COLLOQUIUM MATHEMATICUM

VOL. 111 2008 NO. 2

SOME CRITICAL ALMOST KÄHLER STRUCTURES

BY

TAKASHI OGURO (Saitama) and KOUEI SEKIGAWA (Niigata)

Abstract. We consider the set of all almost Kähler structures (g, J) on a $2n$-dimensional compact orientable manifold M and study a critical point of the functional $\mathscr{F}_{\lambda,\mu}(J, g) = \int_M (\lambda\tau + \mu\tau^*)\, dM_g$ with respect to the scalar curvature τ and the $*$-scalar curvature τ^*. We show that an almost Kähler structure (J, g) is a critical point of $\mathscr{F}_{-1,1}$ if and only if (J, g) is a Kähler structure on M.

1. Introduction. Let M be a compact orientable manifold of dimension m. We denote by $\mathcal{M}(M)$ the set of all Riemannian metrics on M and by $\mathcal{R}(M)$ the set of all Riemannian metrics of a fixed volume form. It is well-known that a Riemannian metric $g \in \mathcal{R}(M)$ is a critical point of the functional $\mathscr{A}$ on $\mathcal{R}(M)$ defined by

$$\mathscr{A}(g) = \int_M \tau\, dM_g \tag{1.1}$$

if and only if g is an Einstein metric, where τ is the scalar curvature of g and dM_g is the volume form of g.

Now, let M be a compact manifold of dimension $m = 2n$ admitting an almost complex structure. We denote by $\mathcal{AH}(M)$ the set of all almost Hermitian structures and by $\mathcal{AH}(M, \Omega)$ the set of all almost Hermitian structures with the same Kähler form Ω. An almost Hermitian manifold $M = (M, J, g)$ with the closed Kähler form Ω ($d\Omega = 0$) is called an *almost Kähler manifold.*

Let $M = (M, J, g)$ be a compact almost Kähler manifold and Ω the corresponding Kähler form. Then we may note that any almost Hermitian structure $(J, g) \in \mathcal{AH}(M, \Omega)$ is an almost Kähler structure on M. In this case, we denote $\mathcal{AH}(M, \Omega)$ by $\mathcal{AK}(M, \Omega)$. In [1], Blair and Ianus studied critical points of the functional $\mathscr{F}$ on $\mathcal{AK}(M, \Omega)$ defined by

$$\mathscr{F}(J, g) = \int_M (\tau^* - \tau)\, dM_g, \tag{1.2}$$

where τ^* is the $*$-scalar curvature of (M, J, g). They proved that (J, g) is

2000 *Mathematics Subject Classification*: 53C15, 53C55.

Key words and phrases: almost Kähler manifold, Kähler manifold.

a critical point of $\mathscr{F}$ on $\mathcal{AK}(M,\Omega)$ if and only if the Ricci tensor ϱ is J-invariant.

We denote by $\mathcal{AK}(M,[\Omega])$ the set of all almost Kähler structures on M with the same Kähler class $[\Omega]$ in the de Rham cohomology group. It is well-known that $\mathcal{AK}(M,\Omega)$ is a contractible Fréchet space. However, the space $\mathcal{AK}(M,[\Omega])$ might be disconnected in general. In [4, 5], Koda studied critical points of the functional $\mathscr{F}_{\lambda,\mu}$ on $\mathcal{AH}(M,\Omega)$ and $\mathcal{AK}(M,[\Omega])$ defined by

$$\mathscr{F}_{\lambda,\mu}(J,g)=\int_M(\lambda\tau+\mu\tau^*)\,dM_g,\qquad(\lambda,\mu)\in\mathbb{R}^2\setminus(0,0).\tag{1.3}$$

and gave a necessary condition for $(J,g)\in\mathcal{AK}(M,[\Omega])$ to be a critical point of $\mathscr{F}_{\lambda,\mu}$. Since the functional $\mathscr{F}_{\lambda,\mu}$ is invariant under the action of the diffeomorphism group $\mathrm{Diff}(M)$ of M, by applying Moser's stability theorem ([6]), we may easily show that critical points of the functional $\mathscr{F}_{\lambda,\mu}$ on $\mathcal{AK}(M,[\Omega])$ coincide with the ones of $\mathscr{F}_{\lambda,\mu}$ on $\mathcal{AK}(M,\Omega)$ corresponding to the Kähler form Ω of (J,g). Therefore, $(J,g)\in\mathcal{AK}(M,[\Omega])$ is a critical point of $\mathscr{F}_{\lambda,\mu}$ if and only if $(\mu-\lambda)\varrho$ is J-invariant [7].

