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ON FINITENESS CONDITIONS FOR SUBALGEBRAS WITH ZERO MULTIPLICATION

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Abstract. Let F be a commutative ring with unit. In this paper, for an associative F-algebra A we study some properties forced by finite length or DCC condition on F-submodules of A that are subalgebras with zero multiplication. Such conditions were considered earlier when F was either a field or the ring of rational integers.

In the final section, we consider algebras with maximal commutative subalgebras of finite length as F-modules and obtain some results parallel to those known for ACC condition or finite generation.

1. Motivation. In this note F stands for a commutative ring with $1 \neq 0$. Associative F-algebras, not necessarily with 1, will be named *algebras* for short. If A is an algebra and $1 \notin A$, then by A^1 we denote the standard extension of A to a unital algebra with the help of F. If $1 \in A$, then let simply $A^1 = A$. In both cases any algebra ideal of A is an ideal of A^1. An algebra is called *reduced* if it has no nontrivial nilpotent elements. Subalgebras with zero multiplication will be named *zero subalgebras*.

In [8, 9], among other results, T. J. Laffey proved the following one:

Theorem 1.1. *Let A be an algebra over a field F. If every zero subalgebra of A is finite-dimensional over F, then either all nilpotent elements of A are contained in a finite-dimensional ideal of A, or A has an infinite-dimensional ideal J such that $J^3 = 0$ and J contains an infinite-dimensional commutative ideal.*

Applying this theorem to semiprime algebras it is not difficult to obtain the following consequence, reproved in [11]:

Theorem 1.2. *Let A be a semiprime algebra over a field F. Then every zero subalgebra of A is finite-dimensional over F if and only if A is a direct sum of a finite-dimensional algebra and a reduced algebra.*

Let $\mathbb{Z}$ denote the ring of rational integers. Then a theorem from [7] can be formulated in the following way.

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Theorem 1.3. *Let $F = \mathbb{Z}$ and let A be a semiprime F-algebra. Then every zero subalgebra of A is an F-module of finite length if and only if A is a direct sum of an ideal of finite length as an F-module and a reduced algebra.*

In this note, we are going to extend the above results. We also show that to have only zero subalgebras of finite length or only with DCC on submodules over F is a very strong condition, not only in the semiprime case and not only over a field. Our investigation was motivated by papers [11, 7], but our approach will rather follow [5, 6, 8, 9]. In particular, we are not going to use generalized polynomial identities.

In the final section we apply the results obtained to study connections between the module structure of an algebra and its commutative subalgebras, in the spirit of [10] and [1].

2. On finiteness conditions. In what follows, F-modules will be named modules, for short. By $\mathcal{F}$ we denote the class of all modules of finite length and by $\mathcal{D}$ the class of all modules with the DCC condition on submodules. Algebras will often be identified with their underlying modules. For example, if A is an algebra with DCC on submodules, then we will often write $A \in \mathcal{D}$.

It is evident that $\mathcal{F} \subseteq \mathcal{D}$ and, if $F \in \mathcal{D}$, then $F \in \mathcal{F}$. In this case we have $\mathcal{F} = \mathcal{D}$ (see [2]). Our further notation and terminology are rather standard, as for example in [5].

In this section we adapt some classical results on rings with finiteness conditions presented for example in [2, 5] to algebras with analogous conditions for submodules.

Lemma 2.1. *Let $A \in \mathcal{D}$ be an algebra and I be the annihilator of A in F. If A is prime, then F/I is a field and A is a simple algebra finite-dimensional over F/I. In particular $A \in \mathcal{F}$ and it is unital.*

Proof. Since A is prime, F/I is a domain and A is torsion free as an F/I-module. Moreover, if $0 \neq a \in A$, then the modules F/I and $(F/I)a$ are isomorphic. Hence, by assumption, $F/I \in \mathcal{D}$ and F/I is a field, being a domain. This means that A is a finite-dimensional algebra over F/I and the result follows by the primeness of A. ∎

Lemma 2.2. *Let $A \in \mathcal{D}$ be a semiprime algebra. Then A is a unital algebra with DCC on left and right ideals. Thus A is a finite direct sum of simple algebras and $A \in \mathcal{F}$. Hence, for every $a \in A$ the modules Aa and aA have the same length.*

Proof. Let I be a minimal annihilator of a central idempotent, say f, in A. Then $I = (1 - f)A \in \mathcal{D}$ and I is a semiprime algebra. Assume that

$I \neq 0$. If J is a minimal algebra ideal of I, then it is easy to check that J is a prime algebra. Hence, by the previous lemma, J has a unit, say e. Thus $0 \neq e$ is a central idempotent in I and also in A. We have $J = eI$ and $(1-e)I$ is an annihilator of e in I. It can be checked that $e+f$ is a central idempotent in A and $(1-e)I = (1-e)(1-f)A = (1-(e+f))A$ is an annihilator in A of $e+f$, properly containing I, a contradiction. Thus f is the unit element of A, and DCC on submodules forces DCC on one-sided ideals. Hence A is a finite direct sum of simple algebras with DCC on submodules, and Lemma 2.1 yields $A \in \mathcal{F}$.

