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NONMEASURABLE ALGEBRAIC SUMS OF SETS OF REALS

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Abstract. We present a theorem which generalizes some known theorems on the existence of nonmeasurable (in various senses) sets of the form $X+Y$. Some additional related questions concerning measure, category and the algebra of Borel sets are also studied.

Sierpiński showed in [14] that there exist two sets $X, Y \subseteq \mathbb{R}$ of Lebesgue measure zero such that their algebraic sum, i.e. the set $X+Y=\{x+y : x\in X,\ y\in Y\}$, is nonmeasurable. The analogous result is also true for the Baire property.

Sierpiński's construction has been generalized to other σ-algebras and σ-ideals of subsets of $\mathbb{R}$. Kharazishvili proves in [10] that for every σ-ideal $\mathcal{I}$ which is not closed under algebraic sums and every σ-algebra $\mathcal{A}$ such that the quotient algebra $\mathcal{A}/\mathcal{I}$ satisfies the countable chain condition, there exist sets $X, Y \in \mathcal{I}$ such that $X+Y \notin \mathcal{A}$. A similar result was proved by Cichoń and Jasiński in [3] for every σ-ideal $\mathcal{I}$ with coanalytic base and the algebra $\mathrm{Bor}[\mathcal{I}]$ (the smallest algebra containing $\mathcal{I}$ and Bor).

Ciesielski, Fejzić and Freiling prove in [4] a stronger version of Sierpiński's theorem. They show that for every set $C \subseteq \mathbb{R}$ such that $C+C$ has positive outer measure there exists $X \subseteq C$ such that $X+X$ is not Lebesgue measurable. In particular, starting with such a set C of measure zero (the "middle third" Cantor set in $[0,1]$ for example), we obtain Sierpiński's example as a corollary.

In Section 1 of our paper we introduce an elementary notion of the Perfect Set Property of pairs $\langle \mathcal{I}, \mathcal{A}\rangle$, where $\mathcal{I}$ is a σ-ideal of subsets of $\mathbb{R}$ and $\mathcal{A} \supseteq \mathcal{I}$ is any family of subsets of $\mathbb{R}$. Using a simple argument, we generalize the results of Sierpiński, Cichoń–Jasiński and Ciesielski–Fejzić–Freiling to pairs with the Perfect Set Property.

The main result of Section 2 is a stronger version of this theorem for measure and category. Namely, we show that if C is a measurable set such that $C+C$ does not have measure zero, then we can find a measure zero set

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$X \subseteq C$ such that $X + X$ is nonmeasurable. The analogue for Baire category is also proved.

In Section 3 similar questions concerning the algebra of Borel sets are studied. Although it is known that this algebra is not closed under taking algebraic sums, we show that there exists an uncountable Borel set $P \subseteq \mathbb{R}$ such that for every pair of Borel sets $A, B \subseteq P$ the set $A + B$ is Borel.

Standard set-theoretic notation and terminology is used throughout the paper. The reader may consult [1] or [9] for basic definitions.

We work in the space $\mathbb{R}$ (viewed as an additive group, with Lebesgue measure). The arguments of Section 1 can be easily generalized to Polish groups which have a structure of a linear space over a countable field. In particular, they remain valid in separable Banach spaces or in the space 2^ω. The results of Section 2 remain valid also in 2^ω, but the author does not know how general they are.

We denote by $\mathcal{M}$ and $\mathcal{N}$ the collections of meager and null sets (the space, its topology and measure should always be clear from the context). Similarly, $\mathcal{M}^*, \mathcal{N}^*$ stand for the collections of co-meager and full measure sets. $\mathcal{BP}$ is the collection of sets with the Baire property and $\mathcal{LM}$ is the collection of Lebesgue measurable sets.

The symbol Bor denotes the σ-algebra of Borel sets. For a σ-ideal $\mathcal{I}$ by $\mathrm{Bor}[\mathcal{I}]$ we denote the smallest σ-algebra containing Bor and $\mathcal{I}$. Observe that $\mathcal{LM} = \mathrm{Bor}[\mathcal{N}]$ and $\mathcal{BP} = \mathrm{Bor}[\mathcal{M}]$. We say that a σ-ideal has a *co-analytic base* if it has a base consisting of $\mathbf{\Pi}^1_1$ sets.

