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THE PROPORTIONALITY CONSTANT FOR THE SIMPLICIAL VOLUME OF LOCALLY SYMMETRIC SPACES

BY

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Abstract. We follow ideas going back to Gromov's seminal article [Publ. Math. IHES 56 (1982)] to show that the proportionality constant relating the simplicial volume and the volume of a closed, oriented, locally symmetric space $M = \Gamma\backslash G/K$ of noncompact type is equal to the Gromov norm of the volume form in the continuous cohomology of G. The proportionality constant thus becomes easier to compute. Furthermore, this method also gives a simple proof of the proportionality principle for arbitrary manifolds.

1. Introduction. We propose here a reformulation of a proof of Gromov [Gr82] of the proportionality principle relating the simplicial volume and the volume of a closed oriented Riemannian manifold in the language of continuous bounded cohomology, a theory whose functorial properties have been established in recent years by Burger and Monod (see [Mo01]).

THEOREM 1 (Proportionality principle). *Let M be a closed, oriented Riemannian manifold. Then there exists a proportionality constant* $\mathrm{c}(\widetilde{M})$ *in* $\mathbb{R}\cup\{+\infty\}$, *depending on the universal cover* $\widetilde{M}$ *of* M *only*, *such that*

$$\|M\| = \frac{\mathrm{Vol}(M)}{\mathrm{c}(\widetilde{M})}.$$

The proportionality principle admits another proof by Strohm-Löh [St05] following an approach sketched by Thurston in [Th78].

This fundamental result shows in particular that within classes of manifolds isometrically covered by a given simply connected Riemannian manifold $\widetilde{M}$ for which $\mathrm{c}(\widetilde{M}) \neq +\infty$, the volume is a topological invariant. For hyperbolic spaces, Gromov showed that the proportionality constant $\mathrm{c}(\mathbb{H}^n)$ is equal to the maximal volume of ideal geodesic simplices in the hyperbolic space and used it as a key ingredient to a simple proof of Mostow's rigidity theorem for $\mathbb{H}^n$ (see [Gr82], [Mu80], [BePe92]).

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We give below a new expression for the proportionality constant of closed, oriented, locally symmetric spaces. While it is easy to see that the simplicial volume of a locally symmetric space whose universal cover admits nontrivial compact or Euclidean factors has to vanish (see Lemma 5) and that the corresponding proportionality constant is consequently equal to $+\infty$, the noncompact type case is more interesting:

THEOREM 2. *Let M be a closed, oriented, locally symmetric space of noncompact type of dimension n. Let G be the connected component of the identity of the isometry group of the universal cover $\widetilde{M}$ of M and let $\omega \in A^n(\widetilde{M})^G \cong H_c^n(G)$ be the volume form. Then*

$$\|M\| = \frac{\mathrm{Vol}(M)}{\|\omega\|_\infty}.$$

It follows from [LaSch06] that $\|\omega\|_\infty < \infty$. Indeed, Lafont and Schmidt answer affirmatively the conjecture of Gromov that the simplicial volume of a closed locally symmetric space of noncompact type is strictly positive (relying on [Gr82] and [Bu05] in the cases where the universal cover of M has an $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ or $\mathrm{SL}(3,\mathbb{R})/\mathrm{SO}(3)$ factor respectively).

The advantage of our approach is that, with the techniques developed in [Mo01], the proportionality constant as expressed in Theorem 2 becomes easier to compute. It is elementary to show that for hyperbolic manifolds, the norm $\|\omega\|_\infty$ is equal to the maximal volume of regular ideal geodesic simplices in the hyperbolic space (Theorem 8 here), so that we obtain Gromov's well known result in Corollary 9. Furthermore, we show in [Bu07] that the Gromov norm of the volume form $\omega_{\mathbb{H}^2\times\mathbb{H}^2}$ on the product $\mathbb{H}^2 \times \mathbb{H}^2$ of two copies of the hyperbolic plane is equal to

$$\|\omega_{\mathbb{H}^2\times\mathbb{H}^2}\|_\infty = \frac{2}{3}\pi^2.$$

We can hence compute the simplicial volume of closed Riemannian manifolds whose universal cover is $\mathbb{H}^2 \times \mathbb{H}^2$. In particular, if Σ_g and Σ_h are surfaces of genus $g \geq 1$ and $h \geq 1$, we obtain

$$\|\Sigma_g \times \Sigma_h\| = \frac{3}{2}\,\|\Sigma_g\|\,\|\Sigma_h\| = 24(g-1)(h-1).$$

Thus, Theorem 2 is the fundamental ingredient in the computation of the first example of a nonvanishing simplicial volume for a manifold not admitting a metric of constant curvature.

Observe that if M is a generic manifold, then the isometry group of its universal cover is discrete and the proportionality principle is trivial. At the other end of the genericity scale, M is locally symmetric and we have a simple, conceptual proof using continuous bounded cohomology theory. This proof generalizes to arbitrary manifolds, but the continuous (bounded) cohomology of the isometry group of the universal cover of M needs to be replaced by some adapted cohomology theories. The proportionality constant which one henceforth obtains, already present in [Gr82], is perhaps, in contrast with the symmetric space case, not so useful for explicit computations. The proof of the proportionality principle for arbitrary manifolds is presented in the last section of this paper. For enlightening comments concerning that section, I am grateful to Clara Strohm-Löh.

Note also that the argument exposed here only applies to closed manifolds. In fact, for open manifolds, the situation is still mysterious. While it is proven in [Gr82] that the proportionality principle holds for all finite volume hyperbolic manifolds, the author also shows that the proportionality principle fails in general for open manifolds, since he proves that the simplicial volume of the Cartesian product of three open manifolds always vanishes [Gr82, p. 59, Example (a)].