The last statement can be proved as an application of Lemma 2.1, in fact by reduction to matrices over fields and over finite-dimensional division algebras. ∎

As another analogue of a classical result on Artin rings (see [5]) we have

COROLLARY 2.3. *Let $A \in \mathcal{D}$ be an algebra with a one-sided unit. Then $A \in \mathcal{F}$.*

Proof. Let $e \in A$ be a left unit. Then all left ideals of A are algebra ideals, and since $A \in \mathcal{D}$, our algebra satisfies DCC on left ideals. Hence it is of finite length as a left A-module by the Hopkins–Levitski theorem.

If $I \subset J$ are left ideals of A such that $M = J/I$ is a simple A-module, then $eM = M$ and, by Lemma 2.1, M is of finite length over F. Hence $A \in \mathcal{F}$.

The case when A has a right unit can be proved analogously. ∎

In what follows, if A is an algebra, then we denote by $N(A)$ the largest nil-ideal of A and by $P(A)$ the smallest semiprime ideal (the prime radical) of A. Clearly $P(A) \subseteq N(A)$ and both are algebra ideals of A.

Now, with the help of Lemma 2.2, we can obtain another analogue of a classical result on Artin rings.

COROLLARY 2.4. *Let $A \in \mathcal{D}$ be an algebra. Then there exists an idempotent $e \in A$ such that $eA, Ae \in \mathcal{F}$, while $(1-e)A$ and $A(1-e)$ are nilpotent algebras.*

Proof. Assume first $P(A) = A$. Evidently, for any $n \geq 1$ the ring A^n is an algebra ideal of A. Hence, by DCC on submodules, there exists $n \geq 1$ such that

$$A^n = A^{n+1} = \cdots = A^{2n} = A^n A^n.$$

Assume that $A^n \neq 0$. Then by DCC there exists a minimal left algebra ideal $J \subseteq A$ such that $A^n J \neq 0$. Hence $J = A^n J$, and there exists $x \in J$ such that $J = A^n x$ and $0 \neq x = ax$ for some $a \in A^n$. Hence, $x = ax = a^2 x = \cdots$. But A^n is a nil-algebra, a contradiction. Thus necessarily $A^n = 0$. In this case it is enough to put $e = 0$.

Now let $P(A) \neq A$. Then, by Lemma 2.2, $A/P(A)$ is a nontrivial unital algebra. Thus it is enough to lift the unit of this algebra to an idempotent $e \in A$ and to apply Corollary 2.3. ∎

As a consequence we can prove the following result for algebras with no finiteness conditions.

PROPOSITION 2.5. *Let $I \in \mathcal{D}$ be a one-sided ideal of an algebra A. Then $I \in \mathcal{F}$ in any of the following cases*:

- *A is semiprime*;
- *I is a finitely generated one-sided ideal of A*;
- *I is a finitely generated two-sided ideal of A.*

Proof. Let A be semiprime and $I \in \mathcal{D}$ be a left ideal. Then, by Corollary 2.4, $I = Ie \oplus I(1-e)$, where $Ie \in \mathcal{F}$, while $I(1-e)$ is a nilpotent left ideal of A. By the semiprimeness of A, this means that $I = Ie \in \mathcal{F}$. The case when I is a right ideal can be checked analogously.

Now let A be an arbitrary algebra, $I \in \mathcal{D}$ be a finitely generated left ideal of A and let $I = A^1 b_1 + \cdots + A^1 b_n$, where $b_i \in I$. Let J_i, $i = 1, \ldots, n$, be the annihilators of b_i in F. Then $F/J_i \simeq Fb_i \in \mathcal{D}$. If J is the intersection of all J_i, then $F/J \in \mathcal{F}$, because $F/J \subseteq \bigoplus_{i=1}^{n} F/J_i \in \mathcal{D}$. Now I is a module over F/J. Hence $I \in \mathcal{F}$, because $I \in \mathcal{D}$.