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1. Perfect Set Property

DEFINITION 1.1. Let $\mathcal{I}$ be a σ-ideal of subsets of $\mathbb{R}$ and let $\mathcal{A} \subseteq \mathcal{P}(\mathbb{R})$ be any family of sets containing $\mathcal{I}$. We say that the pair $\langle \mathcal{I}, \mathcal{A} \rangle$ has the *Perfect Set Property* if every set $X \in \mathcal{A} \setminus \mathcal{I}$ contains a perfect set.

The following theorem is the main result of this section.

THEOREM 1.2. *Suppose that a pair $\langle \mathcal{I}, \mathcal{A} \rangle$ has the Perfect Set Property. Then*

- *for every set $A \subseteq \mathbb{R}$ such that $A + A \notin \mathcal{I}$ there exists a set $X \subseteq A$ such that $X + X \notin \mathcal{A}$,*
- *for every pair of sets $A, B \subseteq \mathbb{R}$ such that $A + B \notin \mathcal{I}$ there exist $X \subseteq A$ and $Y \subseteq B$ such that $X + Y \notin \mathcal{A}$.*

As an immediate corollary we get

COROLLARY 1.3. *Suppose that a pair $\langle\mathcal{I},\mathcal{A}\rangle$ has the Perfect Set Property. Then the following conditions are equivalent*:

- $\exists X\in\mathcal{I}\ \ X+X\notin\mathcal{I}$,
- $\exists X\in\mathcal{I}\ \ X+X\notin\mathcal{A}$,
- $\exists X,Y\in\mathcal{I}\ \ X+Y\notin\mathcal{I}$,
- $\exists X,Y\in\mathcal{I}\ \ X+Y\notin\mathcal{A}$. ■

To prove Theorem 1.2 we need the following simple observation.

LEMMA 1.4. *There exists a subgroup $G\subseteq\mathbb{R}$ such that $|\mathbb{R}/G|=\omega$ and G is a Bernstein set in $\mathbb{R}$.*

Proof. First observe that if a proper subgroup $G\subseteq\mathbb{R}$ intersects every perfect set then G is a Bernstein set. Indeed, if there exists a perfect set $P\subseteq G$ then $(x+P)\cap G=\emptyset$ for any $x\notin G$.

We inductively construct a $\mathbb{Q}$-linear subspace G' of $\mathbb{R}$ which contains $\mathbb{Q}$, intersects every perfect set and does not contain $\sqrt{2}$. Finally, we can extend G' to a maximal subspace G not containing $\sqrt{2}$. It is easy to check that $|\mathbb{R}/G|=\omega$. ■

Proof of Theorem 1.2. First observe that if G is as in Lemma 1.4 then the union of a finite nonempty family $\mathcal{T}$ of cosets of G is a Bernstein set. Indeed, each coset is a Bernstein set, so $\bigcup\mathcal{T}$ intersects every perfect set. On the other hand, as $\mathcal{T}$ is finite, there exists a coset $T\notin\mathcal{T}$. But T is also a Bernstein set and $T\cap\bigcup\mathcal{T}=\emptyset$, so $\bigcup\mathcal{T}$ cannot contain any perfect set.

Now, assume that $A+A\notin\mathcal{I}$ and let G be as above. Let us fix a 1-1 enumeration $\mathbb{R}/G=\{T_n:n\in\omega\}$ and put $A_n=A\cap T_n$. As $A+A=\bigcup_{n,m\in\omega}(A_n+A_m)\notin\mathcal{I}$, there exist $n,m\in\omega$ such that $A_n+A_m\notin\mathcal{I}$. Let $X=A_n\cup A_m$. We have

$$X+X=(A_n+A_n)\cup(A_n+A_m)\cup(A_m+A_m)\notin\mathcal{I}.$$

We see that $X+X$ intersects at most three cosets of G, so it cannot contain a perfect set. As $X+X\notin\mathcal{I}$, by the Perfect Set Property, $X+X\notin\mathcal{A}$.