In the present paper, we may regard (1.3) as a functional on $\mathcal{AK}(M)$, the set of all almost Kähler structures on M, and we give a necessary and sufficient condition for $(J,g)\in\mathcal{AK}(M)$ to be a critical point of $\mathscr{F}_{\lambda,\mu}$.

2. Preliminaries. Let $M=(M,J,g)$ be a $2n$-dimensional compact almost Kähler manifold with almost Hermitian structure (J,g), and Ω be the Kähler form of M defined by $\Omega(X,Y)=g(X,JY)$ for $X,Y\in\mathfrak{X}(M)$. We assume that M is oriented by the volume form $dM_g=((-1)^n/n!)\Omega^n$. We denote by ∇, R, ϱ and τ the Riemannian connection, the curvature tensor, the Ricci tensor and the scalar curvature of M, respectively. The curvature tensor R is defined by $R(X,Y)Z=[\nabla_X,\nabla_Y]Z-\nabla_{[X,Y]}Z$ for $X,Y,Z\in\mathfrak{X}(M)$. A tensor field ϱ^* on M of type $(0,2)$ defined by

$$\varrho^*(x,y)=\mathrm{trace}(z\mapsto R(x,Jz)Jy)=\frac{1}{2}\,\mathrm{trace}(z\mapsto R(x,Jy)Jz)\tag{2.1}$$

is called the *Ricci $*$-tensor*, where $x,y,z\in T_p(M)$ (the tangent space of M at $p\in M$). We denote by τ^* the *$*$-scalar curvature* of M, which is the trace of the linear endomorphism Q^* defined by $g(Q^*x,y)=\varrho^*(x,y)$. We remark that ϱ^* satisfies

$$\varrho^*(JX,JY)=\varrho^*(Y,X)\tag{2.2}$$

for any $X,Y\in\mathfrak{X}(M)$. Thus ϱ^* is symmetric if and only if ϱ^* is J-invariant.

In this paper, for any orthonormal basis (resp. any local orthonormal frame field) $\{e_i\}_{i=1,\dots,2n}$ at any point $p\in M$ (resp. on a neighborhood of p), we shall adopt the following notational convention:

$$(2.3)\qquad \begin{aligned} R_{ijkl} &= g(R(e_i, e_j)e_k, e_l), & R_{\bar{i}\bar{j}\bar{k}\bar{l}} &= g(R(Je_i, Je_j)Je_k, Je_l),\\ \varrho_{ij} &= \varrho(e_i, e_j), & \varrho_{\bar{i}\bar{j}} &= \varrho(Je_i, Je_j),\\ \varrho^*_{ij} &= \varrho^*(e_i, e_j), & \varrho^*_{\bar{i}\bar{j}} &= \varrho^*(Je_i, Je_j),\\ J_{ij} &= g(Je_i, e_j), & \nabla_i J_{jk} &= g((\nabla_{e_i} J)e_j, e_k), \end{aligned}$$

and so on, where the Latin indices run over the range $1, \ldots, 2n$. Then we have

$$(2.4)\qquad J_{ij} = -J_{ji}, \quad \nabla_i J_{jk} = -\nabla_i J_{kj}, \quad \nabla_i J_{\bar{j}\bar{k}} = -\nabla_i J_{jk}.$$

The condition $d\Omega = 0$ is equivalent to

$$(2.5)\qquad \underset{i,j,k}{\mathfrak{S}} \nabla_i J_{jk} = \nabla_i J_{jk} + \nabla_j J_{ki} + \nabla_k J_{ij} = 0.$$