2. Simplicial volume and bounded cohomology. Let M be an n-dimensional oriented closed manifold. On the space of real-valued chains $C_*(M)$ on M consider the L^1-norm with respect to the canonical basis of singular simplices, that is,

$$\Big\|\sum_{i=1}^{r} a_i\sigma_i\Big\|_1 = \sum_{i=1}^{r} |a_i|$$

for $\sum_{i=1}^{r} a_i\sigma_i$ in $C_q(M)$. This induces a seminorm, which we still denote by $\|-\|_1$, on the real-valued homology $H_*(M)$ of M: the seminorm of a homology class is defined as the infimum of the norms of its representatives. The *simplicial volume* of M, denoted by $\|M\|$, is defined to be the seminorm of the real-valued fundamental class $[M] \in H_n(M)$ of M. It is elementary to see that if $f : N \to M$ is a q-covering, then $\|N\| = q\|M\|$. This in particular shows that the simplicial volume of tori is zero. Other examples of vanishing simplicial volumes can be obtained from Lemma 4 below. Of course, we could have considered the L^1-norm in the singular homology of any topological space—although a simplicial volume could not have been defined without a fundamental class.

The dual L^∞-norm (or *Gromov norm*) on the space of real-valued cochains $C^*(M)$ on M is given, for every cochain c in $C^q(M)$, by

$$\begin{aligned}\|c\|_\infty &= \sup\{|c(z)| \mid z \in C_q(M) \text{ with } \|z\|_1 = 1\}\\ &= \sup\{|c(\sigma)| \mid \ \sigma : \Delta^q \to M \text{ continuous}\}.\end{aligned}$$

Define the space of bounded cochains $C^*_{\mathrm{b}}(M)$ on M as the subspace of $C^*(M)$ consisting of those cochains for which the Gromov norm is finite. Clearly, the coboundary operator of $C^*(M)$ restricts to $C^*_{\mathrm{b}}(M)$ and we define the *bounded cohomology* $H^*_{\mathrm{b}}(M)$ of M as the cohomology of the cocomplex $C^*_{\mathrm{b}}(M)$. The inclusion of cocomplexes $C^*_{\mathrm{b}}(M) \subset C^*(M)$ induces a *comparison map* $c : H^*_{\mathrm{b}}(M) \to H^*(M)$ which is in general neither injective nor surjective. The Gromov norm on the space of (bounded) cochains defines a seminorm both on $H^*_{\mathrm{b}}(M)$ and $H^*(M)$ and we continue to denote those by $\| - \|_\infty$. (Note that on $H^*(M)$ we allow the value $+\infty$.) By definition, for any α in $H^q(M)$ we have

$$\|\alpha\|_\infty = \inf\{\|\alpha_{\mathrm{b}}\|_\infty \mid \alpha_{\mathrm{b}} \in H^q_{\mathrm{b}}(M),\, c(\alpha_{\mathrm{b}}) = \alpha\}, \tag{1}$$

where the right hand side of the above equation is understood to be equal to infinity when the infimum is taken over the empty set. Again, observe that the theory of bounded cohomology can more generally be defined for topological spaces.

Let $\beta_M \in H^n(M)$ denote the dual of the fundamental class of M, so that the Kronecker product $\langle \beta_M \mid [M] \rangle$ is equal to 1. From the duality of the L^1- and L^∞-norms, it is easy to show, using the Hahn–Banach theorem, that

$$\|M\| = \frac{1}{\|\beta_M\|_\infty}. \tag{2}$$

For a detailed proof, see for example [BePe92, Proposition F.2.2].

Suppose now that M is a Riemannian manifold, and let $\omega_M \in H^n(M)$ be the image, under the de Rham isomorphism, of the volume form. The Kronecker product $\langle \omega_M \mid [M] \rangle$ is clearly equal to $\mathrm{Vol}(M)$, the volume of M, and $\beta_M = (1/\mathrm{Vol}(M))\omega_M$. In particular, we can rewrite equality (2) as

$$\|M\| = \frac{\mathrm{Vol}(M)}{\|\omega_M\|_\infty}. \tag{3}$$

Note that the proportionality principle now reduces to showing that $\|\omega_M\|_\infty$ only depends on the universal covering of M. This will be done in Corollary 7 in the symmetric space case and in Corollary 13 in the general case.

3. Bounded group cohomology. In this section, we define bounded group cohomology and give the few easy and known properties which we need for our proof of Theorem 2. For more details we invite the reader to consult [Gui80] or [BoWa00] for continuous cohomology and [Mo01] for bounded continuous cohomology.

Let G be a topological group. Recall that the continuous cohomology $H^*_{\mathrm{c}}(G)$ of G can be computed as the cohomology of the cocomplex $C^*_{\mathrm{c}}(G)^G$ endowed with its natural symmetric coboundary operator, where

$$C^q_{\mathrm{c}}(G) = \{c : G^{q+1} \to \mathbb{R} \mid c \text{ is continuous}\}$$

and $C^q_{\mathrm{c}}(G)^G$ denotes the subspace of G-invariant cochains, where the action of G is given by the diagonal action by left multiplication on the Cartesian product G^{q+1}. For c in $C^q_{\mathrm{c}}(G)$, we define its sup norm as

$$\|c\|_\infty = \sup\{|c(g_0, \ldots, g_q)| \mid (g_0, \ldots, g_q) \in G^{q+1}\}.$$

Clearly, the coboundary operator restricts to the cocomplex $C^*_{\mathrm{c,b}}(G)^G$ of continuous bounded cochains, where

$$C^q_{\mathrm{c,b}}(G) = \{c \in C^q_{\mathrm{c}}(G) \mid \|c\|_\infty < \infty\},$$

and the *continuous bounded cohomology* $H^*_{\mathrm{c,b}}(G)$ of G is defined as the cohomology of this cocomplex. The inclusion of cocomplexes $C^*_{\mathrm{c,b}}(G)^G \subset C^*_{\mathrm{c}}(G)^G$ induces a *comparison map* $c : H^*_{\mathrm{c,b}}(G) \to H^*_{\mathrm{c}}(G)$. As in the singular case, the sup norm induces seminorms on both $H^*_{\mathrm{c,b}}(G)$ and $H^*_{\mathrm{c}}(G)$ (where in the latter space we again allow the value $+\infty$) and we have, for any α in $H^q_{\mathrm{c}}(G)$

$$\|\alpha\|_\infty = \inf\{\|\alpha_{\mathrm{b}}\|_\infty \mid \alpha_{\mathrm{b}} \in H^q_{\mathrm{b}}(M),\, c(\alpha_{\mathrm{b}}) = \alpha\}. \tag{4}$$

If Γ is a discrete group, then the continuity condition is void and we omit the term "continuous" and the subscript "c" in the corresponding terminology and notation. Note that the group cohomology $H^*(\Gamma)$ is then nothing else than the standard Eilenberg–MacLane cohomology of Γ.