The case when I is a finitely generated right ideal, or finitely generated two-sided ideal, can be checked in an analogous way. ∎

Let $M, N \in \mathcal{F}$ be modules. Then it is well known (see [2]) that the module $\mathrm{Hom}_F(M, N)$ also belongs to $\mathcal{F}$. However, this is not true for $\mathcal{D}$ in place of $\mathcal{F}$.

EXAMPLE 2.6. Let $F = \mathbb{Z}$ and p be a prime number. Let $M = N$ be the Prüfer p-group C_p^∞. Then it is known (see [4]) that $\mathrm{Hom}_F(M, N) = \mathrm{End}_F(M)$ is the ring $\mathcal{O}_p$ of p-adic integers, which does not belong to $\mathcal{D}$.

Now let $M = N = C_p^\infty \oplus C_p^\infty$. Then the ring $\mathrm{End}_F(M)$ is isomorphic to the matrix ring $\mathrm{M}_2(\mathcal{O}_p)$. This ring contains as a zero subring the additive group of $\mathcal{O}_p$, isomorphic to the upper right corner of $\mathrm{M}_2(\mathcal{O}_p)$.

Having in mind the above example one can only prove the following result.

LEMMA 2.7. *Let $B \in \mathcal{F}$ be a left* (*resp. right*) *ideal of an algebra A, and C be the left* (*resp. right*) *annihilator of B in A. Then C is an ideal of A and the algebra A/C belongs to $\mathcal{F}$.*

In connection with the class $\mathcal{F}$ we have the following observation (see [9] for a special case).

LEMMA 2.8. *Let $B \subseteq A$ be algebras and I be the largest ideal of A contained in B. If the module A/B is in $\mathcal{F}$, then $A/I \in \mathcal{F}$.*

Proof. Since $A/B \in \mathcal{F}$, it is finitely generated. Thus we can find in A^1 elements $x_1 = 1, x_2, \ldots, x_n$ such that the module A^1/B is generated by the cosets $x_i + B$. For any pair $1 \le i, j \le n$ let $V_{ij} = \{b \in B : x_i b x_j \in B\}$. It can be checked that

$$I = \bigcap_{1\le i,j\le n} V_{ij} \quad \text{and} \quad B/I \subseteq \prod_{1\le i,j\le n} A/B.$$

Thus, by assumption, $B/I \in \mathcal{F}$, hence $A/I \in \mathcal{F}$ as well. ∎

Again, the above lemma does not extend to the case of $\mathcal{D}$. Indeed, let $1 < n \in \mathbb{N}$, $A = \mathbb{Z}[1/n]$ and $F = B = \mathbb{Z}$. Then $A/B \in \mathcal{D}$ and $A/(0) \simeq A \notin \mathcal{D}$, but clearly the only ideal of A contained in B is (0).

3. A useful ideal. Let A be an algebra. Adapting an idea from [9] we introduce the ideal $\Delta(A)$ equal to the sum of all the ideals I of A such that $I \in \mathcal{F}$. If F is a field, then $\Delta(A) = y(A)$, where $y(A)$ is defined as in [9]. In his paper Laffey refers to the analogy with the FC-center of a group, often denoted by Δ, and we will follow this analogy in our notation. We have the following characterizations of the elements of $\Delta(A)$.

LEMMA 3.1. *Let A be an algebra and $a \in A$. Then the following conditions are equivalent*:

(i) $a \in \Delta(A)$.
(ii) $A^1 a A^1 \in \mathcal{D}$.
(iii) $A^1 a + a A^1 \in \mathcal{D}$.
(iv) $A^1 a + a A^1 \in \mathcal{F}$.
(v) $A^1 a A^1 \in \mathcal{F}$.

Proof. Clearly (v)⇒(i)⇒(ii)⇒(iii). The implication (iii)⇒(iv) follows by Proposition 2.5.

(iv)⇒(v). Suppose $A^1 a + a A^1 \in \mathcal{F}$. Then, in particular, aA^1 is finitely generated as a module. If $aA^1 = Fax_1 + \cdots + Fax_n$, where $x_i \in A^1$, then it is easy to check that $A^1 a A^1 = A^1 a x_1 + \cdots + A^1 a x_n$. Hence $A^1 a A^1 \in \mathcal{F}$. ∎

Below we collect general properties of the ideal just defined.

