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COMMON HYPERCYCLIC ENTIRE FUNCTIONS FOR MULTIPLES OF DIFFERENTIAL OPERATORS

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Abstract. The purpose of the present note is to prove the existence of common hypercyclic entire functions for the family of differential operators $\{\lambda p(D) : \lambda \in \mathbb{C} \setminus \{0\}\}$, where D is the differentiation operator acting on the space of entire functions and p a non-constant polynomial.

1. Introduction. Let X be a topological vector space. A sequence of continuous linear operators $T_n : X \to X$, $n = 1, 2, \ldots$, is said to be *hypercyclic* if there exists a vector $x \in X$ so that the sequence $\{T_1x, T_2x, \ldots\}$ is dense in X. In this case the vector x will be called hypercyclic for $\{T_n\}$ and the symbol $\mathrm{HC}(\{T_n\})$ stands for the set of hypercyclic vectors for $\{T_n\}$. If the sequence $\{T_n\}$ comes from the iterates of a single operator T, i.e. $T_n = T^n$, $n = 1, 2, \ldots$, then T is called hypercyclic and the set of hypercyclic vectors for T is denoted by $\mathrm{HC}(T)$. There are many examples of such operators, and several deep results in this area have been established during the last decade: see for example the survey articles [7], [15], [16], [18], [21]. It is well known that under some mild assumptions on X, if $T : X \to X$ is hypercyclic then the set $\mathrm{HC}(T)$ is G_δ, i.e. a countable intersection of open sets, and dense in X (see [15]).

In the present article we focus on the problem of common hypercyclic vectors for an uncountable family of operators. Let us briefly explain this problem. Suppose that $T_j : X \to X$ is a hypercyclic operator for every $j \in J$, where J is an index set. If the set J is countable, then since $\mathrm{HC}(T_j)$ is G_δ and dense in X, Baire's category theorem implies that the set $\bigcap_{j\in J} \mathrm{HC}(T_j)$ of common hypercyclic vectors for the family $\{T_j\}_{j\in J}$ is residual in X (i.e. it contains a dense G_δ set), hence non-empty. On the other hand, if J is uncountable, the problem of the existence of common hypercyclic vectors for $\{T_j\}_{j\in J}$ becomes non-trivial. The first result in this direction is due to Abakumov and Gordon (see [1]) and independently Peris [19] (unpub-

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lished). In particular they proved, answering a question of Salas [20], that, if $B : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ is the backward shift operator on the space $l^2(\mathbb{N})$ of square summable sequences then the set $\bigcap_{|\lambda|>1} \mathrm{HC}(\lambda B)$ is residual in $l^2(\mathbb{N})$. Shortly after, many authors established the existence of common hypercyclic vectors for several uncountable families of operators (see [2]–[6], [8]–[13]). Recently, Gallardo-Gutiérrez and Partington provided an alternative version of a common hypercyclicity-universality criterion from [12] and as an application they proved the existence of a residual set of common hypercyclic vectors for a particular family of hypercyclic scalar multiples of the adjoint of a multiplier in the Hardy space. Their results extend those from [1], [3]. They point out that it is still unknown whether the set $\{\lambda M_\phi^\star : |\lambda| > 1/2\}$ has common hypercyclic vectors, where ϕ is the outer function $z \mapsto z+1$ and $M_\phi^\star$ is essentially the operator $\mathrm{Id} + B$, B being the backward shift and Id the identity operator. The purpose of this note is to show that if we replace the backward shift by the differentiation operator acting on the space of entire functions, then common hypercyclic vectors for the family $\{\lambda(\mathrm{Id} + D) : \lambda \in \mathbb{C} \setminus \{0\}\}$ do exist. Actually we shall prove the following stronger theorem.

THEOREM 1.1. *Consider the space $H(\mathbb{C})$ of entire functions, endowed with the topology of uniform convergence on compact subsets of $\mathbb{C}$. Let $D : H(\mathbb{C}) \to H(\mathbb{C})$ be the differentiation operator, i.e. $Df = f'$ for every $f \in H(\mathbb{C})$, and let p be a non-constant polynomial. Then the set $\bigcap_{\lambda\in\mathbb{C}\setminus\{0\}} \mathrm{HC}(\lambda p(D))$ is G_δ and dense in $H(\mathbb{C})$.*

2. Proof of Theorem 1.1. The main ingredient of the proof is the following variant of a result of Bayart and Matheron (see Proposition 4.2 in [6]).

THEOREM 2.1 (Bayart–Matheron). *Let X be a separable Fréchet space and let $T : X \to X$ be a continuous linear operator. Assume that:*

(i) *There exists $A \subset \bigcup_n \operatorname{Ker} T^n$ such that A is dense in X and T has a right inverse $S : A \to X$.*
(ii) *There exists $\lambda_0 \geq 0$ such that for $\lambda > \lambda_0$ and each $u \in A$, the set $\{\lambda^{-n} S^n u : n = 1, 2, \ldots\}$ is bounded in X.*

Then $\bigcap_{\lambda>\lambda_0} \mathrm{HC}(\lambda T)$ is a dense G_δ subset of X.