Let now G be a Lie group and $\Gamma < G$ a cocompact lattice. There is in this context another convenient way to compute the cohomology groups $H^*(\Gamma)$ and $H^*_{\mathrm{b}}(\Gamma)$, namely as the cohomology of the cocomplexes $C^*_{\mathrm{c}}(G)^\Gamma$ and $C^*_{\mathrm{c,b}}(G)^\Gamma$ respectively. Furthermore, the sup norm on $C^q_{\mathrm{c}}(G)$ gives rise to the same seminorms in $H^*_{\mathrm{c}}(\Gamma)$ and $H^*_{\mathrm{c,b}}(\Gamma)$, as proven in [Mo01, Corollary 7.4.10].

The restriction maps $\mathrm{res} : H^*_{\mathrm{c}}(G) \to H^*(\Gamma)$ and $\mathrm{res}_{\mathrm{b}} : H^*_{\mathrm{c,b}}(G) \to H^*_{\mathrm{b}}(\Gamma)$ induced by the inclusion $\Gamma < G$ can now be realized at the cochain level by the inclusions $C^*_{\mathrm{c}}(G)^G \subset C^*_{\mathrm{c}}(G)^\Gamma$ and $C^*_{\mathrm{c,b}}(G)^G \subset C^*_{\mathrm{c,b}}(G)^\Gamma$ respectively. On the other hand, starting with a Γ-invariant cochain $c : G^{q+1} \to \mathbb{R}$ in $C^q_{\mathrm{c}}(G)^\Gamma$, we can construct a G-invariant cochain $\overline{c} \in C^q_{\mathrm{c}}(G)^G$ by integrating c over a fundamental domain $F \subset G$ for $\Gamma \setminus G$:

$$\overline{c}(g_0, \dots, g_q) = \frac{1}{\mu(F)} \int c(fg_0, \dots, fg_q)\, d\mu(f),$$

where $g_i \in G$ and μ is the Haar measure on G. This induces *transfer maps* $\mathrm{trans} : H^*(\Gamma) \to H^*_{\mathrm{c}}(G)$ and $\mathrm{trans}_{\mathrm{b}} : H^*_{\mathrm{b}}(\Gamma) \to H^*_{\mathrm{c,b}}(G)$. Because the compositions $\mathrm{trans} \circ \mathrm{res}$ and $\mathrm{trans}_{\mathrm{b}} \circ \mathrm{res}_{\mathrm{b}}$ are clearly the identity at the cochain level, we obtain the commutative diagram

$$\begin{array}{ccccc}
 & & \overset{\mathrm{Id}}{\curvearrowright} & & \\
H^*_{\mathrm{c}}(G) & \xhookrightarrow{\mathrm{res}} & H^*(\Gamma) & \xrightarrow{\mathrm{trans}} & H^*_{\mathrm{c}}(G) \\
\uparrow c & & \uparrow c & & \uparrow c \\
H^*_{\mathrm{c,b}}(G) & \xhookrightarrow{\mathrm{res}_{\mathrm{b}}} & H^*_{\mathrm{b}}(\Gamma) & \xrightarrow{\mathrm{trans}_{\mathrm{b}}} & H^*_{\mathrm{c,b}}(G) \\
 & & \underset{\mathrm{Id}}{\smile} & &
\end{array} \tag{5}$$

Note that for bounded cohomology the above construction carries through also for noncocompact lattices $\Lambda < G$ since it is possible to integrate *bounded* functions over $\Lambda \backslash G$. This is however not the case in the unbounded setting: for noncocompact lattices, $H^*(\Lambda)$ vanishes in top dimension, so that the present argument cannot be applied to open manifolds.

Because the sup norms on the G-invariant (respectively Γ-invariant) continuous (bounded) cochains $G^{q+1} \to \mathbb{R}$ realize the seminorms on the respective cohomology groups, it is immediate that the maps res, trans, $\mathrm{res}_{\mathrm{b}}$ and $\mathrm{trans}_{\mathrm{b}}$ do not increase seminorms: for every α in $H^q_{\mathrm{c}}(G)$, β in $H^q_{\mathrm{b}}(\Gamma)$, α_{b} in $H^q_{\mathrm{c,b}}(G)$ and β_{b} in $H^q_{\mathrm{b}}(\Gamma)$ we have

$$\|\mathrm{res}(\alpha)\|_\infty \le \|\alpha\|_\infty, \qquad \|\mathrm{trans}(\beta)\|_\infty \le \|\beta\|_\infty,$$
$$\|\mathrm{res}_{\mathrm{b}}(\alpha_{\mathrm{b}})\|_\infty \le \|\alpha_{\mathrm{b}}\|_\infty, \quad \|\mathrm{trans}_{\mathrm{b}}(\beta_{\mathrm{b}})\|_\infty \le \|\beta_{\mathrm{b}}\|_\infty.$$

Furthermore, since $\mathrm{trans}_{\mathrm{b}} \circ \mathrm{res}_{\mathrm{b}}$ is the identity of $H^q_{\mathrm{c,b}}(G)$, the restriction map $\mathrm{res}_{\mathrm{b}} : H^*_{\mathrm{c,b}}(G) \to H^*_{\mathrm{b}}(\Gamma)$ is an isometric embedding:

$$\|\beta_{\mathrm{b}}\|_\infty = \|\mathrm{trans}_{\mathrm{b}} \circ \mathrm{res}_{\mathrm{b}}(\beta_{\mathrm{b}})\|_\infty \le \|\mathrm{res}_{\mathrm{b}}(\beta_{\mathrm{b}})\|_\infty \le \|\beta\|_\infty,$$

hence

$$\|\mathrm{res}_{\mathrm{b}}(\beta_{\mathrm{b}})\|_\infty = \|\beta_{\mathrm{b}}\|_\infty.$$

For more details, consult the original result [Mo01, Proposition 8.6.2].