LEMMA 3.2. *Let A be an algebra. Then*:

(i) $\Delta(\Delta(A)) = \Delta(A)$.
(ii) *If $B \subseteq A$ is a subalgebra, then $B \cap \Delta(A) \subseteq \Delta(B)$.*
(iii) *If $B \subseteq A$ is a subalgebra with $A/B \in \mathcal{D}$, then $B \cap \Delta(A) = \Delta(B)$.*
(iv) *If I is an ideal of A, then $(\Delta(A) + I)/I \subseteq \Delta(A/I)$.*
(v) *If $\Delta(A) \in \mathcal{D}$, then $\Delta(A/\Delta(A)) = 0$.*

Proof. Claims (i) and (ii) are evident. Let $B \subseteq A$ be an algebra with $A/B \in \mathcal{D}$ and let $b \in \Delta(B)$. Assume, for simplicity, that $A = A^1$ and $1 \in B$. Then, by the above lemma, $bB \in \mathcal{D}$. If U is the right annihilator of b in A, then $B \cap U$ is the right annihilator of b in B and $bB \simeq B/(B \cap U) \simeq (B+U)/U \in \mathcal{D}$. On the other hand, by assumption, $A/B \in \mathcal{D}$, so $A/(B+U) \in \mathcal{D}$. Hence $bA \simeq A/U \in \mathcal{D}$. In a similar way one can see that $Ab \in \mathcal{D}$. Hence $A^1b + bA^1 \in \mathcal{D}$, because $1 \in A$.

If $1 \notin A$ or B is not a unital subalgebra of A, then similar considerations can be applied after going to A^1 and $B' = B+F1$. In this way, by Lemma 3.1, $b \in \Delta(A)$, $\Delta(B) \subseteq \Delta(A)$ and claim (ii) finishes the argument.

Now let $\Delta(A) = D \in \mathcal{D}$ and $a \in A$ be such that $a + D \in \Delta(A/D)$. This means that $A^1aA^1/(D \cap A^1aA^1) \simeq (A^1aA^1 + D)/D \in \mathcal{D}$. By assumption $D \cap A^1aA^1 \in \mathcal{D}$. Hence $A^1aA^1 \in \mathcal{D}$. In this way, by the above lemma we have shown that $a \in \Delta(A) = D$ and, as a consequence, that $\Delta(A/\Delta(A)) = 0$. ∎

The last conclusion of the above lemma does not extend to the case when $\Delta(A) \notin \mathcal{D}$. Indeed, let F be a field and A be the direct sum of infinitely many copies of F considered as an algebra with pointwise multiplication. It is easy to check that $\Delta(A^1) = A$ but $\Delta(A^1/A) = A^1/A \neq 0$.

It turns out that the condition $A = \Delta(A)$ is a very strong finiteness condition on the algebra A.

LEMMA 3.3. *Let N be a nil-algebra and I be a maximal zero ideal of N. Put*

$$J = \{x \in N \mid xI = Ix = 0\} \quad \textit{and} \quad N_k = \{x \in N \mid xN^k = N^kx = 0\}$$

for any $k \geq 1$. If $N = \Delta(N)$, then:

(i) $N = P(N)$, *because N is the union of the nilpotent ideals N_k.*
(ii) *If $I \in \mathcal{D}$, then N is nilpotent.*
(iii) *If $I \in \mathcal{F}$, then $N/J \in \mathcal{F}$.*

Proof. If $x \in N$, then, by assumption, $xN \in \mathcal{F}$. Thus, there exists $m \geq 1$ with $xN^m = xN^{m+1}$. Hence, $xN^m = 0$, because N is a nil-algebra. In a similar way one can check that $N^lx = 0$ for some $l \geq 1$. In this way we have proved that $x \in N_k$, where $k = \max(m, l)$, and $N = \bigcup_{k=1}^{\infty} N_k$. This means that $N = P(N)$, because $N_k^{k+1} = 0$.

By our definition, $I \subseteq J$ and J is an ideal of N. Suppose that $J^3 \neq 0$. Then there are elements $a, b, c \in J$ such that $abc \neq 0$. Let K be the ideal of N generated by a, b and c. Then, as we already know, $K \subseteq N = \Delta(N)$ and $K^3 \neq 0$. However, by Lemma 3.1 and Corollary 2.4, $K^m = 0$ for some $m > 3$. We can assume that m is minimal with this property. Let $L = K^{m-2}$. Then clearly

$$L^2 = K^{2m-4} = 0 = IL = LI, \quad \text{thus} \quad (I + L)^2 = 0.$$

By the maximality of I, we then have $L \subseteq I$. Hence

$$K^{m-1} = LK \subseteq IK \subseteq IJ = 0.$$

This contradicts the choice of m, and so the assumption $J^3 \neq 0$. Hence $J^3 = 0$.