The proof of the second part is similar. ■

As the pairs $\langle\mathcal{N},\mathcal{LM}\rangle$ and $\langle\mathcal{M},\mathcal{BP}\rangle$ have the Perfect Set Property, we immediately obtain the following corollaries.

COROLLARY 1.5 (Ciesielski–Fejzić–Freiling). *If $A\subseteq\mathbb{R}$ is a set such that $A+A$ has positive outer measure then there exists $X\subseteq A$ such that $X+X$ is nonmeasurable.*

COROLLARY 1.6. *If $A\subseteq\mathbb{R}$ is a set such that $A+A$ is nonmeager then there exists $X\subseteq A$ such that $X+X$ does not have the Baire property.*

REMARK 1.1. To prove only the preceding two corollaries concerning measure and category, a simpler argument can be used. In these cases, instead of the group constructed in Lemma 1.4 one can use any dense subgroup of $\mathbb{R}$ of countable index. Such a group can be easily obtained using a Hamel basis. The further part of the proof follows the same pattern; the only observation needed is the fact that the union of finitely many cosets has inner measure zero (does not contain a nonmeager Borel set, respectively).

As corollaries we can also obtain a little stronger versions of the main theorem of [3].

COROLLARY 1.7. *Suppose that $\mathcal{I} \subseteq \mathcal{P}(\mathbb{R})$ is a σ-ideal with a co-analytic base, containing all singletons. Then*

- *for every set $A \subseteq 2^\omega$ such that $A+A \notin \mathcal{I}$ there exists $X \subseteq A$ such that $X + X \notin \mathrm{Bor}[\mathcal{I}]$,*
- *for every pair of sets A, B such that $A+B \notin \mathcal{I}$ there exist $X \subseteq A$ and $Y \subseteq B$ such that $X + Y \notin \mathrm{Bor}[\mathcal{I}]$.*

COROLLARY 1.8. *Let $\mathcal{I} \subseteq \mathcal{P}(\mathbb{R})$ be a σ-ideal with a co-analytic base, containing all singletons. Then the following conditions are equivalent:*

(1) $\exists A \in \mathcal{I}\ \ A + A \notin \mathcal{I}$,
(2) $\exists A \in \mathcal{I}\ \ A + A \notin \mathrm{Bor}[\mathcal{I}]$,
(3) $\exists A, B \in \mathcal{I}\ \ A + B \notin \mathcal{I}$,
(4) $\exists A, B \in \mathcal{I}\ \ A + B \notin \mathrm{Bor}[\mathcal{I}]$.

Proof. It is enough to observe that the pair $\langle \mathcal{I}, \mathrm{Bor}[\mathcal{I}]\rangle$ has the Perfect Set Property. Let $X \in \mathrm{Bor}[\mathcal{I}] \setminus \mathcal{I}$. We can find a Borel set B such that $B \triangle X \in \mathcal{I}$; obviously $B \notin \mathcal{I}$. Let $C \in \mathcal{I}$ be a $\mathbf{\Pi}^1_1$ set such that $B \triangle X \subseteq C$. Then $B \setminus C$ is a $\mathbf{\Sigma}^1_1$ set which is not in $\mathcal{I}$. In particular, this set is uncountable, so it contains a perfect set P. As P is disjoint from $B \triangle X$, we have $P \subseteq X$. ∎

Another application of Theorem 1.2 concerns Marczewski measurable sets.

DEFINITION 1.9. A set X is

- *Marczewski null* ($X \in (s_0)$) if for every perfect set P there exists a perfect set $Q \subseteq P$ such that $Q \cap X = \emptyset$,
- *Marczewski measurable* ($X \in (s)$) if for every perfect set P there exists a perfect set $Q \subseteq P$ such that either $Q \cap X = \emptyset$, or $Q \subseteq X$.