Further, since M is a quasi-Kähler manifold and a semi-Kähler manifold, we have

$$(2.6)\qquad \nabla_i J_{jk} + \nabla_{\bar{i}} J_{\bar{j}k} = 0, \quad \sum_a \nabla_a J_{ai} = 0.$$

The following curvature identity is due to Gray ([3]):

$$(2.7)\qquad \begin{aligned} 2\sum_a (\nabla_a J_{ij})\nabla_a J_{kl} = {} & R_{ijkl} - R_{ij\bar{k}\bar{l}} - R_{\bar{i}\bar{j}kl} + R_{\bar{i}\bar{j}\bar{k}\bar{l}}\\ & + R_{\bar{i}j\bar{k}l} + R_{\bar{i}jk\bar{l}} + R_{i\bar{j}\bar{k}l} + R_{i\bar{j}k\bar{l}}. \end{aligned}$$

From this equality, we have

$$(2.8)\qquad \varrho^*_{ij} + \varrho^*_{ji} - \varrho_{ij} - \varrho_{\bar{i}\bar{j}} = \sum_{a,b} (\nabla_a J_{ib})\nabla_a J_{jb},$$

and further

$$(2.9)\qquad \|\nabla J\|^2 = 2(\tau^* - \tau).$$

Therefore, M is a Kähler manifold if and only if $\tau^* = \tau$.

3. Critical points of $\mathscr{F}_{\lambda,\mu}$. Let M be a $2n$-dimensional compact orientable manifold and $\mathcal{AK}(M)$ the set of all almost Kähler structures on M. For a point $(J, g) \in \mathcal{AK}(M)$, consider a curve $(J(t), g(t))$ in $\mathcal{AK}(M)$ with $(J(0), g(0)) = (J, g)$. We denote by Ω and $\Omega(t)$ the Kähler forms of (J, g) and $(J(t), g(t))$, respectively. Then $\alpha(t) = \Omega(t) - \Omega$ is a 1-parameter family of closed 2-forms and $\alpha(0) = 0$.

We denote by $\nabla^{(t)}$, $R(t)$, $\varrho(t)$, $\varrho^*(t)$, $\tau(t)$ and $\tau^*(t)$ the Riemannian connection, the curvature tensor, the Ricci tensor, the Ricci $*$-tensor, the scalar curvature and the $*$-scalar curvature of $(M, J(t), g(t))$, respectively.

Let $(U; x_1, \dots, x_{2n})$ be a local coordinate system on a coordinate neighborhood U of M. With respect to the natural frame $\{\partial_i = \partial/\partial x_i\}_{i=1,\dots,2n}$, we put

$$\begin{aligned} &g(t)(\partial_i, \partial_j) = g(t)_{ij}, && J(t)(\partial_i) = J(t)_i{}^j \partial_j, \\ &(\nabla^{(t)}_{\partial_i} J(t))\partial_j = (\nabla^{(t)}_i J(t)_j{}^k)\partial_k, && R(t)(\partial_i, \partial_j)\partial_k = R(t)_{ijk}{}^l \partial_l, \\ &\varrho(t)(\partial_i, \partial_j) = \varrho(t)_{ij}, && \varrho^*(t)(\partial_i, \partial_j) = \varrho^*(t)_{ij}, \\ &\alpha(t)(\partial_i, \partial_j) = \alpha(t)_{ij}, \end{aligned}$$

and $(g(t)^{ij}) = (g(t)_{ij})^{-1}$. In particular, we have $g(0)_{ij} = g_{ij}$, $J(0)_i{}^j = J_i{}^j$, $\nabla^{(0)}_i J(0)_j{}^k = \nabla_i J_j{}^k$, $R(0)_{ijk}{}^l = R_{ijk}{}^l$, $\varrho(0)_{ij} = \varrho_{ij}$, $\varrho^*(0)_{ij} = \varrho^*_{ij}$ and $\alpha(0)_{ij} = 0$.