If Γ is a cocompact lattice in G, then trans $\circ$ res is the identity on $H^q_{\mathrm{c}}(G)$ and removing all the subscripts "b" in the above implication, we immediately see that the restriction map is an isometric embedding also on the unbounded cohomology groups, as expressed in Theorem 3 below. (By isometric embedding, we mean that the restriction map preserves the seminorms, which possibly take the value $+\infty$.) Note that Theorem 3 does not hold when Γ is not cocompact (since in this case the restriction map is not even injective).

THEOREM 3. *Let G be a Lie group and Γ be a cocompact lattice in G. The restriction map*

$$\mathrm{res} : H^*_{\mathrm{c}}(G) \hookrightarrow H^*(\Gamma)$$

is an isometric embedding.

Let now M be a manifold and Γ denote its fundamental group. As for (unbounded) singular cohomology, the natural map $M \to B\Gamma$, where $B\Gamma$ denotes the classifying space of Γ-bundles, induces a natural map in bounded cohomology $H^*_{\mathrm{b}}(B\Gamma) \to H^*_{\mathrm{b}}(M)$, and as in the unbounded setting, $H^*_{\mathrm{b}}(B\Gamma)$ is canonically isomorphic to $H^*_{\mathrm{b}}(\Gamma)$. There is however a fundamental difference for bounded cohomology, namely the remarkable result of Gromov that the natural map $H^*_{\mathrm{b}}(\Gamma) \to H^*_{\mathrm{b}}(M)$ is an isometric isomorphism (which more generally holds if one replaces M by any CW-complex). Note that, in contrast to the unbounded case, there is no assumption that M be aspherical.

For a proof, see [Gr82, Section 3.1] or [Iv85, Theorem 4.1]. To summarize, we have the commutative diagram

$$\begin{array}{ccc} H^*(\Gamma) & \longrightarrow & H^*(M) \\ {\scriptstyle c}\big\uparrow & & \big\uparrow{\scriptstyle c} \\ H^*_{\mathrm{b}}(\Gamma) & \xrightarrow{\cong} & H^*_{\mathrm{b}}(M) \end{array} \tag{6}$$

LEMMA 4. *Let M be an n-dimensional closed, oriented manifold, and let Γ denote its fundamental group. If $H^n(\Gamma)=0$, then $\|M\|=0$.*

Proof. From (2) it is clear that $\|M\|>0$ if and only if β_M is in the image of the comparison map $c: H^n_{\mathrm{b}}(M)\to H^n(M)$. But since the diagram (6) commutes, this comparison map factors through $H^n(\Gamma)=0$, and is hence the zero map. ∎

4. Locally symmetric spaces. Let M be a closed, connected, oriented, locally symmetric space. Its universal cover can be decomposed as a product $U\times\mathbb{R}^k\times N$, where U is a compact symmetric space, k is a nonnegative integer and N is a symmetric space of noncompact type. Before concentrating on the more interesting purely noncompact case, let us get the case of nontrivial compact and Euclidean factors out of the way.

LEMMA 5. *Let M be a closed, connected, oriented, locally symmetric space whose universal cover has a nontrivial compact or Euclidean factor. Then $\|M\|=0$.*

In particular, the proportionality constant is in this case equal to $+\infty$.

Proof. Let G denote the isometry group of the universal cover $\widetilde{M}$ of M and let K be a maximal compact subgroup of G. Set $n=\dim M=\dim\widetilde{M}$ and observe that the fundamental group Γ of M sits in G as a cocompact lattice. It is well known (for a proof, see for example [BoWa00, Theorem VII.2.2]) that the cohomology of Γ is isomorphic to the cohomology of the cocomplex of Γ-invariant differential forms on G/K. But if $\widetilde{M}$ has a nontrivial compact factor, then the dimension of G/K is strictly smaller than n, because the isometry group of the compact factor is a compact Lie group. In particular $H^n(\Gamma)=0$, and by Lemma 4 we get $\|M\|=0$.

Suppose now that $\widetilde{M}$ has no nontrivial compact factor, so that we are in the case when M has nonpositive curvature. If $\widetilde{M}$ has a factor $\mathbb{R}^k$, then it follows from the work of Eberlein (see for example [Eb83, Corollary 2]) that there exists a finite q-covering $\overline{M}\to M$ of M such that $\overline{M}$ is diffeomorphic to the product of a k-torus and some closed manifold. If $k>0$, then $\overline{M}$ admits self-coverings of degree greater than 2, so that its simplicial volume has to vanish, and hence also $\|M\|=(1/q)\|\overline{M}\|=0$. ∎

Let from now on M be a closed, connected, oriented, locally symmetric space of noncompact type of dimension n. Its universal cover can be written as the quotient $\widetilde{M} = G/K$, where G is a connected semisimple Lie group of noncompact type and K is a maximal compact subgroup of G. Denote by Γ the fundamental group of M, and observe that Γ sits in G as a cocompact lattice.

Because M has no higher homotopy, we have $H^*(M) \cong H^*(\Gamma)$. Furthermore, by the commutativity of the diagram (6) and the fact that the corresponding isomorphism $H^*_{\mathrm{b}}(M) \cong H^*_{\mathrm{b}}(\Gamma)$ between the bounded cohomology groups is an isometry, it is clear that the isomorphism $H^*(M) \cong H^*(\Gamma)$ is also an isometry. Denote by $\varphi : H^*_{\mathrm{c}}(G) \to H^*(M)$ the composition of the restriction map and the natural isomorphism $H^*(\Gamma) \cong H^*(M)$ and observe that Theorem 3 now admits the following reformulation:

THEOREM 6. *The injection*

$$\varphi : H^*_{\mathrm{c}}(G) \hookrightarrow H^*(M)$$

is an isometric embedding.

