$, such that

$$\varrho_{B^\phi}\left(\frac{\widetilde{f} - P_m}{4}\right) \to 0 \quad \text{as } m \to \infty \qquad \text{(cf. [5]).} \tag{3.1}$$

We now state some fundamental convergence results that will be used below (cf. [5]–[7]).

LEMMA 3.1. *Let* $\{f_k\}_{k\geq 1}\subset B^{\phi}(\mathbb{R})$.

(1) *If there exists* $f\in B^{\phi}(\mathbb{R})$ *such that* $\lim_{k\to\infty}\varrho_{B^\phi}(f_k-f)=0$ *and there exists* $g\in B^{\phi}$*-a.p. satisfying* $\max(|f_k|,|f|)\leq g$, *then* $\lim_{k\to\infty}\varrho_{B^\phi}(f_k) = \varrho_{B^\phi}(f)$.
(2) *If* $f\in B^{\phi}$*-a.p. and* $\{P_n\}$ *is the sequence of Bochner–Fejér polynomials associated to* f, *then* $\lim_{n\to\infty}\varrho_{B^\phi}(P_n)=\varrho_{B^\phi}(f)$.
(3) *If* $f\in B^{\phi}$*-a.p. and* $\lim_{n\to\infty}\varrho_{B^\phi}(f_n-f)=0$, *then*
   (a) $\underline{\lim}_{k\to\infty}\varrho_{B^\phi}(f_k)\geq\varrho(f)$.
   (b) $\{f_k\}_{k\geq 1}$ *is* $\overline{\mu}$*-convergent to* f.

4. Auxiliary results

LEMMA 4.1. *Let* $f\in B^{\phi}$*-a.p.* $f\neq 0$ *and let* $\{f_n\}_{n\geq 1}$ *be modular convergent to* f. *Then there exist constants* $\alpha_1,\beta_1,\theta_1$ *with* $\theta_1\in\,]0,1[$, $0<\alpha_1<\beta_1$, *and* $n_0\in\mathbb{N}$ *such that* $\overline{\mu}(G_n)\geq\theta_1$ *for all* $n\geq n_0$, *where* $G_n=\{t\in\mathbb{R}:\alpha_1\leq |f_n(t)|\leq\beta_1\}$.

Proof. It is known from [5] that there exist α,β,θ with $\theta\in\,]0,1[$ and $0<\alpha<\beta$ such that $\overline{\mu}(G)\geq\theta$, where $G=\{t\in\mathbb{R}:\alpha\leq|f(t)|\leq\beta\}$. Take $\alpha_1=\alpha/2$, $\beta_1=\alpha/2+\beta$ and $\theta_1=\theta/2$. Then, since $\{f_n\}_{n\geq 1}$ is modular convergent to f, it is also $\overline{\mu}$-convergent to f (cf. Lemma 3.1(3)(b)) and so

$$\overline{\mu}\{t\in\mathbb{R}:|f_n(t)-f(t)|\geq\alpha/2\}<\theta/2\qquad\text{for } n\geq n_0.$$

Putting $G_n'=\{t\in\mathbb{R}:|f_n(t)-f(t)|\geq\alpha/2\}$, we have $G\setminus G_n'\subset G_n$ for all $n\geq n_0$. Indeed, if $t\in G\setminus G_n'$ then $\alpha\leq|f(t)|\leq\beta$ and $|f_n(t)-f(t)|\leq\alpha/2$, from which it follows that $\alpha_1\leq|f_n(t)|\leq\beta_1$ for all $n\geq n_0$, and so $t\in G_n$ for all $n\geq n_0$.

Finally, $\overline{\mu}(G_n)\geq\overline{\mu}(G\backslash G_n')\geq\overline{\mu}(G)-\overline{\mu}(G_n')\geq\theta-\theta/2=\theta_1$ for all $n\geq n_0$. ∎

LEMMA 4.2. *Let* $f\in B^{\phi}$*-a.p. and* $E\in\Sigma$. *Then the function*

$$F:\,]0,\infty[\,\to\mathbb{R},\qquad F(\lambda)=\varrho_\phi(f\chi_E/\lambda),$$

is continuous on $]0,\infty[$.

Proof. Let $\lambda_0>0$ and let $\lambda_n\to\lambda_0$ as $n\to\infty$. We have

$$\varrho_{B^\phi}\left[\left(\frac{1}{\lambda_n}-\frac{1}{\lambda_0}\right)f\chi_E\right]\leq\left|\frac{1}{\lambda_n}-\frac{1}{\lambda_0}\right|\varrho_{B^\phi}(f\chi_E)\to 0\qquad\text{as } n\to\infty.$$