Now, put

$$\left.\frac{d}{dt}\right|_{t=0} g(t)_{ij} = h_{ij}, \qquad \left.\frac{d}{dt}\right|_{t=0} J(t)_i{}^j = K_i{}^j, \qquad \left.\frac{d}{dt}\right|_{t=0} \alpha(t)_{ij} = A_{ij}. \tag{3.1}$$

Then $A = (A_{ij})$ is a closed 2-form, $h = (h_{ij})$ is a symmetric $(0,2)$-tensor on M and

$$\left.\frac{d}{dt}\right|_{t=0} g(t)^{ij} = -h^{ij}, \tag{3.2}$$

where we may use the standard notational convention of tensor analysis; thus h^{ij} means $h^{ij} = g^{ia}g^{jb}h_{ab}$. We denote by $dM_{g(t)}$ the volume form of M with respect to $g(t)$. Then

$$\left.\frac{d}{dt}\right|_{t=0} dM_{g(t)} = \frac{1}{2}\,(g^{ij}h_{ij})\,dM_g. \tag{3.3}$$

By (3.2), the connection coefficients $\varGamma(t)^k_{ij}$ of $\nabla^{(t)}$ satisfy

$$\left.\frac{d}{dt}\right|_{t=0} \varGamma(t)^k_{ij} = \frac{1}{2}\, g^{ka}(\nabla_i h_{aj} + \nabla_j h_{ia} - \nabla_a h_{ij}).$$

Therefore, the derivatives of $R(t)_{ijk}{}^l$, $\varrho(t)_{ij}$ and $\tau(t)$ at $t = 0$ are ([9])

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} R(t)_{ijk}{}^l = \frac{1}{2}\,(&-R_{ijk}{}^a h_a{}^l + R_{ija}{}^l h_k{}^a \\ &+ \nabla_i\nabla_k h_j{}^l - \nabla_j\nabla_k h_i{}^l - \nabla_i\nabla^l h_{jk} + \nabla_j\nabla^l h_{ik}), \end{aligned} \tag{3.4}$$

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \varrho(t)_{ij} = \frac{1}{2}\,(&-R_{aij}{}^b h_b{}^a + \varrho_{ia} h_j{}^a \\ &+ \nabla_a\nabla_j h_i{}^a - \nabla_i\nabla_j h_a{}^a - \nabla^a\nabla_a h_{ij} + \nabla_i\nabla_a h_j{}^a), \end{aligned} \tag{3.5}$$

$$\left.\frac{d}{dt}\right|_{t=0} \tau(t) = -\varrho_{ij}h^{ij} + \nabla^i\nabla^j h_{ij} - \nabla^i\nabla_i h, \tag{3.6}$$

where $h = h_a{}^a$. Further, since $(J(t), g(t)) \in \mathcal{AK}(M)$, we have

$$K_a{}^i J_j{}^a + J_a{}^i K_j{}^a = 0, \tag{3.7}$$

$$h_{ij} = h_{ab} J_i{}^a J_j{}^b + K_{ia} J_j{}^a + J_{ia} K_j{}^a, \tag{3.8}$$

$$K_j{}^i = -h_a{}^i J_j{}^a - A_j{}^i, \tag{3.9}$$

$$h_{ij} = -h_{ab} J_i{}^a J_j{}^b + J_i{}^a A_{aj} + J_j{}^a A_{ai}. \tag{3.10}$$

Conversely, let (h, A) be the pair of a symmetric $(0,2)$-tensor h and a closed 2-form A satisfying (3.10) and define a $(1,1)$-tensor K by (3.9); then the equalities (3.7) and (3.8) hold. This means that for a given point $(J, g) \in \mathcal{AK}(M)$ and the pair (h, A) satisfying (3.10), we obtain a curve $(J(t), g(t)) \in \mathcal{AK}(M)$ whose tangent vector at $t = 0$ is (K, h).