Additive properties of Marczewski measurable sets have been studied in the literature. It is known that the σ-ideal (s_0) is not closed under algebraic sums. This is probably folklore; it follows easily from some results of [11]. Filipów and Dorais in [5] construct a set $X \in (s_0)$ such that $X + X \notin (s)$.

As the pair $\langle (s_0), (s)\rangle$ clearly has the Perfect Set Property, we obtain the following.

Corollary 1.10.

- *For every set $A \subseteq \mathbb{R}$ such that $A + A \notin (s_0)$ there exists $X \subseteq A$ such that $X + X \notin (s)$.*
- *For every pair of sets $A, B \subseteq \mathbb{R}$ such that $A + B \notin (s_0)$ there exist $X \subseteq A$ and $Y \subseteq B$ such that $X + Y \notin (s)$.*

This argument can also be generalized to some other σ-ideals and σ-algebras having similar definitions. For instance, the pair $\langle (cr_0), (cr) \rangle$ of completely Ramsey-null and completely Ramsey subsets of 2^ω (see [9]) has the Perfect Set Property.

2. More on measure and category. The starting point of our paper was Sierpiński's example: there exist two measure zero sets X, Y such that $X + Y$ is nonmeasurable. In the previous section we showed that given any pair of sets A, B such that $A + B$ has positive measure we can find $X \subseteq A$ and $Y \subseteq B$ such that $X+Y$ is nonmeasurable (and the analogue for category as well). One may ask whether we can strengthen our theorems to obtain measure zero (or meager) subsets X, Y of given sets A, B such that $A + B$ has positive outer measure (is nonmeager, respectively). This turns out to be false with an obvious counterexample of $A = \mathbb{Q}$ and $B = \mathbb{R}$. Also, under CH, a Sierpiński set X such that $X + X = \mathbb{R}$ is a counterexample for measure and a Lusin set with this property is a counterexample for category (see [1] for constructions of such sets). In this section, we obtain a positive answer by imposing some additional restrictions on A and B.

Our underlying space will still be $\mathbb{R}$. One can easily see from the proofs that the arguments work in the Cantor space as well.

We begin with the results for Baire category.

Theorem 2.1. *Let A, B be nonmeager sets with the Baire property. Then there exist meager sets $X \subseteq A$ and $Y \subseteq B$ such that $X + Y$ does not have the Baire property.*

Proof. We will need the following lemma:

Lemma 2.2. *Let $G \subseteq \mathbb{R}$ be co-meager. Then there exist meager sets $F_0, F_1 \subseteq G$ such that $F_0 + F_1 = \mathbb{R}$.*

Proof. Without loss of generality we can assume that $G = G + \mathbb{Z}$. By a routine argument, this allows us to work in $[0, 1]$ with addition mod 1 instead of $\mathbb{R}$.

Let $\varphi : 2^\omega \to [0, 1]$ be given by the formula

$$\varphi(x) = \sum_{n=0}^{\infty} \frac{x(n)}{2^{n+1}}.$$

Then $G' = \varphi^{-1}[G]$ is a co-meager subset of 2^ω. Thus, by Lemma 2.2.4 of [1] there exists a partition of ω into consecutive finite intervals $\langle I_n : n \in \omega\rangle$ and $x_G \in 2^\omega$ such that

$$G' \supseteq \{x \in 2^\omega : \exists^\infty n\ \ x{\restriction}I_n = x_G{\restriction}I_n\}.$$

Let $F_0^* = \{x \in 2^\omega : \forall n \in \omega\ \ x{\restriction}I_{2n} = x_G{\restriction}I_{2n}\}$ and $F_1^* = \{x \in 2^\omega : \forall n \in \omega\ \ x{\restriction}I_{2n+1} = x_G{\restriction}I_{2n+1}\}$. We put $F_0 = \varphi[F_0^*]$ and $F_1 = \varphi[F_1^*]$.