It follows that $\{(1/\lambda_n)f\chi_E\}$ is modular convergent to $(1/\lambda_0)f\chi_E$. Moreover,

$$\max\left(\frac{1}{|\lambda_n|}\,|f|\chi_E,\frac{1}{|\lambda_0|}\,|f|\chi_E\right)\leq A|f|\in B^{\phi}\text{-a.p.}$$

for some constant $A > 0$. From Lemma 3.1, it follows directly that

$$\lim_{n\to\infty} \varrho_{B^\phi}\left(\frac{f\chi_E}{\lambda_n}\right) = \varrho_{B^\phi}\left(\frac{f\chi_E}{\lambda_0}\right),$$

which means that F is continuous at λ_0. ∎

Lemma 4.3. *If $f \in B^\phi$-a.p., then:*

(1) $|||f|||_{B^\phi} = \inf\left\{\frac{1}{k}(1 + \varrho_{B^\phi}(kf)) : k > 0\right\}$ *and the set*

$$K(f) = \left\{k > 0 : |||f|||_{B^\phi} = \frac{1}{k}\,(1 + \varrho_{B^\phi}(kf))\right\}$$

is not empty.

(2) $\varrho_{B^\phi}(f/|||f|||_{B^\phi}) \le 1$ *if* $|||f|||_{B^\phi} \neq 0$.

(3) $\|f\|_{B^\phi} \le |||f|||_{B^\phi} \le 2\|f\|_{B^\phi}$ *for any* $f \in B^\phi$*-a.p.*

Proof. Note that by arguments similar to those used in the Orlicz space case, we may show that

$$|||f|||_{B^\phi} \le 2\|f\|_{B^\phi}. \tag{4.1}$$

(1) From the Young inequality we have

$$M(|fg|) = \frac{1}{k}\,M(|kfg|) \le \frac{1}{k}\,[\varrho_{B^\phi}(kf) + \varrho_{B^\psi}(g)] \quad \text{for all } k > 0,$$

and therefore

$$|||f|||_{B^\phi} \le \inf_{k>0}\left\{\frac{1}{k}\,(1 + \varrho_{B^\phi}(kf))\right\}. \tag{4.2}$$

For the opposite inequality, we proceed in several steps:

(a) We suppose first that the derivative ϕ' is continuous, and prove that if $P \in \mathcal{P}$ then there exists $k_0 \in\,]0,\infty[$ such that

$$|||P|||_{B^\phi} = \frac{1}{k_0}\,(1 + \varrho_{B^\phi}(k_0P)).$$

Define

$$F : [0,\infty[\; \to [0,\infty[, \qquad F(k) = \varrho_{B^\psi}[\phi'(k|P|)].$$

Then $\lim_{k\to\infty} F(k) = \infty$. Indeed, if $P \neq 0$, then from Lemma 4.1, there exist α, β, θ with $\beta > \alpha > 0$ and $\theta \in (0,1)$ such that $\overline{\mu}(G) \ge \theta$, where $G = \{t \in \mathbb{R} : \alpha \le |P(t)| \le \beta\}$. It follows that

$$\varrho_{B^\psi}[\phi'(k|P|)] \ge \overline{\lim_{T\to\infty}}\; \frac{1}{2T} \int\limits_{[-T,T]\cap G} \psi(\phi'(k|P(x)|))\,dx \ge \theta\psi[\phi'(k\alpha)].$$

Now, since an Orlicz function increases to infinity with its derivative (cf. [2], [7]), we get $\lim_{k\to\infty} F(k) = \infty$.

Let us show that F is continuous. Let $k_n \to k_0 \in\,]0,\infty[$. Trigonometric polynomials being uniformly bounded, we put $\|P\|_\infty = M$. Let $\varepsilon > 0$ be arbitrary. Since ϕ' is uniformly continuous on $[k_0M/2, 3k_0M/2]$, there exists n_0 such that

$$n \geq n_0 \;\Rightarrow\; |\phi'(k_n|P|) - \phi'(k_0|P|)| \leq \psi^{-1}(\varepsilon).$$

Hence

$$\varrho_{B^\psi}[\phi'(k_n|P|) - \phi'(k_0|P|)] \leq \varepsilon. \tag{4.3}$$

Set $f_n = \phi'(k_n|P|)$ and $f = \phi'(k_0|P|)$. Then $f_n, f \in \{\text{u.a.p.}\}$. Since ϕ' is increasing, we have moreover $f_n \leq \phi'(2k_0|P|)$. Now, (4.3) implies $\lim_{n\to\infty} \varrho_{B^\psi}(f_n - f) = 0$. Finally, in view of Lemma 3.1(1),

$$\lim_{n\to\infty} \varrho_{B^\psi}(f_n) = \varrho_{B^\psi}(f)$$

and thus F is continuous at k_0.