Recall that $H^*_{\mathrm{c}}(G)$ is isomorphic to $A^*(G/K)^G$, the G-invariant differential forms on G/K, which is in top degree one-dimensional, generated by the volume form ω on the Riemannian manifold G/K. Let us abuse notation and denote also by ω the corresponding cohomology class in $H^n_{\mathrm{c}}(G)$. Similarly, let ω_M stand for both the Riemannian volume form on M and the corresponding cohomology element in $H^n(M)$. We claim that

$$\varphi(\omega) = \omega_M.$$

To see this, observe once again that the cohomology of Γ is isomorphic to the cohomology of the cocomplex $A^*(G/K)^\Gamma$ ([BoWa00, Theorem VII.2.2]). The map $\varphi : H^*_{\mathrm{c}}(G) \to H^*(M)$ can hence be represented, at the cochain level, by the composition of the natural maps

$$A^n(G/K)^G \to A^n(G/K)^\Gamma \xrightarrow{\cong} A^n(M),$$

which clearly maps ω to ω_M. An immediate consequence of Theorem 6 is:

COROLLARY 7. $\|\omega_M\|_\infty = \|\omega\|_\infty$.

Theorem 2 now obviously follows from equation (3) and Corollary 7.

5. Hyperbolic manifolds. As an application of our method, let us give an elementary proof of Gromov's result that the proportionality constant for hyperbolic n-manifolds is

$$\begin{aligned} v_n &= \sup\{|\mathrm{Vol}(\sigma)| \mid \sigma : \Delta^n \to \mathbb{H}^n \text{ a geodesic simplex}\} \\ &= |\text{volume of the regular ideal geodesic simplex in } \mathbb{H}^n|, \end{aligned}$$

where the last equality follows from [HaMu81]. Let thus G denote the group of orientation preserving isometries of the n-dimensional hyperbolic space $\mathbb{H}^n$.

THEOREM 8. *Let $\omega \in H_{\mathrm{c}}^n(G)$ be the volume form. Then*

$$\|\omega\|_\infty = v_n.$$

In view of Theorem 2, Gromov's result is now immediate:

COROLLARY 9 (Gromov). *Let M be an n-dimensional closed hyperbolic manifold. Then*

$$\|M\| = \frac{\mathrm{Vol}(M)}{v_n}.$$

Proof of Theorem 8. The key is to choose appropriate cocomplexes for $H_{\mathrm{c,b}}^*(G)$ and $H_{\mathrm{c}}^*(G)$ in order to compute the seminorm of the volume form ω. From [Gui80, Chapitre III, Proposition 2.3] and [Mo01, Corollary 7.4.10] we know that those cohomology groups can be obtained as the cohomology of the cocomplexes $C_{\mathrm{c}}^*(\mathbb{H}^n)_{\mathrm{alt}}^G$ and $C_{\mathrm{c,b}}^*(\mathbb{H}^n)_{\mathrm{alt}}^G$ respectively, where $C_{\mathrm{c}}^q(\mathbb{H}^n)_{\mathrm{alt}}$ denotes the alternating, continuous real-valued functions on $(\mathbb{H}^n)^{q+1}$ and $C_{\mathrm{c,b}}^q(\mathbb{H}^n)_{\mathrm{alt}}^G$ is its subcomplex of bounded functions. The action of G is given by the natural diagonal action on the Cartesian product $(\mathbb{H}^n)^{q+1}$. Monod shows moreover that the sup norm on $C_{\mathrm{c,b}}^*(\mathbb{H}^n)_{\mathrm{alt}}^G$ gives rise to the canonical seminorm on $H_{\mathrm{c,b}}^*(G)$.

By Dupont's expression for the Van Est isomorphism $A^*(\mathbb{H}^n)^G \cong H_{\mathrm{c}}^*(G)$ we can represent the volume form $\omega \in H_{\mathrm{c}}^*(G)$ by the alternating, continuous cocycle ν sending the $(n+1)$-tuple $(x_0, \dots, x_n)$ to the (signed) volume of the geodesic simplex with vertices $(x_0, \dots, x_n)$. It follows immediately that

$$\|\omega\|_\infty \leq \|\nu\|_\infty = v_n.$$

For the other inequality we need to show that if $\tau : (\mathbb{H}^n)^n \to \mathbb{R}$ is any G-invariant, continuous, alternating cochain, then $\|\nu + \delta\tau\|_\infty \geq \|\nu\|_\infty$. But such a τ, being invariant and alternating, has to vanish on regular n-tuples $(y_0, \dots, y_{n-1})$, i.e. n-tuples for which any permutation $y_i \mapsto y_{\sigma(i)}$, for $\sigma \in \mathrm{Sym}(n)$, can be realized by an isometry. It is now easy to construct a sequence of $(n+1)$-tuples $(y_0^k, \dots, y_n^k)$ such that each of the faces $(y_0^k, \dots, \widehat{y_i^k}, \dots, y_n^k)$ is regular and $\lim_{k\to\infty} |\nu(y_0^k, \dots, y_n^k)| = v_n$. Assume for a second that such a sequence has been constructed. We then have

$$\begin{aligned}
\|\nu + \delta\tau\|_\infty &= \sup\{|(\nu + \delta\tau)(y_0, \dots, y_n) \mid (y_0, \dots, y_n) \in (\mathbb{H}^n)^{n+1}\} \\
&\geq \sup\{|(\nu + \delta\tau)(y_0^k, \dots, y_n^k) \mid k \in \mathbb{N}\} \\
&= \lim_{k\to\infty} |\nu(y_0^k, \dots, y_n^k)| = v_n.
\end{aligned}$$