Let $z \in [0,1]$ and $y = z - \varphi(x_G)$. Fix $y^* \in 2^\omega$ such that $\varphi(y^*) = y$. We define $x_0^* \in F_0^*$ and $x_1^* \in F_1^*$ as follows:

$$x_0^*{\restriction}I_n = \begin{cases} y^*{\restriction}I_n & \text{for } n \text{ odd},\\ x_G^*{\restriction}I_n & \text{for } n \text{ even},\end{cases}$$

$$x_1^*{\restriction}I_n = \begin{cases} x_G^*{\restriction}I_n & \text{for } n \text{ odd},\\ y^*{\restriction}I_n & \text{for } n \text{ even}.\end{cases}$$

It is easy to check that

$$\varphi(x_0^*) + \varphi(x_1^*) = \varphi(y^*) + \varphi(x_G) = y + \varphi(x_G) = z. \ \blacksquare$$

Returning to the proof of Theorem 2.1, first observe that we can assume that A, B are co-meager. Indeed, let us consider $A' = A + \mathbb{Q}$, $B' = B + \mathbb{Q}$. If we can find meager sets $X' \subseteq A'$ and $Y' \subseteq B'$ such that $X' + Y'$ is nonmeasurable, we put $X_q = X' \cap (A + q)$, $Y_q = Y' \cap (B + q)$, for $q \in \mathbb{Q}$. Then there exist $p, q \in \mathbb{Q}$ such that $X_p + Y_q$ is nonmeasurable. We put $X = X_p - p$ and $Y = Y_q - q$.

Assuming that A, B are co-meager, we apply Lemma 2.2 to $G = A \cap B$. We obtain meager sets $F_0 \subseteq A$ and $F_1 \subseteq B$ such that $F_0 + F_1 = \mathbb{R}$. Now we apply Theorem 1.2 to F_0 and F_1 to obtain X and Y as needed. $\blacksquare$

COROLLARY 2.3. *Suppose that a set $A \subseteq \mathbb{R}$ has the Baire property and $A + A$ is nonmeager. Then there exists a meager set $X \subseteq A$ such that $X + X$ does not have the Baire property.*

Proof. If A is meager we simply apply Corollary 1.6. If not, from the previous theorem we get two meager sets $X_0, X_1 \subseteq A$ such that $X_0 + X_1$ is nonmeager. Then $X' = X_0 \cup X_1 \subseteq A$ is meager as well and $X' + X'$ is nonmeager. Now apply Corollary 1.6 to X'. $\blacksquare$

Now we are going to prove the analogue of Theorem 2.1 for measure. The proof is more complicated than for category. We will use the following theorem.

THEOREM 2.4 (Carlson, [2]). *Let $M \models$ ZFC and let $c \in \mathbb{R}$ be a Cohen real over M. Then in $M[c]$ there exists a full measure set $D \subseteq \mathbb{R}$ such that $\forall t \in \mathbb{R}\ \ |(t + D) \cap (\mathbb{R})^M| \le \omega$.*

The following fact follows from Lemma 9 of [6] as well as from Lemma 3 of [12].

LEMMA 2.5. *For every measure zero set $N \subseteq \mathbb{R}$ there exists a perfect set $P \subseteq \mathbb{R}$ such that $0 \in P$ and $P + N \in \mathcal{N}$.*

THEOREM 2.6. *Let A, B be measurable sets with positive measure. Then there exist null sets $X \subseteq A$ and $Y \subseteq B$ such that $X + Y$ is nonmeasurable.*

Proof. The proof is analogous to the category case. We only need to prove a (weaker) analogue of Lemma 2.2.

LEMMA 2.7. *For every set $G \in \mathcal{N}^*$ there exists a measure zero set $H \subseteq G$ such that $H + H$ has full measure.*

Proof. Without loss of generality we can assume that $G + \mathbb{Q} = G$ and $G = -G$. Let M be a countable transitive model of ZFC* such that $G \in M$ (precisely: we assume that G is a full measure Borel set coded in M). Let $P \in M$ be a perfect set such that $0 \in P$ and $(\mathbb{R} \setminus G) - P \in \mathcal{N}$.