Consequently, since $F(0) = 0$ and $\lim_{k\to\infty} F(k) = \infty$, there exists $k_0 \in]0, \infty[$ for which $\varrho_{B^\psi}[\phi'(k_0|P|)] = 1$. Considering the case of equality in the Young inequality, we get

$$\begin{aligned}
\|\!|P|\!\|_{B^\phi} &\geq \frac{1}{k_0}\, M(|k_0P| \cdot \phi'(k_0|P|)) \\
&\geq \frac{1}{k_0}\, (\varrho_{B^\phi}(k_0P) + \varrho_{B^\psi}[\phi'(k_0|P|)]) \\
&\geq \frac{1}{k_0}\, (\varrho_{B^\phi}(k_0P) + 1)
\end{aligned}$$

and finally, combining this with (4.2), it follows that

$$\|\!|P|\!\|_{B^\phi} = \inf_{k>0}\left\{\frac{1}{k}(\varrho_{B^\phi}(kP) + 1)\right\} = \frac{1}{k_0}(\varrho_{B^\phi}(k_0P) + 1).$$

We now show that this result remains true for $f \in B^\phi$-a.p. For, let $\{P_n\}$ be the sequence of Bochner–Fejér polynomials that converge to f. We have seen that for each $n \geq 1$ there exists $k_n \in]0, \infty[$ such that

$$\|\!|P_n|\!\|_{B^\phi} = \left\{\frac{1}{k_n}\, (1 + \varrho_{B^\phi}(k_nP_n))\right\}. \tag{4.4}$$

From (4.1) and the Bochner–Fejér approximation property (see (1) of 2.2), we get

$$1/k_n \leq \|\!|P_n|\!\|_{B^\phi} \leq 2\|P_n\|_{B^\phi} \leq 2\|f\|_{B^\phi}$$

and thus $k_n \geq 1/2\|f\|_{B^\phi} = c_1 > 0$. Let us show that $k_n \leq c_2$ for all $n \geq 1$, for some constant c_2. Indeed, if this is not the case, there exists a subsequence, denoted again by $\{k_n\}$, increasing to infinity and such that

$$
\begin{aligned}
1 = \varrho_{B^\psi}[\phi'(k_n|P_n|)] &\geq \overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{-T}^{T} \psi(\phi'(k_n|P_n(x)|))\,dx \\
&\geq \overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{G_n} \psi(\phi'(k_n|P_n(x)|))\,dx \\
&\geq \overline{\lim_{T\to\infty}} \frac{1}{2T} \int_{G_n} \psi(\phi'(k_n\alpha_1))\,dt \geq \theta_1\psi[\phi'(k_n\alpha_1)] \to \infty \quad \text{as } n\to\infty,
\end{aligned}
$$

where G_n, θ_1, α_1 are defined in Lemma 4.1. A contradiction.

Now, the sequence $\{k_n\}$ being bounded, there exists a subsequence denoted again by $\{k_n\}$ that converges to some k_0 with $0 < k_0 < \infty$. Let us show that

$$\lim_{n\to\infty} \varrho_{B^\phi}(k_nP_n) = \varrho_{B^\phi}(k_0f).$$

Indeed, by (1) of 2.2 we have

$$
\begin{aligned}
\varrho_{B^\phi}(k_nP_n - k_0f) &\leq \tfrac{1}{2}\varrho_{B^\phi}(2(k_n-k_0)P_n) + \tfrac{1}{2}\varrho_{B^\phi}(2k_0(P_n-f)) \\
&\leq |k_n-k_0|\varrho_{B^\phi}(f) + \tfrac{1}{2}\varrho_{B^\phi}(2k_0(P_n-f))
\end{aligned}
$$

and so $\lim_{n\to\infty} \varrho_{B^\phi}(k_nP_n - k_0f) = 0$. Now, in view of Lemma 3.1(3)(a),

$$\underline{\lim_{n\to\infty}}\, \varrho_{B^\phi}(k_nP_n) \geq \varrho_{B^\phi}(k_0f).$$

On the other hand, from the inequality $\varrho_{B^\phi}(k_nP_n) \leq \varrho_{B^\phi}(k_nf)$, we have

$$\overline{\lim_{n\to\infty}}\, \varrho_{B^\phi}(k_nP_n) \leq \overline{\lim_{n\to\infty}}\, \varrho_{B^\phi}(k_nf) = \lim_{n\to\infty} \varrho_{B^\phi}(k_nf) = \varrho_{B^\phi}(k_0f)$$

and thus

$$\overline{\lim_{n\to\infty}}\, \varrho_{B^\phi}(k_nP_n) \leq \varrho_{B^\phi}(k_0f) \leq \underline{\lim_{n\to\infty}}\, \varrho_{B^\phi}(k_nP_n),$$

i.e. $\lim_{n\to\infty} \varrho_{B^\phi}(k_nP_n) = \varrho_{B^\phi}(k_0f)$.

Finally, letting $n\to\infty$ in (4.4) we get

$$\|\!|f|\!\|_{B^\phi} = \frac{1}{k_0}\,(\varrho_{B^\phi}(k_0f) + 1). \tag{4.5}$$

(b) Consider now the case of ϕ' discontinuous. From [3], we know that for each $\varepsilon > 0$ there exists an equivalent Orlicz function ϕ_ε with continuous derivative, more precisely

$$(1-\varepsilon)\phi(x) \leq \phi_\varepsilon(x) \leq \phi(x), \quad x \geq 0. \tag{4.6}$$

We also have B^ϕ-a.p. $= B^{\phi_\varepsilon}$-a.p. as sets and one sees easily that

$$(1-\varepsilon)\varrho_{B^\phi}(f) \leq \varrho_{B^{\phi_\varepsilon}}(f) \leq \varrho_{B^\phi}(f), \quad f \in B^\phi\text{-a.p.} \tag{4.7}$$

The same inequality holds for the corresponding norms.