As for the sequence $(y_0^k, \dots, y_n^k)$, fix an ideal regular simplex with vertices $\xi_0, \dots, \xi_n$ in the boundary of $\mathbb{H}^n$. Let $b \in \mathbb{H}^n$ be its barycenter (defined as the unique point of $\mathbb{H}^n$ fixed by all the isometries permuting the ξ_i's) and let $\gamma_i : (-\infty, +\infty) \to \mathbb{H}^n$ be the unique geodesics with $\gamma_i(0) = b$ and $\gamma_i(+\infty) = \xi_i$ for $i = 0, \dots, n$. Then the sequence

$$(y_0^k, \dots, y_n^k) = (\gamma_0(k), \dots, \gamma_n(k))$$

is as claimed. ∎

6. The proportionality principle for arbitrary manifolds. In this last section, we present a proof of the proportionality principle for arbitrary manifolds. This proof is essentially based on the one in [Gr82], but we give it here for the sake of completeness, as it naturally and straightforwardly generalizes the argument we give above for the symmetric space case. To do so, we need to replace the continuous and continuous bounded cohomology of G by appropriate cohomology theories. Those theories will be defined in Subsections 6.1 and 6.3 respectively, and the comparison map relating them in Subsection 6.4. Note that the cohomology theory of Subsection 6.3 already appears in [Gr82].

Observe that we cannot work with the continuous cohomology of the isometry group of the universal cover of M since this cohomology group can be zero, while we want it to contain some universal (nonzero) volume form (see Subsection 6.2).

Throughout this section, we establish the elementary properties which we need our present cohomology theories to satisfy in order to conclude in Subsection 6.5 that the analogue of Corollary 7, namely Corollary 13, also holds for arbitrary manifolds.

6.1. *Continuous singular cohomology.* Let $S_q(M)$ denote the set of all smooth singular simplices $\sigma : \Delta^q \to M$ and endow it with the compact-open topology. From now on we suppose that $C^*(M)$ denotes the space of cochains on smooth singular simplices (as opposed to merely continuous singular simplices). This restriction has no consequence for the cohomology of the cocomplex. Define the space of *continuous singular q-cochains* on M as

$$C_{\mathrm{c}}^q(M) = \{c \in C^q(M) \mid c|_{S_q(M)} : S_q(M) \to \mathbb{R} \text{ continuous}\}.$$

The coboundary operator clearly restricts to $C_{\mathrm{c}}^*(M)$. Now, for "reasonable spaces", and in particular for manifolds, the cohomology of this cocomplex is nothing else than the usual singular cohomology, as can be read in [Bo75]:

THEOREM 10. *The inclusion of cocomplexes $C_{\mathrm{c}}^*(M) \subset C^*(M)$ induces an isomorphism*

$$H^*(C_{\mathrm{c}}^*(M)) \cong H^*(M).$$

Let G denote the isometry group of the universal cover $\widetilde{M}$ of the Riemannian manifold M and note that the fundamental group Γ of M sits as a cocompact lattice in the Lie group G. As a replacement for $H_{\mathrm{c}}^*(G)$ we will consider the cohomology of the cocomplex of G-invariant continuous singular cochains on $\widetilde{M}$. The inclusion of cocomplexes

$$C_{\mathrm{c}}^q(\widetilde{M})^G \hookrightarrow C_{\mathrm{c}}^q(\widetilde{M})^\Gamma \cong C_{\mathrm{c}}^q(M)$$

induces a *restriction map*

$$\mathrm{res}: H^*(C_{\mathrm{c}}^*(\widetilde{M})^G) \to H^*(C_{\mathrm{c}}^*(\widetilde{M})^\Gamma) \cong H^*(M).$$

Let μ denote the Haar measure on G. Choose a fundamental domain $F \subset G$ for $\Gamma \setminus G$ and define

$$\mathrm{trans}: C_{\mathrm{c}}^q(\widetilde{M})^\Gamma \to C_{\mathrm{c}}^q(\widetilde{M})^G$$

as

$$\mathrm{trans}(c)(\sigma) = \frac{1}{\mu(F)} \int_F c(f \cdot \sigma)\, d\mu(f),$$

for c in $C_{\mathrm{c}}^q(\widetilde{M})^\Gamma$ and σ in $S_q(\widetilde{M})$. To see that the map trans is well defined, we need to check that $\mathrm{trans}(c)$ is G-invariant and that its restriction to $S_q(\widetilde{M})$ is continuous:

G-invariance: Pick g in G. We have three partitions of G: two given by the tiling of G by F and Fg respectively, and the third one as the intersection of the former two partitions, namely,

$$G = \coprod_{\delta,\gamma\in\Gamma} \delta F \cap \gamma F g.$$

Because F has compact closure, it admits a partition $F = \coprod_{i=1}^r F_i$ into finitely many $F_i = F \cap \gamma_i^{-1} F g$, for γ_i in Γ. But then, since $\gamma_i F_i \cap \gamma_j F_j = \emptyset$ whenever $i \neq j$, and $\gamma_i F_i \subset Fg$, it follows that Fg admits the finite partition $Fg = \coprod_{i=1}^r \gamma_i F_i$.