Working in M, consider the set $\bigcap_{p \in P}(G - p) \subseteq G$. By the choice of P this set has full measure, so we can find a full measure Borel set $\widetilde{G} \subseteq \bigcap_{p \in P}(G - p) \subseteq G$. Observe that for every $p \in P$ we have $p + \widetilde{G} \subseteq G$. Indeed, if $x \in \widetilde{G}$ then $x \in G - p$ so $p + x \in G$. Moreover, for a fixed $p \in P \cap M$ this property of the sets $\widetilde{G}, G$ holds in every generic extension of M.

Let $c \in \mathbb{R}$ be a Cohen real over M. Working in $M[c]$, let $D \in \mathcal{N}^*$ be as in Theorem 2.4. We put $H_0 = G \setminus D$ and $H_1 = G^M = G \cap (\mathbb{R})^M$. Both sets have measure zero, the first as a difference of two full measure sets, the second as a subset of $(\mathbb{R})^M$ which is known to have measure zero in $M[c]$ (see [1]).

We want to show that (in $M[c]$) the set $H_0 + H_1 = \{t \in \mathbb{R} : (t + H_0) \cap H_1 \neq \emptyset\}$ has full measure. Observe that from the choice of D and H_1 we have $\{t \in \mathbb{R} : (t + G) \cap H_1 \geq \omega_1\} \subseteq \{t \in \mathbb{R} : (t + H_0) \cap H_1 \neq \emptyset\}$, so it is sufficient to show that the first set has full measure.

To see this, fix $x \in \widetilde{G}^M = \widetilde{G} \cap M$ and take any $t \in x - \widetilde{G}$. Observe that $x \in (t + \widetilde{G}) \cap G^M$, so for every $p \in P^M$ we have $x + p \in (t + G) \cap G^M$. Thus $x + P^M \subseteq (t + G) \cap G^M$. As $M[c] \models |P^M| > \omega$, this shows that for every $t \in x - \widetilde{G}$ the set $(t + G) \cap H_1$ is uncountable, so

$$\begin{aligned} H_0 + H_1 = \{t \in \mathbb{R} : (t + H_0) \cap H_1 \neq \emptyset\} &\supseteq \{t \in \mathbb{R} : (t + G) \cap H_1 \geq \omega_1\} \\ &\supseteq x - \widetilde{G} \in \mathcal{N}^*. \end{aligned}$$

Finally, we put $H = H_0 \cup H_1$. Enlarging H if necessary we may assume that it is Borel and $\mathbb{Q}$-invariant, i.e. $\mathbb{Q} + H = H$. Clearly $H + H$ is analytic and $\mathbb{Q}$-invariant, thus $M[c] \models H + H \in \mathcal{N}^*$. As having positive measure is a $\mathbf{\Sigma}^1_1$ property of a code of an analytic set (by the Kondô–Tugué theorem applied to a universal $\mathbf{\Sigma}^1_1$ set, see [9]), also in $\mathbf{V}$ we have $H + H \in \mathcal{N}^*$. ■

This completes the proof of Theorem 2.6. ■

Corollary 2.8. *Suppose that $A \subseteq \mathbb{R}$ is a measurable set. Then there exists a measure zero set $X \subseteq A$ such that $X + X$ is nonmeasurable.*

Proof. Analogous to the proof of Corollary 2.3. ■

Remark 2.1. The proof of Lemma 2.7 seems to be too complicated compared with Lemma 2.2. We have not been able to give a simpler general argument.

In particular we could not find any variation of Sierpiński's argument from [14] which would work in this case. To give an example of measure zero sets X, Y such that $X + Y$ is nonmeasurable, Sierpiński considers a Hamel basis which has measure zero. He finds such a basis as a maximal linearly independent subset of a measure zero set N such that $N + N = \mathbb{R}$. To adapt this argument for our purposes, we would need N to be a subset of a given full measure set G—but this is even stronger than the lemma we are proving. A similar problem appears when one tries to modify the arguments from [3].