Now, since (4.5) is true for ϕ_ε, using (4.7) we get

$$\begin{aligned} \inf_{k>0}\left\{\frac{1}{k}\left(\varrho_{B^\phi}(kf)+1\right)\right\} &\le \inf_{k>0}\left\{\frac{1}{k}\left(\frac{1}{1-\varepsilon}\,\varrho_{B^{\phi_\varepsilon}}(kf)+1\right)\right\} \\ &\le \frac{1}{1-\varepsilon}\inf_{k>0}\left\{\frac{1}{k}\left(\varrho_{B^{\phi_\varepsilon}}(kf)+(1-\varepsilon)\right)\right\} \\ &\le \frac{1}{1-\varepsilon}\,\|\!|f|\!\|_{B^{\phi_\varepsilon}} \le \frac{1}{1-\varepsilon}\,\|\!|f|\!\|_{B^\phi}. \end{aligned} \tag{4.8}$$

Finally, $\varepsilon>0$ being arbitrary and recalling (4.2), this proves that $\|\!|f|\!\|_{B^\phi} = \inf\left\{\frac{1}{k}(\varrho_{B^\phi}(kf)+1) : k>0\right\}$.

(c) It remains to show that if $f\in B^\phi$-a.p. then

$$\|\!|f|\!\|_{B^\phi} = \frac{1}{k_0}\left(\varrho_{B^\phi}(k_0 f)+1\right) \quad \text{for some } k_0>0.$$

For $\varepsilon>0$, let ϕ_ε be the associated smooth function satisfying (4.6). We have

$$\begin{aligned} \inf_{k>0}\left\{\frac{1}{k}\left(\varrho_{B^\phi}(kf)+1\right)\right\} &\le \frac{1}{1-\varepsilon}\inf_{k>0}\left\{\frac{1}{k}\left(\varrho_{B^{\phi_\varepsilon}}(kf)+(1-\varepsilon)\right)\right\} \\ &\le \frac{1}{1-\varepsilon}\inf_{k>0}\left\{\frac{1}{k}\left(\varrho_{B^{\phi_\varepsilon}}(kf)+1\right)\right\} \\ &\le \frac{1}{1-\varepsilon}\,\frac{1}{k_\varepsilon}\left(\varrho_{B^{\phi_\varepsilon}}(k_\varepsilon f)+1\right) \\ &\le \frac{1}{1-\varepsilon}\,\frac{1}{k_\varepsilon}\left(\varrho_{B^\phi}(k_\varepsilon f)+1\right). \end{aligned}$$

On the other hand,

$$\begin{aligned} \inf_{k>0}\left\{\frac{1}{k}\left(\varrho_{B^\phi}(kf)+1\right)\right\} &\ge \inf_{k>0}\left\{\frac{1}{k}\left(\varrho_{B^{\phi_\varepsilon}}(kf)+1\right)\right\} \\ &\ge \frac{1}{k_\varepsilon}\left(\varrho_{B^{\phi_\varepsilon}}(k_\varepsilon f)+1\right) \\ &\ge \frac{1-\varepsilon}{k_\varepsilon}\left(\varrho_{B^\phi}(k_\varepsilon f)+1\right). \end{aligned}$$

Consequently,

$$\begin{aligned} (1-\varepsilon)\,\frac{1}{k_\varepsilon}\left(\varrho_{B^\phi}(k_\varepsilon f)+1\right) &\le \inf_{k>0}\left\{\frac{1}{k}\left(\varrho_{B^\phi}(kf)+1\right)\right\} \\ &\le \frac{1}{1-\varepsilon}\,\frac{1}{k_\varepsilon}\left(\varrho_{B^\phi}(k_\varepsilon f)+1\right). \end{aligned} \tag{4.9}$$

We claim that the sequence $\{k_\varepsilon\}$ is bounded. Indeed, otherwise a subsequence, denoted also by $\{k_\varepsilon\}$, increases to infinity, and then

$$\begin{aligned}1 = \varrho_{B^\psi}[\varphi_\varepsilon(k_\varepsilon|f|)] &\geq \overline{\lim_{T\to\infty}} \frac{1}{2T} \int_G \psi(\varphi_\varepsilon(k_\varepsilon|f(x)|))\,dx \\ &\geq \theta_1 \psi[\varphi_\varepsilon(k_\varepsilon \alpha_1)] \to \infty \quad \text{ as } k_\varepsilon \to \infty,\end{aligned}$$

where G, θ_1, α_1 are defined in Lemma 4.1. A contradiction.

We can show easily that $k_\varepsilon \geq c_1 > 0$ for some $c_1 > 0$.