We first prove the following

PROPOSITION 2.2. *The set $\bigcap_{\lambda\in(0,\infty)} \mathrm{HC}(\lambda p(D))$ is G_δ and dense in $H(\mathbb{C})$.*

Proof. Let $s = \deg(p)$ and suppose without loss of generality that the leading coefficient of p is 1. Hence p decomposes as $p(z) = (\lambda_1+z)\cdots(\lambda_s+z)$, where $-\lambda_1, \dots, -\lambda_s \in \mathbb{C}$ are the roots of p ignoring multiplicities. For any $\mu \in \mathbb{C}$ consider the differential operator $T_\mu = \mu\mathrm{Id} + D$. It is easy to check that the operator $S_\mu : H(\mathbb{C}) \to H(\mathbb{C})$ defined by

$$S_\mu(f)(z) = e^{-\mu z} \int_0^z f(w) e^{\mu w}\, dw$$

for $f \in H(\mathbb{C})$ is a right inverse for T_μ, i.e.

$$T_\mu \circ S_\mu = \mathrm{Id}.$$

A direct calculation gives that

$$(\mu\,\mathrm{Id} + D)^n (e^{-\mu z} q(z)) = 0 \quad \text{for every polynomial } q \text{ with } \deg(q) < n. \tag{1}$$

Hence a right inverse for $p(D) = T_{\lambda_1} \circ \dots \circ T_{\lambda_s}$ is

$$S = S_{\lambda_s} \circ \dots \circ S_{\lambda_1}$$

and (1) implies that

$$A := \{e^{-\lambda_1 z} q(z) : q \text{ polynomial}\} \subset \bigcup_n \mathrm{Ker}\, p(D)^n.$$

Observe also that the set A is dense in $H(\mathbb{C})$. For every positive integer k, every $\mu \in \mathbb{C}$ and every $f \in H(\mathbb{C})$ we get

$$\sup_{|z| \le k} |S_\mu(f)(z)| \le k \left(\sup_{|z| \le k} |e^{\mu z}|\right)^2 \sup_{|z| \le k} |f(z)|, \tag{2}$$

which yields

$$\sup_{|z| \le k} |S_\mu^n(f)(z)| \le \left[k \left(\sup_{|z| \le k} |e^{\mu z}|\right)^2\right]^n \sup_{|z| \le k} |f(z)| \tag{3}$$

for every positive integer n. Observe that for $\mu_1, \mu_2 \in \mathbb{C}$ the operators S_{μ_1}, S_{μ_2} commute. An easy way to see this is to check first that we have $S_{\mu_1} \circ S_{\mu_2}(e^{bz}) = S_{\mu_2} \circ S_{\mu_1}(e^{bz})$ for every $b \in \mathbb{C}$. Then fix an infinite subset B of $\mathbb{C}$ having at least one limit point in $\mathbb{C}$. It is well known that the closure of the linear span of the set $\{e^{bz} : b \in B\}$ is the whole space $H(\mathbb{C})$ of entire functions and since $S_{\mu_1} \circ S_{\mu_2}, S_{\mu_2} \circ S_{\mu_1}$ coincide on $\{e^{bz} : b \in B\}$, using (2) we conclude that S_{μ_1}, S_{μ_2} commute.

We shall now prove that conditions (i), (ii) of Theorem 2.1 are satisfied for the operators $p(D)$, S and the set A. It is clear that (i) is satisfied. Observe that it suffices to verify (ii) for the function $e^{-\lambda_1 z} z^m$, where m is any given non-negative integer. It is straightforward that

$$S_{\lambda_1}^n (e^{-\lambda_1 z} z^m) = \frac{e^{-\lambda_1 z} z^{m+n}}{(m+1) \cdots (m+n)}. \tag{4}$$

Fix any positive integer k. By (3), (4) and since S_{μ_1}, S_{μ_2} commute for every

$\mu_1, \mu_2 \in \mathbb{C}$, we arrive at

$$\sup_{|z|\le k} |S^n(e^{-\lambda_1 z} z^m)| = \sup_{|z|\le k} |S^n_{\lambda_s} \circ \cdots \circ S^n_{\lambda_1}(e^{-\lambda_1 z} z^m)|$$
$$\le k^{(s-1)n} \Big(\prod_{j=2}^{s} \sup_{|z|\le k} |e^{\lambda_j z}|\Big)^{2n} \frac{\sup_{|z|\le k} |e^{\lambda_1 z}| k^{m+n}}{(m+1)\cdots(m+n)}.$$

This implies that $\lambda^{-n} \sup_{|z|\le k} |S^n(e^{-\lambda_1 z} z^m)| \to 0$ as $n \to \infty$ for every $\lambda > 0$ and therefore condition (ii) of Theorem 2.1 is satisfied for every function in A. This completes the proof of the proposition. ∎

The following result due to León-Saavedra and Müller [17] is needed to finish the proof of Theorem 1.1.

PROPOSITION 2.3 (León-Saavedra–Müller). *Let $T : X \to X$ be a continuous linear operator acting on a complex topological vector space X. If T is hypercyclic then for every $\theta \in [0,1]$ the operator $e^{2\pi i\theta}T$ is hypercyclic and in addition T and $e^{2\pi i\theta}T$ have the same set of hypercyclic vectors*, *i.e.* $\mathrm{HC}(T) = \mathrm{HC}(e^{2\pi i\theta}T)$.

We would like to mention that León-Saavedra and Müller stated their result in the context of a complex Banach space, but, as the anonymous referee kindly informed us, their argument still works on any complex topological vector space.

Last step of the proof of Theorem 1.1. By Proposition 2.3, for every $r > 0$ and every $\theta \in [0,1]$ we have $\mathrm{HC}(rp(D)) = \mathrm{HC}(re^{2\pi i\theta}p(D))$. This yields $\bigcap_{\lambda\in(0,\infty)} \mathrm{HC}(\lambda p(D)) = \bigcap_{\lambda\in\mathbb{C}\setminus\{0\}} \mathrm{HC}(\lambda p(D))$ and Proposition 2.2 implies the desired result. The proof of Theorem 1.1 is complete.

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