By (3.9) and (3.10), we have

$$K_j{}^i = (h_{cd} J_a{}^c J^{id} + A^{ic} J_{ac} + A_a{}^c J^i{}_c) J_j{}^a - A_j{}^i = h_j{}^a J_a{}^i - A_b{}^a J_a{}^i J_j{}^b. \tag{3.11}$$

By (3.9)–(3.11), we have

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} J(t)^{ij} &= \left.\frac{d}{dt}\right|_{t=0} (g(t)^{ia} J(t)_a{}^j) \\ &= -h^{ia} J_a{}^j + g^{ia} K_a{}^j = A_{ab} J^{ia} J^{jb}, \end{aligned} \tag{3.12}$$

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \varrho^*(t)_{ij} &= \frac{1}{2} \left.\frac{d}{dt}\right|_{t=0} (J(t)_j{}^u R(t)_{iua}{}^b J(t)_b{}^a) \\ &= \varrho^*_{ia} h_j{}^a - \frac{1}{2} R_{iua}{}^b J_j{}^u J^{ac} h_{bc} \\ &\quad - \frac{1}{2} J^{ab} J_j{}^c \nabla_i \nabla_a h_{bc} + \frac{1}{2} J^{ab} J_j{}^c \nabla_c \nabla_a h_{bi} \\ &\quad + \frac{1}{2} (2 J_j{}^q \varrho_i^{*p} - J_j{}^u J^{pa} J^{qb} R_{iuab}) A_{pq}, \end{aligned} \tag{3.13}$$

$$\left.\frac{d}{dt}\right|_{t=0} \tau^*(t) = \varrho^*_{ab} h^{ab} - J^{ia} J^{jb} \nabla_a \nabla_b h_{ij} - 2 J^{ip} \varrho^*_{iq} A_p{}^q, \tag{3.14}$$

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \nabla_i^{(t)} J(t)_j{}^k &= \nabla_i K_j{}^k + J_j{}^a \left.\frac{d}{dt}\right|_{t=0} \Gamma(t)^k_{ia} - J_a{}^k \left.\frac{d}{dt}\right|_{t=0} \Gamma(t)^a_{ij} \\ &= -h_a{}^k \nabla_i J_j{}^a + \frac{1}{2} J_j{}^a (\nabla_a h_i{}^k - \nabla_i h_a{}^k - \nabla^k h_{ia}) \\ &\quad - \frac{1}{2} J_a{}^k (\nabla_i h_j{}^a + \nabla_j h_i{}^a - \nabla^a h_{ij}) - \nabla_i A_j{}^k. \end{aligned} \tag{3.15}$$

We are ready to compute the first variation of (1.3) on $\mathcal{AK}(M)$. We shall use the notational convention (2.3) with respect to a (local) orthonormal frame field $\{e_i\}_{i=1,\ldots,2n}$. By (3.3), (3.6) and (3.14), we have

$$(3.16)\quad \left.\frac{d}{dt}\right|_{t=0} \mathscr{F}_{\lambda,\mu}(J(t), g(t))$$
$$= \left.\frac{d}{dt}\right|_{t=0} \int_M (\lambda\tau(t) + \mu\tau^*(t))\, dM_{g(t)}$$
$$= \int_M \sum_{i,j} \left(-\lambda\varrho_{ij} + \mu\varrho^*_{ij} + \frac{1}{2}(\lambda\tau + \mu\tau^*)\delta_{ij} \right) h_{ij}\, dM_g$$
$$- \mu \int_M \sum_{i,j,a,b} J_{ia}J_{jb}\nabla_a\nabla_b h_{ij}\, dM_g + 2\mu \int_M \sum_{i,j} \varrho^*_{\bar{i}j} A_{ij}\, dM_g$$
$$= \int_M \sum_{i,j} \left(-\lambda\varrho_{ij} + \mu\varrho^*_{ij} - \mu \sum_{a,b} \nabla_a\nabla_b(J_{ia}J_{jb}) + \frac{1}{2}(\lambda\tau + \mu\tau^*)\delta_{ij} \right) h_{ij}\, dM_g$$
$$+ 2\mu \int_M \sum_{i,j} \varrho^*_{\bar{i}j} A_{ij}\, dM_g$$

Here, we have

$$(3.17)\quad \sum_{a,b} \nabla_a\nabla_b(J_{ia}J_{jb}) = \sum_{a,b}(\nabla_a\nabla_b J_{ia})J_{jb} + \sum_{a,b}(\nabla_b J_{ia})\nabla_a J_{jb}.$$