Now, the sequence $\{k_\varepsilon\}$ being bounded, there exists a subsequence denoted again by $\{k_\varepsilon\}$ that converges to some $0 < k_0 < \infty$. Finally, letting $\varepsilon \to 0$ in (4.9) and using Lemma 4.2, we get

$$|||f|||_{B^\phi} = \inf_{k>0}\left\{\frac{1}{k}\left(\varrho_{B^\phi}(kf)+1\right)\right\} = \frac{1}{k_0}\left(\varrho_{B^\phi}(k_0 f)+1\right).$$

(2) Suppose first that ϕ' is continuous. Let $f \in \{\text{u.a.p.}\}$, $f \neq 0$ and $g \in B^\psi$-a.p. Then

(a) if $\varrho_{B^\psi}(g) \leq 1$, we have $M(|fg|) \leq |||f|||_{B^\phi}$,
(b) if $\varrho_{B^\psi}(g) > 1$, we have

$$\varrho_{B^\psi}\left(\frac{g}{\varrho_{B^\psi}(g)}\right) \leq \frac{1}{\varrho_{B^\psi}(g)}\,\varrho_{B^\psi}(g) = 1$$

and so $M(|fg/\varrho_{B^\psi}(g)|) \leq |||f|||_{B^\phi}$.

It follows that in all cases we have,

$$M(|fg|) \leq \max(1, \varrho_{B^\psi}(g)) \cdot |||f|||_{B^\phi}.$$

Defining now $g = \phi'(f/|||f|||_{B^\phi})$, we have $g \in \{\text{u.a.p.}\}$ and using the case of equality in the Young inequality and the fact that in this case the limit exists, we have

$$M\left(\left|\frac{f}{|||f|||_{B^\phi}}\,g\right|\right) = \varrho_{B^\phi}\left(\frac{f}{|||f|||_{B^\phi}}\right) + \varrho_{B^\psi}(g) \leq \max(1, \varrho_{B^\psi}(g))$$

so that $\varrho_{B^\phi}(f/|||f|||_{B^\phi}) \leq 1$.

To consider the general case of $f \in B^\phi$-a.p., let $\{P_n\}_{n=1}^\infty$ be the sequence of Bochner–Fejér polynomials approximating f. Then

$$\varrho_{B^\phi}\left(\frac{P_n}{|||P_n|||_{B^\phi}}\right) \leq 1, \qquad \forall n \geq 1.$$

But, using Lemma 4.3(1) and (1) of 2.2, we can write

$$|||P_n|||_{B^\phi} = \inf_{k>0}\left\{\frac{1}{k}\left(1+\varrho_{B^\phi}(kP_n)\right)\right\} \leq \inf_{k>0}\left\{\frac{1}{k}\left(1+\varrho_{B^\phi}(kf)\right)\right\} = |||f|||_{B^\phi},$$

so that
$$\varrho_{B^\phi}\left(\frac{P_n}{\|\!|f|\!\|_{B^\phi}}\right) \le \varrho_{B^\phi}\left(\frac{P_n}{\|\!|P_n|\!\|_{B^\phi}}\right) \le 1,$$
and thus $\varrho_{B^\phi}(f/\|\!|f|\!\|_{B^\phi}) \le 1$ by Lemma 3.1(2).

In the general case of a discontinuous ϕ', we use the inequalities (4.6) to obtain
$$\varrho_{B^\phi}\left(\frac{f}{\|\!|f|\!\|_{B^\phi}}\right) \le \varrho_{B^\phi}\left(\frac{f}{\|\!|f|\!\|_{B^{\phi_\varepsilon}}}\right) \le \frac{1}{1-\varepsilon}\varrho_{B^{\phi_\varepsilon}}\left(\frac{f}{\|\!|f|\!\|_{B^{\phi_\varepsilon}}}\right) \le \frac{1}{1-\varepsilon},$$
and since ε is arbitrary, we get $\varrho_{B^\phi}(f/\|\!|f|\!\|_{B^\phi}) \le 1$, which is the desired result.

(3) We have $\varrho_{B^\phi}(f/\|\!|f|\!\|_{B^\phi}) \le 1$ and so $\|f\|_{B^\phi} \le \|\!|f|\!\|_{B^\phi}$. Finally, in view of (4.1), we get $\|f\|_{B^\phi} \le \|\!|f|\!\|_{B^\phi} \le 2\|f\|_{B^\phi}$. ∎

LEMMA 4.4. *Let $f \in E^\phi([0,1])$, where $E^\phi([0,1])$ is the Orlicz class of functions, i.e.*
$$E^\phi([0,1]) = \{f \text{ measurable} : \varrho_\phi(\lambda f) < \infty, \forall \lambda > 0\},$$
ϱ_ϕ being the usual Orlicz modular. Then:

- (i) *If $\widetilde{f}$ is the periodic extension of f to the whole $\mathbb{R}$ (with period 1), then $\widetilde{f} \in B^\phi$-a.p.*
- (ii) *The injection $i : E^\phi([0,1]) \to B^\phi$-a.p., $i(f) = \widetilde{f}$, is an isometry with respect to the modulars and also for the respective Orlicz norms.*

Proof. Let $f = \sum_{i=1}^n a_i\chi_{A_i}$, $A_i \cap A_j = \emptyset$ if $i \ne j$ and $\bigcup_{i=1}^n A_i \subset [0,\alpha]$, $0 < \alpha < 1$ and let $m \in \mathbb{N}^*$. Then $\sum_{i=1}^n \phi(ma_i)\mu(A_i) < \infty$ and using (3.1) we assert that there exists $P_m \in \mathcal{P}$ (the set of generalized trigonometric polynomials) for which
$$\varrho_{B^\phi}\left(\frac{m}{4}\,(\widetilde{f} - P_m)\right) \le \frac{1}{m},$$
where $\widetilde{f}$ is the 1-periodic extension of f to the whole $\mathbb{R}$.