Now assume that $I \in \mathcal{D}$. Then, by Theorem 1.5 of [3] and the definition of J, one can see that N/J is nilpotent and $N^k \subseteq J$ for some $k \geq 1$. It follows that N is nilpotent, because $J^3 = 0$.

Now assume that $I \in \mathcal{F}$. Then, by Lemma 2.7 and the definition of J, one can see that $N/J \in \mathcal{F}$. ■

The above result cannot be essentially generalized. Indeed, let F be a field. For any $m \geq 1$ the algebra $A_m = F[t]/(t^m)$ is in $\mathcal{F}$ and its prime radical is nilpotent of index m. Hence $A = \bigoplus_{m=1}^{\infty} A_m$ is equal to $\Delta(A)$ and $N(A) = \Delta(N(A))$ is not nilpotent.

One might expect that in the above lemma, if $I \in \mathcal{F}$, then $N(A)$ belongs at least to $\mathcal{D}$. One might also expect that in the above lemma the condition $J^3 = 0$ can be replaced by $J^2 = 0$. However, this is not true in general. Indeed, as a special case of Example 3.11 of [9] we have

Proposition 3.4. *Let F be a linearly ordered field. Then there exists an infinite-dimensional F-algebra A with $\Delta(A) = A$ and $A^3 = 0$, containing an ideal I of dimension one which is the only nontrivial zero subalgebra of A.*

In a semiprime algebra A any element of $\Delta(A)$ can be characterized with the help of only one one-sided ideal. The result below extends also Theorem 2.2 of [12] and Theorems 2.12, 2.13 of [11].

Theorem 3.5. *Let A be a semiprime algebra and let $a \in A$ be such that either $aA \in \mathcal{D}$ or $Aa \in \mathcal{D}$. Then $a \in \Delta(A)$. Moreover, both modules aA and Aa have the same length.*

Proof. Let $aA \in \mathcal{D}$. Then, by Proposition 2.5, we have $aA \in \mathcal{F}$ and $aA = eA$, where e is an idempotent.

Now let B be the right annihilator of aA and C be the left annihilator of B in A. Then B is an ideal of A and by Lemma 2.7, $A/B \in \mathcal{F}$. Clearly $A^1aA^1 \subseteq C$, because $e \in aA \subseteq C$ and C is an ideal of A. By the semiprimeness of A we have $B \cap C = 0$. Hence $A^1aA^1 \subseteq C \in \mathcal{F}$ and $a \in \Delta(A)$.

The second conclusion follows from Lemma 2.2, because A^1aA^1 is a direct summand of A and we can restrict ourselves to this ideal. ■

In the above theorem, the assumption of the semiprimeness of A is essential. Indeed, suppose F fails DCC on ideals, M is a maximal ideal of F, $I = t^3F[t] + Mt^2$ is an ideal of the polynomial algebra $F[t]$, $A = tF[t]/I$ and $a = t + I$. Then $Aa = Fa^2 \simeq F/M$ is a simple module, while $A = A^1a =$

$A^1aA^1 \simeq F \oplus F/M$ as a module, so A does not satisfy DCC on submodules and $a \notin \Delta(A)$.

Let F be a field, and $A = F[a_1, a_2, \ldots]$ be an algebra with relations $a_i a_j = 0$ for all $i \geq j$. Then $Aa_1 \simeq F \in \mathcal{F}$, while $a_1 A$ is infinite-dimensional. In this case also $a_1 \notin \Delta(A)$.

In any semiprime algebra A the ideal $\Delta(A)$ is closely related to the socle $\operatorname{soc}(A)$. As a consequence of Lemmas 2.1, 2.2 and Theorem 3.5 we have the following extension of some results of [11] and [12].

THEOREM 3.6. *Let A be a semiprime algebra with* $\operatorname{soc}(A) = \bigoplus_{s \in S} I_s$, *where I_s are homogeneous components. Then $\Delta(A)$ is the direct sum of all ideals I_s belonging to $\mathcal{F}$. In particular*:

- *If A is prime and $\Delta(A) \neq 0$, then $A = \Delta(A) \in \mathcal{F}$ and A is a simple unital algebra.*
- *If $\Delta(A) \in \mathcal{D}$, then $A = \Delta(A) \oplus B$, where $\Delta(B) = 0$.*

Proof. It is well known (see Theorem 4.3.1 in [6]) that in our notation any I_s is a simple algebra. If $I_s \in \mathcal{F}$, then, by definition, $I_s \subseteq \Delta(A)$. On the other hand, if $a \in \Delta(A)$, then, by Lemma 3.1, $A^1aA^1 \in \mathcal{F}$ and, by assumption, it is a semiprime algebra. Hence, by Lemma 2.2, a belongs to a finite direct sum of ideals I_s belonging to $\mathcal{F}$. Now the final conclusions follow easily. ∎

4. DZS-algebras. Adopting the terminology from [11, 7], we say that an algebra A is an *FZS-algebra* if every zero subalgebra of A belongs to $\mathcal{F}$, and A is a *DZS-algebra* if every zero subalgebra of A belongs to $\mathcal{D}$. FZS-algebras are certainly DZS-algebras. However, by considering modules as zero algebras it can be observed that in general these two classes are different. Here we rather concentrate on DZS-algebras.