The first term of (3.17) becomes

$$\sum_{a,b}(\nabla_a\nabla_b J_{ia})J_{jb} = -\sum_{a,b,s}(R_{abis}J_{sa} + R_{abas}J_{is})J_{jb}$$
$$= -\frac{1}{2}\sum_{a,b,s}(R_{abis} - R_{sbia})J_{sa}J_{ib} + \varrho_{\bar{i}\bar{j}} = -\varrho^*_{ij} + \varrho_{\bar{i}\bar{j}}.$$

Further, by (2.8), the second term of (3.17) becomes

$$\sum_{a,b}(\nabla_b J_{ia})\nabla_a J_{jb} = -\sum_{a,b}(\nabla_i J_{ab} + \nabla_a J_{bi})\nabla_a J_{jb}$$
$$= -\frac{1}{2}\sum_{a,b} \nabla_i J_{ab}(\nabla_a J_{jb} - \nabla_b J_{ja}) + \sum_{a,b}(\nabla_a J_{ib})\nabla_a J_{jb}$$
$$= -\frac{1}{2}\sum_{a,b}(\nabla_i J_{ab})\nabla_j J_{ab} + \varrho^*_{ij} + \varrho^*_{ji} - \varrho_{ij} - \varrho_{\bar{i}\bar{j}}.$$

Therefore, by (3.16), we obtain

$$(3.18)\quad \left.\frac{d}{dt}\right|_{t=0} \mathscr{F}_{\lambda,\mu}(J(t), g(t)) = \int_M \sum_{i,j}(T_{ij}h_{ij} + 2\mu\varrho^*_{\bar{i}j}A_{ij})\, dM_g,$$

where

$$(3.19)\quad T_{ij} = (\mu - \lambda)\varrho_{ij} + \frac{1}{2}(\lambda\tau + \mu\tau^*)\delta_{ij} + \frac{\mu}{2}\sum_{a,b}(\nabla_i J_{ab})\nabla_j J_{ab}.$$

We note that T_{ij} defines a symmetric $(0,2)$-tensor fields T on M. Summing up the above argument, we obtain

THEOREM 1. *Let M be a $2n$-dimensional compact orientable manifold. Then $(J,g) \in \mathcal{AK}(M)$ is a critical point of the functional $\mathscr{F}_{\lambda,\mu}$ if and only if*

$$\int_M \sum_{i,j} \big(T_{ij}h_{ij} + 2\mu\, \varrho^*_{ij} A_{ij}\big)\, dM_g = 0 \tag{3.20}$$

for any pair (h,A) of a symmetric $(0,2)$-tensor h and a closed 2-form $A = (A_{ij})$ satisfying (3.10), where $T = (T_{ij})$ is a symmetric $(0,2)$-tensor field given by (3.19).

We recall the following fact due to Blair and Ianus:

LEMMA 2 ([1]). *Let B be a symmetric $(0,2)$-tensor on M. Then*

$$\int_M \sum_{i,j} B_{ij} D_{ij}\, dM_g = 0$$

for all symmetric $(0,2)$-tensors D satisfying $DJ + JD = 0$ if and only if B is J-invariant.

Let M be a $2n$-dimensional compact orientable manifold and suppose that $(J,g) \in \mathcal{AK}(M)$ is a critical point of the functional $\mathscr{F}_{\lambda,\mu}$. Then, by Theorem 1, if $A = 0$, we have $\int_M \sum_{i,j} T_{ij}h_{ij}\, dM_g = 0$ for any symmetric $(0,2)$-tensor h satisfying $hJ + Jh = 0$. Thus, by virtue of Lemma 2, we conclude that $T = (T_{ij})$ is J-invariant. By (3.19), we observe that the J-invariance of T and $(\mu-\lambda)\varrho$ are equivalent. On the one hand, consider the pair $(h_{ij}, A_{ij}) = (\delta_{ij}, \Omega_{ij})$ which satisfies (3.10). Then, by (3.19),

$$\sum_{i,j} (T_{ij}h_{ij} + 2\mu\, \varrho^*_{ij} A_{ij}) = (n-1)(\lambda\tau + \mu\tau^*).$$