Let $\lambda > 0$ be arbitrary. If $m_0 \in \mathbb{N}^*$ is such that $\lambda \le m_0/4$ then
$$\varrho_{B^\phi}(\lambda(\widetilde{f} - P_m)) \le \varrho_{B^\phi}\left(\frac{m}{4}\,(\widetilde{f} - P_m)\right) \le \frac{1}{m}, \quad \forall m \ge m_0.$$
This means that $\lim_{m\to\infty} \|\widetilde{f} - P_m\|_{B^\phi} = 0$, i.e. $\widetilde{f} \in B^\phi$-a.p.

Consider now the general case of $f \in E^\phi([0,1])$. It is known (see [3]) that the step functions are dense in $E^\phi([0,1])$ and hence, given $\varepsilon > 0$, there is $g_\varepsilon = \sum_{i=1}^n a_i\chi_{A_i}$ for which $\|g_\varepsilon - f\|_\phi \le \varepsilon/4$. Here $\|\cdot\|_\phi$ is the usual Luxemburg norm in $E^\phi([0,1])$.

Since f is absolutely continuous, choose $\delta > 0$ such that
$$\mu(A) \le \delta \ \Rightarrow\ \|f\chi_A\|_\phi \le \varepsilon/4.$$

Take $\alpha > 0$ such that $1 - \alpha \le \delta$, put $A_i^\alpha = A_i \cap [0, \alpha]$, $i = 1, \ldots, n$, and let $g_\varepsilon^\alpha = \sum_{i=1}^n a_i \chi_{A_i^\alpha}$. Then $g_\varepsilon^\alpha \in E^\phi([0,1])$.

Let $\widetilde{f}$ and $\widetilde{g}_\varepsilon^\alpha$ be the 1-periodic extensions of f and g_ε^α respectively. We have

$$\begin{aligned}\|\widetilde{f} - \widetilde{g}_\varepsilon^\alpha\|_{B^\phi} &= \|f - g_\varepsilon^\alpha\|_\phi \le \|(f - g_\varepsilon^\alpha)\chi_{[0,\alpha]}\|_\phi + \|(f - g_\varepsilon^\alpha)\chi_{[\alpha,1]}\|_\phi \\ &\le \|f - g_\varepsilon\|_\phi + \|f\chi_{[\alpha,1]}\|_\phi \le \varepsilon/4 + \varepsilon/4 = \varepsilon/2.\end{aligned}$$

Now, since $\widetilde{g}_\varepsilon^\alpha \in B^\phi$-a.p., there exists $P_\varepsilon \in \mathcal{P}$ for which $\|\widetilde{g}_\varepsilon^\alpha - P_\varepsilon\|_{B^\phi} \le \varepsilon/2$. Finally,

$$\|\widetilde{f} - P_\varepsilon\|_{B^\phi} \le \|\widetilde{f} - \widetilde{g}_\varepsilon^\alpha\|_{B^\phi} + \|\widetilde{g}_\varepsilon^\alpha - P_\varepsilon\|_{B^\phi} \le \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

i.e. $\widetilde{f} \in B^\phi$-a.p.

It is clear that $i : E^\phi([0,1]) \to B^\phi$-a.p. is a modular isometry. It is also immediate that it is an isometry for the Orlicz norms. Indeed,

$$|\!|\!| f |\!|\!|_\phi = \inf_{k>0}\left\{\frac{1}{k}\,(1 + \varrho_\phi(kf))\right\} = \inf_{k>0}\left\{\frac{1}{k}\,(1 + \varrho_{B^\phi}(k\widetilde{f}))\right\} = |\!|\!| \widetilde{f} |\!|\!|_{B^\phi}. \ \blacksquare$$

Lemma 4.5. (1) *Let ϕ be of Δ_2-type. Then*

$$\inf\{k \in K(f) : |\!|\!| f |\!|\!|_{B^\phi} = 1,\ f \in B^\phi\text{-}a.p.\} = d > 1.$$

(2) *If ψ, the conjugate to ϕ, is of Δ_2-type, then for each $a, b > 0$, the set*

$$Q = \{k \in K(f) : a \le |\!|\!| f |\!|\!|_{B^\phi} \le b,\ f \in B^\phi\text{-}a.p.\}$$

is bounded.