Reduced algebras and algebras with DCC condition on submodules are certainly DZS-algebras. One can also easily deduce the following observation.

PROPOSITION 4.1. *Let A be an algebra.*

- *If A is a DZS-algebra* (*resp. an FZS-algebra*) *and $B \subseteq A$ is a subalgebra, then B is a DZS-algebra* (*resp. an FZS-algebra*).
- *If $I \subseteq A$ is an algebra ideal such that I and A/I are DZS-algebras* (*resp. FZS-algebras*), *then A is a DZS-algebra* (*resp. an FZS-algebra*).

If F is a domain and A is a free nonunital algebra with infinitely many free generators, then A is a domain, thus it is an FZS-algebra. Now the algebra A/A^2 is a free module of infinite rank with zero multiplication. Hence it is not even a DZS-algebra. Thus the classes of DZS-algebras and FZS-algebras are not closed under homomorphic images. Even if $A = \Delta(A)$

and A^2 is very small, then, by Proposition 3.4, the algebra A^2 need not be a DZS-algebra.

Some properties of DZS-algebras are similar to those of semiprime algebras. For example, as an analogue of a part of Theorem 3.5, some additional characterizations of elements from $\Delta(A)$ for an arbitrary DZS-algebra A can be given.

LEMMA 4.2. *Let A be a DZS-algebra and $a \in A$. If Aa or aA belongs to $\mathcal{D}$, then $a \in \Delta(A)$.*

Proof. Suppose $a \in A$ and $Aa \in \mathcal{D}$. Let I be the annihilator of a in F and J be the annihilator of a^2 in F. By definition $a^2 \in Aa$, hence, by assumption, $F/J \simeq Fa^2 \in \mathcal{D}$. Moreover, $(Ja)^2 \subseteq Ja^2 = 0$. Thus, by the DZS property, $Ja \simeq J/I \in \mathcal{D}$. Hence $F/I \simeq Fa \in \mathcal{D}$ and $A^1a = Aa + Fa \in \mathcal{D}$.

Now let $V = \{x \in A^1 : xa = 0\}$. Then the module $A^1/V \simeq A^1a$ belongs to $\mathcal{D}$. Furthermore, $aV \subseteq A$ and $(aV)^2 = 0$. If $W = \{x \in V : ax = 0\}$, then, by DZS we have $V/W \simeq aV \in \mathcal{D}$ and consequently, $A^1/W \in \mathcal{D}$. Clearly aA^1 as a module is a homomorphic image of A^1/W, hence $aA^1 \in \mathcal{D}$ and $A^1a + aA^1 \in \mathcal{D}$. By Lemma 3.1, this means that $a \in \Delta(A)$.

If $aA \in \mathcal{D}$, the proof is analogous. ∎

The following generalization of Lemma 3.5 of [9] is very useful for our further considerations.

LEMMA 4.3. *Let A be a DZS-algebra. Then the algebra $A/\Delta(A)$ is reduced. In particular, $\Delta(A)$ contains all nilpotent elements of A. Hence every nil-subalgebra of A is nilpotent.*

Proof. Let $a^2 \in \Delta(A)$ and $V = \{x \in A^1 : a^2x = 0\}$. Then, by assumption, $A^1/V \simeq a^2A^1 \in \mathcal{D}$. Also, by the definition of V, we have $(aVa)^2 = 0$. If $W = \{x \in V : axa = 0\}$, then, by the DZS property, $V/W \in \mathcal{D}$. Hence, $A^1/W \in \mathcal{D}$. If X is the left annihilator of a in W, then $Wa \simeq W/X$ and $(Wa)^2 = 0$. Thus W/X, and hence A/X, belongs to $\mathcal{D}$. Now A^1a as a module is a homomorphic image of A^1/X and also belongs to $\mathcal{D}$. By the lemma above we then have $a \in \Delta(A)$.