Therefore, we have

COROLLARY 3. *Let M be a $2n\,(\geq 4)$-dimensional compact orientable manifold. If $(J,g) \in \mathcal{AK}(M)$ is a critical point of the functional $\mathscr{F}_{\lambda,\mu}$, then $(\mu-\lambda)\varrho$ is J-invariant and $\mathscr{F}_{\lambda,\mu}(J,g) = 0$.*

In particular, for $(\lambda,\mu) = (-1,1)$, taking account of (2.9), we obtain

COROLLARY 4. *Let M be a $2n\,(\geq 4)$-dimensional compact orientable manifold. Then $(J,g) \in \mathcal{AK}(M)$ is a critical point of the functional $\mathscr{F}_{-1,1}$ if and only if (J,g) is a Kähler structure on M.*

We remark that if we restrict the functional $\mathscr{F}_{-1,1}$ to the space $\mathcal{AK}(M,\Omega)$, then (J,g) is a critical point of $\mathscr{F}_{-1,1}$ if and only if the Ricci tensor ϱ is J-invariant ([1]). Thus, in particular, if a compact manifold M admits an almost Kähler Einstein structure (J,g), then it is a critical point of the functional $\mathscr{F}_{-1,1}$ on the space $\mathcal{AK}(M,\Omega)$, where Ω is the Kähler form corresponding to the almost Kähler structure (J,g). In [3], Goldberg conjectured that a compact almost Kähler Einstein manifold is integrable. This conjecture is true in the case where the scalar curvature is non-negative ([8]). However, it is still open in the remaining case. It is evident that any critical point (J, g) of the functional $\mathscr{F}_{-1,1}$ on the space $\mathcal{AK}(M)$ is necessarily a critical point of the same functional $\mathscr{F}_{-1,1}$ restricted to the subspace $\mathcal{AK}(M, \Omega)$ if $(J, g) \in \mathcal{AK}(M, \Omega)$. However, it does not seem clear in general whether the converse is also valid or not. So, we cannot confirm the Goldberg conjecture using only the result of Corollary 4. The present paper is mainly motivated by the conjecture. The arguments and the results obtained here suggest that it might be effective to discuss the variational problem for suitable functionals $\mathscr{F}$ on the space $\mathcal{AK}(M)$ in the study of the Goldberg conjecture.

REFERENCES

[1] D. E. Blair and S. Ianus, *Critical associated metrics on symplectic manifolds*, in: Contemp. Math. 51, Amer. Math. Soc., Providence, RI, 1986, 23–29.

[2] S. I. Goldberg, *Integrability of almost Kähler manifolds*, Proc. Amer. Math. Soc. 21 (1969), 96–100.

[3] A. Gray, *Curvature identities for Hermitian and almost Hermitian manifolds*, Tôhoku Math. J. 28 (1976), 601–612.

[4] T. Koda, *Critical almost Hermitian structures*, Indian J. Pure Appl. Math. 26 (1995), 679–690.

[5] —, *Almost Kähler structures with a fixed Kähler class*, Math. J. Toyama Univ. 27 (2004), 125–131.

[6] J. Moser, *On the volume elements on a manifold*, Trans. Amer. Math. Soc. 120 (1965), 286–294.

[7] T. Oguro, K. Sekigawa and A. Yamada, *Some critical almost Kähler structures with a fixed Kähler class*, in: Proc. 8th Internat. Workshop on Complex Structures and Vector Fields, K. Sekigawa and S. Dimiev (eds.), World Sci., to appear.

[8] K. Sekigawa, *On some compact Einstein almost Kähler manifolds*, J. Math. Soc. Japan 39 (1987), 677–684.

[9] S. Tanno, *Deformations of Riemannian metrics on 3-dimensional manifolds*, Tôhoku Math. J. 27 (1975), 437–444.

Department of Mathematical Sciences
School of Science and Engineering
Tokyo Denki University
Saitama, 350-0394, Japan
E-mail: oguro@r.dendai.ac.jp

Department of Mathematics
Faculty of Science
Niigata University
Niigata, 950-2181, Japan
E-mail: sekigawa@math.sc.niigata-u.ac.jp

Received 23 March 2007;
revised 8 June 2007 (4889)