Proof. The arguments are exactly as those used in the Orlicz space case (see [3], [9]) so we omit the proof. $\blacksquare$

5. Uniform convexity of B^ϕ-a.p. We now state the main result of this paper.

Theorem 5.1. *The space $(B^\phi\text{-}a.p., |\!|\!| f |\!|\!|_{B^\phi})$ is uniformly convex if and only if ϕ is uniformly convex and it is of Δ_2-type.*

Proof. Sufficiency. The proof of the sufficiency follows by the arguments developed in the Orlicz space case. We sketch it here for completeness. Recall that a Banach space $(X, \|\cdot\|)$ is uniformly convex iff

$$\forall \{x_n\}, \{y_n\} \subset B(X), \qquad \lim_{n\to\infty} \|x_n + y_n\| = 2 \ \Rightarrow\ \lim_{n\to\infty} \|x_n - y_n\| = 0$$

(see [3]), $B(X)$ being the unit ball of X.

Let $\{f_n\}_{n\ge1}, \{g_n\}_{n\ge1}$ be two sequences in the unit sphere of $(B^\phi\text{-a.p.}, |\!|\!| \cdot |\!|\!|_{B^\phi})$. Let $\{k_n\}_{n\ge1}, \{h_n\}_{n\ge1}$ be the sequences of scalars defined by (see Lemma 4.3)

$$|\!|\!| f_n |\!|\!|_{B^\phi} = \frac{1}{k_n}\,(1 + \varrho_{B^\phi}(k_n f_n)), \qquad |\!|\!| g_n |\!|\!|_{B^\phi} = \frac{1}{h_n}\,(1 + \varrho_{B^\phi}(h_n g_n)), \qquad n \ge 1.$$

For given $\alpha > 0$ and $0 < \varepsilon < 1/2$, we define, for each $n \geq 1$, the sets

$$\begin{aligned} G_n &= \{t \in \mathbb{R} : \max(|k_n f_n(t)|, |h_n g_n(t)|) < \alpha\}, \\ E_n &= \{t \in \mathbb{R} : |k_n f_n(t) - h_n g_n(t)| < \varepsilon \max(|k_n f_n(t)|, |h_n g_n(t)|)\}, \\ F_n &= \{t \in \mathbb{R} : |k_n f_n(t) - h_n g_n(t)| \geq \varepsilon \max(|k_n f_n(t)|, |h_n g_n(t)|) \geq \varepsilon\alpha\}. \end{aligned}$$

We have the following estimates:

$$\varrho_{B^\phi}((k_n f_n - h_n g_n)\chi_{G_n}) \leq \varrho_{B^\phi}(2\alpha\chi_{G_n}) \leq \phi(2\alpha) \tag{5.1}$$

and

$$\begin{aligned} \varrho_{B^\phi}((k_n f_n - h_n g_n)\chi_{E_n}) &\leq \varrho_{B^\phi}(\varepsilon(|k_n f_n| + |h_n g_n|)\chi_{E_n}) \\ &\leq 2\varepsilon\varrho_{B^\phi}\left(\frac{|k_n f_n| + |h_n g_n|}{2}\,\chi_{E_n}\right) \\ &\leq \varepsilon(\varrho_{B^\phi}(k_n f_n \chi_{E_n}) + \varrho_{B^\phi}(h_n g_n \chi_{E_n})). \end{aligned}$$

Now, since

$$\varrho_{B^\phi}(k_n f_n) + \varrho_{B^\phi}(h_n g_n) = k_n + h_n - 2,$$

we get (see Lemma 4.5)

$$\varrho_{B^\phi}((k_n f_n - h_n g_n)\chi_{E_n}) \leq \varepsilon(k_n + h_n - 2) \leq \varepsilon(2d - 2) \leq 2\varepsilon d. \tag{5.2}$$

Put

$$a = \inf\left\{\frac{k_n}{k_n + h_n}, \frac{h_n}{k_n + h_n} : n \geq 1\right\}, \quad b = \sup\left\{\frac{k_n}{k_n + h_n}, \frac{h_n}{k_n + h_n} : n \geq 1\right\}.$$

Then from Lemma 4.5, $[a, b] \subset]0, 1[$.

Now, using condition (2.1), it is easily seen that for $t \in F_n$, we have

$$\begin{aligned} &\phi\left(\frac{k_n h_n}{k_n + h_n}\,(f_n(t) + g_n(t))\right) \\ &\qquad \leq (1 - \delta)\left[\frac{h_n}{k_n + h_n}\,\phi(k_n f_n(t)) + \frac{k_n}{k_n + h_n}\,\phi(h_n g_n(t))\right] \end{aligned} \tag{5.3}$$