Let $B \subseteq A$ be a nil-subalgebra. Then, by the first part of the proof, $B \subseteq \Delta(A)$. Hence, by Lemma 3.2, $B = \Delta(B)$. Now it is enough to apply Lemma 3.3. ∎

Now we turn to results connected with direct decompositions of DZS-algebras and FZS-algebras.

THEOREM 4.4. *Let A be a semiprime DZS-algebra, B be the ideal of A generated by all nilpotent elements, and C the annihilator of B in A. Then $A = B \oplus C$, $B \in \mathcal{F}$ and C is a reduced algebra.*

Proof. By Lemma 4.3 and Theorem 3.6 we have $B \subseteq \Delta(A) \subseteq \mathrm{soc}(A)$. Hence we can write $B = \bigoplus_{s\in S} B_s$, where every B_s is a simple unital algebra belonging to the class $\mathcal{F}$ and contains a nonzero nilpotent element. By the DZS property the above sum is finite. Thus $B \in \mathcal{F}$ and it is a unital algebra. Hence $A = B \oplus C$ and C is reduced, by the definition of B. ∎

Under some assumption not including semiprimeness, a similar direct decomposition can also be obtained. Following an idea from [9] we can prove

LEMMA 4.5. *Let A be a DZS-algebra with $\Delta(A) = I$ and $A/I = B$. If $I \in \mathcal{F}$ and B is a unital algebra having no ideals J with $B/J \in \mathcal{F}$, then $A = I \oplus C$, where C is an ideal, unital and reduced as algebra.*

Proof. Let H be the intersection of the left and the right annihilator of I in A. Clearly H is an ideal of A and, by Lemma 2.7, $A/H \in \mathcal{F}$, and $A/(I+H) \in \mathcal{F}$. Hence, by the assumption, $I+H = A$. Thus, for $L = I\cap H$, we have $H/L \simeq (H+I)/I = A/I = B$. In this way, by assumption, H/L is a unital algebra. By definition $L^2 = 0$, and we can take $e \in H$ as a lifting of the unit of H/L. Then, by Peirce decomposition, we have

$$H = eHe \oplus eH(1-e) \oplus (1-e)He \oplus (1-e)H(1-e)$$

and by the choice of e and $L \subseteq I \in \mathcal{F}$ we deduce that the module $H/(eHe)$ belongs to $\mathcal{F}$. Hence also $A/(eAe) \in \mathcal{F}$. This means, by Lemma 2.8, that for any $a \in A$ the one-sided ideals $(a-ea)A$ and $A(a-ae)$ belong to $\Delta(A)$ and consequently, $A = I + eAe$. Let C be a maximal ideal of A contained in eAe. By Lemma 2.8, $A/C \in \mathcal{F}$ so, by assumption, $A = I + C$.

On the other hand, $I \cap C = 0$, because e acts on C as the unit and annihilates I, by definition. In this way we have $A = I \oplus C$. ∎

For further use let us say that, if A is an algebra and $a \in A$, then a is *algebraic* (*over* F) if the subalgebra $F(a)$ generated by a belongs to $\mathcal{F}$. One can check that any element of $\Delta(A)$ is algebraic. The following observation is rather easy.

PROPOSITION 4.6. *Let A be a reduced agebraic algebra with no infinite set of orthogonal idempotents. Then A is a unital algebra and a direct sum of finitely many division algebras.*

Now we have almost proved some generalizations of theorems formulated in Section 1 and some results of [9]. It is enough to add the following one.

THEOREM 4.7. *Let A be a DZS-algebra, algebraic over F. If $\Delta(A) \in \mathcal{F}$ and A has no infinite set of orthogonal idempotents, then $A = \Delta(A) \oplus C$, where C is a finite direct sum of division rings algebraic over F.*

Proof. Let I, H and L have the same meaning as in the proof of Lemma 4.5. In this case, by assumption, H/L is an algebraic algebra with no infinite set of orthogonal idempotents, because $L^2 = 0$. By Lemma 3.2, $\Delta(A/I) = 0$ and A/I is an algebraic algebra. Hence, by the above proposition, A/I has no ideals J with factor algebra of finite length. Hence A/I is also a unital algebra. Now it is enough to apply the previous two results. ∎

In general, even for FZS-algebras, the assumption $\Delta(A) \in \mathcal{F}$ does not imply a possibility to decompose the algebra as above. Indeed, if F is a field and we take matrices of the form

$$R = \begin{bmatrix} F & F[t] \\ 0 & F[t] \end{bmatrix}, \qquad I = \begin{bmatrix} 0 & tF[t] \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad A = R/I,$$

then $\Delta(A) \in \mathcal{F}$, A is not reduced and has no central idempotents. Hence A is not a direct sum of nonzero ideals.