Using successively the fact that c is Γ-invariant, and that the Haar measure μ is G- and hence Γ-invariant, we compute

$$\int_{F_i} c(f\sigma)\, d\mu(f) = \int_{F_i} c(\gamma_i f\sigma)\, d\mu(f) = \int_{\gamma_i F_i} c(\bar{f}\sigma)\, d\mu(\bar{f}). \tag{7}$$

Finally, we conclude, using the G-invariance of the Haar measure, equality

(7) and the two described partitions of F and Fg, that

$$\begin{aligned}\operatorname{trans}(c)(g\sigma) &= \frac{1}{\mu(F)}\int_F c(f\cdot g\sigma)\,d\mu(f) = \frac{1}{\mu(F)}\int_{Fg} c(\bar f\cdot\sigma)\,d\mu(\bar f)\\ &= \frac{1}{\mu(F)}\sum_{i=1}^r\int_{\gamma_i F_i} c(\bar f\cdot\sigma)\,d\mu(\bar f) = \frac{1}{\mu(F)}\sum_{i=1}^r\int_{F_i} c(f\sigma)\,d\mu(f)\\ &= \frac{1}{\mu(F)}\int_F c(f\cdot\sigma)\,d\mu(f) = \operatorname{trans}(c)(\sigma).\end{aligned}$$

Continuity: Pick σ in $S_q(\widetilde{M})$ and fix $\varepsilon > 0$. Consider the continuous map $\varrho : G \to C_{\mathrm{c}}^q(\widetilde{M})$ sending an element g in G to the continuous singular cochain mapping the singular simplex $\tau : \Delta^q \to \widetilde{M}$ to $\varrho(g)(\tau) = c(g\tau)$. Because the closure $\overline{F}$ of F is compact, so is $\varrho(\overline{F}) \subset C_{\mathrm{c}}^q(\widetilde{M})$. By the Arzelà–Ascoli theorem, this now implies that the family of maps $\{\varrho(f)\}_{f\in\overline{F}}$, and by restriction also $\{\varrho(f)\}_{f\in F}$, are equicontinuous. This means that there exists a neighborhood U of σ such that, for every σ' in U and f in F, we have

$$|\varrho(f)(\sigma) - \varrho(f)(\sigma')| = |c(f\sigma) - c(f\sigma')| < \varepsilon,$$

whence

$$\begin{aligned}|\operatorname{trans}(c)(\sigma) - \operatorname{trans}(c)(\sigma')| &= \left|\frac{1}{\mu(F)}\int_F (c(f\cdot\sigma) - c(f\cdot\sigma'))\,d\mu(f)\right|\\ &\le \frac{1}{\mu(F)}\int_F |c(f\cdot\sigma) - c(f\cdot\sigma')|\,d\mu(f) < \varepsilon.\end{aligned}$$

The map trans is obviously a map of cocomplexes, so that it induces the *transfer map*

$$\operatorname{trans} : H^*(M) \to H^*(C_{\mathrm{c}}^*(\widetilde{M})^G).$$

Note that the composition trans ∘ res is the identity on $H^*(C_{\mathrm{c}}^*(\widetilde{M})^G)$ since it is the identity at the cochain level.

6.2. *The volume form.* Suppose that M and hence $\widetilde{M}$ are n-dimensional Riemannian manifolds and let ω_M and ω be their Riemannian volume forms in $A^n(M)$ and $A^n(\widetilde{M})$ respectively. The form ω gives rise to a cocycle

$$\sigma \mapsto \int_\sigma \omega$$

in $C_{\mathrm{c}}^n(\widetilde{M})^G$; we denote the corresponding cohomology class in $H^n(C_{\mathrm{c}}^*(\widetilde{M})^G)$ also by ω. Note that for this cocycle to be continuous for the compact-open topology, it is essential that the volume form is a top-dimensional form. The same construction for ω_M gives a cohomology class ω_M in $H^n(M)$ which is nothing else than the image of ω_M under the de Rham isomorphism. We clearly have

$$\mathrm{res}(\omega) = \omega_M.$$

Because $H^n(M)$ is 1-dimensional, generated by ω_M, the restriction map is an isomorphism in top degree, with inverse the transfer map.

6.3. *A cocomplex for $H^*_{\mathrm{b}}(M)$.* In this subsection, we introduce a cocomplex from which we can compute the bounded cohomology of M. Probably, we could instead have used the cocomplex of continuous, bounded cochains on M, in which case we would not have had to redefine the restriction and transfer maps in the bounded context as we do below. However, we prefer to work with the following cocomplex, so that we can simply refer to a particular case of [Mo01, Theorem 7.4.5] for a proof of Theorem 11. Set

$$B^q(\widetilde{M}) = \{h : \widetilde{M}^{q+1} \to \mathbb{R} \mid h \text{ continuous and bounded}\}$$

and observe that the natural symmetric coboundary operator turns $B^*(\widetilde{M})$ into a cochain complex. Endow $B^q(\widetilde{M})$ with its natural sup norm.

THEOREM 11. *The cohomology of the cocomplex*

$$B^0(\widetilde{M})^\Gamma \to B^1(\widetilde{M})^\Gamma \to B^2(\widetilde{M})^\Gamma \to \cdots$$

is canonically isometrically isomorphic to $H^*_{\mathrm{b}}(M)$.

As a replacement for $H^*_{\mathrm{c,b}}(G)$ we will here consider the cohomology of the cocomplex $B^*(\widetilde{M})^G$. The inclusion of cocomplexes

$$B^q(\widetilde{M})^G \hookrightarrow B^q(\widetilde{M})^\Gamma$$

induces a *restriction map*

$$\mathrm{res}_{\mathrm{b}} : H^*(B^*(\widetilde{M})^G) \to H^*(B^*(\widetilde{M})^\Gamma) \cong H^*_{\mathrm{b}}(M).$$

Clearly, for every α_{b} in $H^*(B^*(\widetilde{M})^G)$, we have

$$\|\mathrm{res}_{\mathrm{b}}(\alpha_{\mathrm{b}})\|_\infty \le \|\alpha_{\mathrm{b}}\|_\infty \tag{8}$$

and we will see below that this inequality is in fact an equality.