Corollaries 2.8 and 2.3 seem to give us the full picture when considering algebraic sums of the form $A + A$. In Theorems 2.1 and 2.6 we assume that both sets A, B considered are positive (nonmeager or of positive measure). Of course, if both sets are meager (for category) or null (for measure) then Theorem 1.2 gives us the same conclusion. There remains, however, the case when both sets are measurable (or have the Baire property) but only one is positive. The obvious example of $A = \mathbb{Q}$ and $B = \mathbb{R}$ shows that we need to assume that both sets are large in some sense. It is a reasonable conjecture that it is enough to assume that one of the sets is positive and the other contains a perfect set. This motivates the following questions.

Question 1. Suppose that $A \subseteq \mathbb{R}$ is a non-meager set with the Baire property and P is perfect. Do there exist meager sets $X \subseteq A$ and $Y \subseteq P$ such that $X + Y$ is nonmeager?

Question 2. Suppose that $A \subseteq \mathbb{R}$ is a measurable set with positive measure and P is perfect. Do there exist measure zero sets $X \subseteq A$ and $Y \subseteq P$ such that $X + Y$ is not null?

Recall also that it is not trivial to prove that for every perfect set P there exists a (closed) measure zero set H such that $P + H = \mathbb{R}$ (see [6]). This result may suggest that affirmative answers to these questions are to be expected. A natural attempt at such an answer would be to construct H as a subset of a given positive set A; it is not clear, however, how to modify the proof from [6] for this purpose.

3. Borel sets. One might ask whether some analogous results are true for the algebra of Borel sets. Erdős and Stone in [7] and, independently, Rogers in [13] gave an example of two Borel sets whose algebraic sum is not Borel. This topic was also considered recently by Cichoń and Jasiński in [3].

Obviously, if sets A, B are Borel, nonempty, and one of them is uncountable then there exist $X \subseteq A$ and $Y \subseteq B$ such that $X + Y$ is not Borel. Assuming that B is uncountable, simply take $X = \{x\}$ for any $x \in A$ and Y any non-Borel subset of B. Using Theorem 1.2 we can also show

PROPOSITION 3.1. *For every uncountable set $A \subseteq \mathbb{R}$ there exists a set $X \subseteq A$ such that $X + X$ is not analytic.*

Proof. The pair $\langle [\mathbb{R}]^{\leq\omega}, \mathbf{\Sigma}^1_1 \rangle$ has the Perfect Set Property. ∎

The main disadvantage of this proposition is that it does not give us any information on the descriptive complexity of X, even if we assume that A is Borel. One might, however, conjecture that for every pair A, B of uncountable Borel sets there exist Borel $X \subseteq A$ and $Y \subseteq B$ such that $X + Y$ is not Borel. We show that this is not the case. Our argument is largely inspired by arguments of Recław from [12] (similar arguments are also used in [3]).

PROPOSITION 3.2. *There exists a perfect set $P \subseteq \mathbb{R}$ such that for every pair of Borel sets $A, B \subseteq P$ the set $A + B$ is Borel. In particular, for every Borel $A \subseteq P$ the set $A + A$ is Borel.*

Proof. Let $P \subseteq \mathbb{R}$ be a perfect set linearly independent over $\mathbb{Q}$. It is a matter of simple calculation that for all pairs $\langle p_0, q_0 \rangle$, $\langle p_1, q_1 \rangle$ in P^2, if $p_0 + q_0 = p_1 + q_1$ then either $\langle p_0, q_0 \rangle = \langle p_1, q_1 \rangle$, or $\langle p_0, q_0 \rangle = \langle q_1, p_1 \rangle$.

Let $C^* = \{\langle p, q \rangle \in P^2 : p \leq q\}$ and $C_* = \{\langle p, q \rangle \in P^2 : p > q\}$. If $A, B \subseteq P$ are Borel, then

$$A + B = \{a + b : \langle a, b \rangle \in (A \times B) \cap C_*\} \cup \{a + b : \langle a, b \rangle \in (A \times B) \cap C^*\}.$$

As the function $\langle p, q \rangle \mapsto p + q$ restricted to C^*, as well as to C_*, is 1-1, $A + B$ is the union of two 1-1 continuous images of Borel sets, thus Borel. ∎

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