and hence

$$\begin{aligned} 2 - \|f_n + g_n\|_{B^\phi} &\geq \frac{1}{k_n}\,(1 + \varrho_{B^\phi}(k_n f_n)) + \frac{1}{h_n}\,(1 + \varrho_{B^\phi}(h_n g_n)) \\ &\quad - \frac{k_n + h_n}{k_n h_n}\left(1 + \varrho_{B^\phi}\left(\frac{k_n h_n}{k_n + h_n}\,(f_n + g_n)\right)\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{k_n+h_n}{k_nh_n}\,\varrho_{B^1}\bigg(\frac{h_n}{k_n+h_n}\,\phi(k_nf_n)+\frac{k_n}{k_n+h_n}\,\phi(h_ng_n)\\
&\qquad\qquad\qquad\qquad\qquad\qquad - \phi\bigg(\frac{k_nh_n}{k_n+h_n}(f_n+g_n)\bigg)\bigg)\\
&\geq \frac{k_n+h_n}{k_nh_n}\,\varrho_{B^1}\bigg(\bigg[\frac{h_n}{k_n+h_n}\,\phi(k_nf_n)+\frac{k_n}{k_n+h_n}\,\phi(h_ng_n)\\
&\qquad\qquad\qquad\qquad\qquad\qquad - \phi\bigg(\frac{k_nh_n}{k_n+h_n}(f_n+g_n)\bigg)\bigg]\chi_{F_n}\bigg)\\
&\geq \frac{k_n+h_n}{k_nh_n}\,\varrho_{B^1}\bigg(\delta\bigg(\frac{h_n}{k_n+h_n}\,\phi(k_nf_n)+\frac{k_n}{k_n+h_n}\,\phi(h_ng_n)\bigg)\chi_{F_n}\bigg)\\
&\geq \delta\varrho_{B^1}\bigg(\bigg(\frac{1}{k_n}\,\phi(k_nf_n)+\frac{1}{h_n}\,\phi(h_ng_n)\bigg)\chi_{F_n}\bigg)\\
&\geq \delta\varrho_{B^1}\bigg(\frac{1}{d}\,(\phi(k_nf_n)+\phi(h_ng_n))\chi_{F_n}\bigg)\\
&\geq \frac{2\delta}{d}\,\varrho_{B^\phi}\bigg(\frac{k_nf_n-h_ng_n}{2}\,\chi_{E_n}\bigg) \geq \frac{2\delta k}{2}\,\varrho_{B^\phi}((k_nf_n-h_ng_n)\chi_{E_n}),
\end{aligned}
$$

k being the constant from the Δ_2-condition on ϕ.

On the other hand, using (5.1) and (5.2), we obtain

$$
\begin{aligned}
\varrho_{B^\phi}(k_nf_n-h_ng_n) &\leq \varrho_{B^\phi}((k_nf_n-h_ng_n)\chi_{G_n})+\varrho_{B^\phi}((k_nf_n-h_ng_n)\chi_{E_n})\\
&\quad+\varrho_{B^\phi}((k_nf_n-h_ng_n)\chi_{F_n})\\
&\leq \phi(2\alpha)+2\varepsilon d+\frac{d}{2\delta k}\,(2-\|f_n+g_n\|_{B^\phi}).
\end{aligned}
$$

Suppose now that $\|f_n+g_n\|_{B^\phi}\to 2$ as $n\to\infty$. We have

$$\overline{\lim_{n\to\infty}}\,\varrho_{B^\phi}(k_nf_n-h_ng_n)\leq\phi(2\alpha)+2\varepsilon d.$$

But, since α and ε are arbitrarily small, it follows that

$$\lim_{n\to\infty}\|k_nf_n-h_ng_n\|_{B^\phi}=0.$$

We now show that in fact we have $\lim_{n\to\infty}\|f_n-g_n\|_{B^\phi}=0$. Indeed, this comes from the inequalities

$$
\begin{aligned}
\|f_n-g_n\|_{B^\phi} &\leq \|k_nf_n-k_ng_n\|_{B^\phi} \leq \|k_nf_n-h_ng_n\|_{B^\phi}+\|h_ng_n-k_ng_n\|_{B^\phi}\\
&\leq \|k_nf_n-h_ng_n\|_{B^\phi}+|h_n-k_n|\\
&\leq \|k_nf_n-h_ng_n\|_{B^\phi}+|\,\|k_nf_n\|_{B^\phi}-\|h_ng_n\|_{B^\phi}|\\
&\leq 2\|k_nf_n-k_ng_n\|_{B^\phi}.
\end{aligned}
$$

Necessity. Suppose the Banach space (B^{ϕ}-a.p., $\|\!|f|\!\|_{B^{\phi}}$) is uniformly convex. Then by a classical result it is reflexive. But we know that the Δ_2-condition on ϕ is necessary for the reflexivity of $E^{\phi}([0,1])$ (cf. [1]); using Lemma 4.4, we deduce that it is also necessary for the reflexivity of B^{ϕ}-a.p.

Now, since ϕ is of Δ_2-type, the mapping

$$i : (L^{\phi}([0,1]), \|\!|\cdot|\!\|_{\phi}) \to (B^{\phi}\text{-a.p.}, \|\!|\cdot|\!\|_{B^{\phi}})$$

is a modular isometry for the respective norms (see Lemma 4.4). Then from the uniform convexity of $L^{\phi}([0,1])$ it follows that ϕ must be uniformly convex. ■

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