Again, let μ denote the Haar measure on G and let $F \subset G$ be a fundamental domain for $\Gamma \setminus G$. Define a map

$$\mathrm{trans}_{\mathrm{b}} : B^q(\widetilde{M})^\Gamma \to B^q(\widetilde{M})^G$$

as

$$\mathrm{trans}_{\mathrm{b}}(h)(x_0, \ldots, x_q) = \frac{1}{\mu(F)} \int_F h(fx_0, \ldots, fx_q)\, d\mu(f)$$

for h in $B^q(\widetilde{M})^\Gamma$ and $(x_0, \ldots, x_q)$ in $\widetilde{M}^{q+1}$. To see that the map $\mathrm{trans}_{\mathrm{b}}$ is well defined, we need to check that $\mathrm{trans}_{\mathrm{b}}(h)$ is G-invariant, continuous and bounded. The former two properties are checked identically to the unbounded case treated in the previous subsection: just replace c by h and σ

by $(x_0, \dots, x_q)$. As for the boundedness, we have

$$|\mathrm{trans}_{\mathrm{b}}(h)(x_0, \dots, x_q)| \le \sup_{f \in F} |h(fx_0, \dots, fx_q)|,$$

and hence

$$\|\mathrm{trans}_{\mathrm{b}}(h)\|_\infty \le \|h\|_\infty. \tag{9}$$

It is obvious that $\mathrm{trans}_{\mathrm{b}}$ is a map of cocomplexes, so that it induces the *transfer map*

$$\mathrm{trans}_{\mathrm{b}} : H^*_{\mathrm{b}}(M) \to H^*(B^*(\widetilde{M})^G).$$

From (9) it is immediate that

$$\|\mathrm{trans}_{\mathrm{b}}(\alpha_{\mathrm{b}})\|_\infty \le \|\alpha_{\mathrm{b}}\|_\infty \tag{10}$$

for every α_{b} in $H^q_{\mathrm{b}}(M)$. Note that the composition $\mathrm{trans}_{\mathrm{b}} \circ \mathrm{res}_{\mathrm{b}}$ is the identity on $H^*(B^*(\widetilde{M})^G)$ since it is the identity at the cochain level. This fact, together with inequalities (8) and (10), immediately implies, as in the symmetric space case, that the inclusion $\mathrm{res}_{\mathrm{b}} : H^*(B^*(\widetilde{M})^G) \to H^*_{\mathrm{b}}(M)$ is isometric.

6.4. *The comparison map.* Let $e_0, \dots, e_q$ be the vertices of the standard simplex Δ^q and consider the cocomplex map $c : B^q(\widetilde{M}) \to C^q_{\mathrm{c}}(\widetilde{M})$ given by

$$c(h)(\sigma) = h(\sigma(e_0), \dots, \sigma(e_q)).$$

Note that c clearly restricts to a cocomplex map $c : B^q(\widetilde{M})^H \to C^q_{\mathrm{c}}(\widetilde{M})^H$ for any subgroup H of the isometry group G of $\widetilde{M}$. The comparison map $H^*_{\mathrm{b}}(M) \to H^*(M)$ is given, at the cochain level and for the cocomplexes $B^*(\widetilde{M})^\Gamma$ and $C^*_{\mathrm{c}}(\widetilde{M})^\Gamma$ respectively, by

$$c : B^q(\widetilde{M})^\Gamma \to C^q_{\mathrm{c}}(\widetilde{M})^\Gamma.$$

Indeed, it is a standard fact from homological algebra that all the cochain maps $B^*(\widetilde{M}) \to C^*_{\mathrm{c}}(\widetilde{M})$ extending $c : B^0(\widetilde{M}) \to C^0_{\mathrm{c}}(\widetilde{M})$ are chain homotopic, where the homotopy can be made up of Γ-equivariant maps. In degree zero, $B^0(\widetilde{M})$ and $C^0_{\mathrm{c}}(\widetilde{M})$ are isomorphic to the space of continuous bounded, respectively continuous functions on $\widetilde{M}$ and it is not hard to see that the comparison map $H^0_{\mathrm{b}}(M) \to H^0(M)$ has to be induced by the natural inclusion, which is precisely what the cocomplex map c amounts to in degree zero.

Similarly, we let $c : B^q(\widetilde{M})^G \to C^q_{\mathrm{c}}(\widetilde{M})^G$ induce a *comparison map*

$$c : H^*(B^*(\widetilde{M})^G) \to H^*(C^*_{\mathrm{c}}(\widetilde{M})^G).$$

In analogy to (1), we define

$$\|\alpha\|_\infty = \inf\{\|\alpha_{\mathrm{b}}\|_\infty \mid \alpha_{\mathrm{b}} \in H^q(B^*(\widetilde{M})^G),\ c(\alpha_{\mathrm{b}}) = \alpha\} \tag{11}$$

for every α in $H^q(C^*_{\mathrm{c}}(\widetilde{M})^G)$.

6.5. *Conclusion.* To summarize, we have a commutative diagram

$$\begin{array}{ccccc} H^*(C^*_{\mathrm{c}}(\widetilde{M})^G) & \xhookrightarrow{\mathrm{res}} & H^*(M) & \xrightarrow{\mathrm{trans}} & H^*(C^*_{\mathrm{c}}(\widetilde{M})^G) \\ \uparrow{\scriptstyle \mathrm{c}} & & \uparrow{\scriptstyle \mathrm{c}} & & \uparrow{\scriptstyle \mathrm{c}} \\ H^*(B^*(\widetilde{M})^G) & \xhookrightarrow{\mathrm{res_b}} & H^*_{\mathrm{b}}(M) & \xrightarrow{\mathrm{trans_b}} & H^*(B^*(\widetilde{M})^G) \end{array}$$

with the additional properties that $\mathrm{res_b}$ and $\mathrm{trans_b}$ do not increase seminorms. Those are the only ingredients which entered the proof of Theorem 3, so that we immediately obtain the analogous result:

THEOREM 12. *The restriction map*

$$\mathrm{res} : H^*(C^*_{\mathrm{c}}(\widetilde{M})^G) \hookrightarrow H^*(M)$$

is an isometric embedding.

COROLLARY 13. $\|\omega_M\|_\infty = \|\omega\|_\infty$.

In view of the relation $\|M\| = \mathrm{Vol}(M)/\|\omega_M\|_\infty$ exhibited in (3) and the fact that the constant $\|\omega\|_\infty$ only depends on the universal cover of M, the proportionality principle is now proven in all generality.

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