One could try to introduce the class $\mathcal{A}$ of all F-modules, then the ideal $\Delta_{\mathcal{A}}(A)$ and AZS-algebras, and try to extend the results of Section 1 to AZS-algebras. But this is impossible.

Example 4.8. Let K be a field and $F = K[t]$ be the ring of polynomials of one variable t. Put $A = tK[[t]] \oplus M_2(tK[t])$. Then the algebras A and A^1 are semiprime AZS-algebras, but A^1 is not a direct sum of a reduced algebra and an algebra with ACC as an F-module.

5. Commutative subalgebras. Some results on DZS-algebras can be used in investigation of commutative subalgebras. As an application of Lemma 3.3 and Theorem 3.6, we can prove the following result.

Proposition 5.1. *Let A be an algebra such that $\Delta(A) \notin \mathcal{F}$. Then $\Delta(A)$ contains either an infinite sequence of orthogonal idempotents, or a commutative subalgebra B with $B^3 = 0$, but $B \notin \mathcal{F}$.*

Proof. Set $\Delta(A) = D$ and $N(D) = N$. By assumption $D \notin \mathcal{F}$ and, by Lemma 3.2, $\Delta(D/N) = D/N$.

If $D/N \notin \mathcal{F}$, then, by Theorem 3.6, D/N contains an infinite sequence of orthogonal idempotents. It is well known ([6]) that this sequence can be lifted to a sequence of orthogonal idempotents in D, because N is a nil-ideal.

Suppose $D/N \in \mathcal{F}$. Then, by assumption, $N \notin \mathcal{F}$. If N is not an FZS-algebra, then it contains a zero subalgebra, say B, not in $\mathcal{F}$ and we are done. Hence, we can assume that N is an FZS-algebra. Applying Lemma 3.3 we can take an ideal $J \subseteq N$ with $J \notin \mathcal{F}$, but $J^3 = 0$.

Let $B_1 = J^2$. Then clearly $B_1^2 = 0$ and $B_1 \in \mathcal{F}$, by the FZS property. Thus B_1 is a commutative ideal of J and $B_1 \neq 0$, because $J \notin \mathcal{F}$. In particular, the length of B_1 is at least one.

Now assume that we have constructed a commutative ideal B_n of J containing B_1, with $B_n \in \mathcal{F}$ and of length at least n. Let

$$V_n = \{x \in J : xB_n = B_n x = 0\}.$$

Then clearly V_n is an ideal of J and, by Lemma 2.7, the module J/V_n is of finite length. Thus, $V_n \notin \mathcal{F}$, by the assumption on J. Hence we can find an element $x_{n+1} \in V_n \setminus B_n$. Put $B_{n+1} = B_n + Fx_{n+1}$. Then it can be checked that B_{n+1} is a commutative ideal of J, belongs to $\mathcal{F}$, and its length is at least $n+1$. Thus the ideal $B = \sum_{n=1}^{\infty} B_n$ satisfies our requirements. ∎

As a consequence we have an extension of a theorem from [9], parallel to the main results of [10] and [1].

THEOREM 5.2. *Let A be an algebra. If $A \notin \mathcal{F}$, then A contains a commutative subalgebra also not belonging to $\mathcal{F}$.*

Proof. If A is not an FZS-algebra, then, by definition, A contains a zero subalgebra $B \notin \mathcal{F}$. Of course, B is commutative. If $a \in A$ is not algebraic over F, then, by definition, $F(a)$ is a commutative subalgebra and $F(a) \notin \mathcal{F}$. If A contains an infinite sequence of orthogonal idempotents, then, as is easy to check, this sequence generates a commutative subalgebra also not belonging to $\mathcal{F}$. If $\Delta(A) \notin \mathcal{F}$, then we can apply Proposition 5.1.

In this way we can restrict ourselves to an algebra A satisfying all assumptions of Theorem 4.7. In this case, at least one direct summand of A is a division algebra not in $\mathcal{F}$ and its maximal subfield is a commutative subalgebra not belonging to $\mathcal{F}$. ∎

QUESTION 1. *Can an analogue of the above theorem be proved for algebras not belonging to $\mathcal{D}$?*

In the case of unital algebras the answer is "yes". Even a stronger statement follows from the above theorem, because maximal commutative subalgebras of unital algebras are unital. We can also use Corollary 2.3.

COROLLARY 5.3. *Let A be an algebra with a one-sided unit. If $A \notin \mathcal{D}$ (or $A \notin \mathcal{F}$), then A contains a commutative subalgebra not belonging to $\mathcal{D}$.*

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