COLLOQUIUM MATHEMATICUM

VOL. 111 2008 NO. 2

THE MULTIPLICITY PROBLEM FOR INDECOMPOSABLE DECOMPOSITIONS OF MODULES OVER DOMESTIC CANONICAL ALGEBRAS

BY

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Dedicated to Professor Helmut Lenzing

Abstract. Given a module M over a domestic canonical algebra Λ and a classifying set $\boldsymbol{X}$ for the indecomposable Λ-modules, the problem of determining the vector $m(M) = (m_x)_{x\in\boldsymbol{X}} \in \mathbb{N}^{\boldsymbol{X}}$ such that $M \cong \bigoplus_{x\in\boldsymbol{X}} X_x^{m_x}$ is studied. A precise formula for $\dim_k \operatorname{Hom}_\Lambda(M, X)$, for any postprojective indecomposable module X, is computed in Theorem 2.3, and interrelations between various structures on the set of all postprojective roots are described in Theorem 2.4. It is proved in Theorem 2.2 that a general method of finding vectors $m(M)$ presented by the authors in Colloq. Math. 107 (2007) leads to algorithms with the complexity $\mathcal{O}((\dim_k M)^4)$. A precise description of algorithms determining the multiplicities $m(M)_x$ for postprojective roots $x \in \boldsymbol{X}$ is given (Algorithms 6.1, 6.2 and 6.3).

INTRODUCTION

The problem of effective decomposition into a direct sum of indecomposable objects for modules over a fixed algebra of finite or tame representation type seems to be a very natural and interesting question. It was intensively studied in modular representation theory of groups. In representation theory of finite-dimensional algebras over a field, it seems to be a method to obtain classifications of indecomposable modules, rather than an independent research task (see [17, 14, 11, 20, 21, 7]). In the last thirty years, several other powerful research methods have been invented. Consequently the problem of determining an efficient decomposition lost its importance, in some sense, and not so many new results concerning this topic have been obtained. On the other hand, the tools developed were oriented mainly towards the categorical approach, not quite adjusted to attack this kind of task.

2000 *Mathematics Subject Classification*: 16G20, 16G60, 16G70, 68Q99.

Key words and phrases: canonical algebra, module, tame representation type, tube, decomposition, algorithm, multiplicity vector.

The paper was prepared on the occasion of the conference “Representations of Algebras and their Geometry”, organized in honour of Helmut Lenzing.

This paper is devoted to a question closely related to that discussed above; namely, to its weaker version asking for a "normal form" of a module. The paper is a natural continuation of [9], where this problem was precisely formulated. Below, we recall this formulation in a slightly more general setting.

Assume that a complete classification of all pairwise nonisomorphic indecomposable Λ-modules is already known and it is given by means of a fixed pair

$$\boldsymbol{X} = (\boldsymbol{X}, \varepsilon),$$

where $\boldsymbol{X}$ is a so-called *classifying set* (of invariants for indecomposable Λ-modules), $\varepsilon : \boldsymbol{X} \to \operatorname{ind} \Lambda/\cong$ a bijection between $\boldsymbol{X}$ and the set of isomorphism classes of all indecomposable finite-dimensional Λ-modules. Now we can formulate the problem as follows:

Given a Λ-module M, we want to determine the sequence

$$m(M) = (m_x) \in \mathbb{N}^{\boldsymbol{X}}$$

such that $M \cong \bigoplus_{x \in \boldsymbol{X}} X_x^{m_x}$, where X_x is a module from the isomorphism class $\varepsilon(x)$ for every $x \in \boldsymbol{X}$.

The sequence $m(M) = (m_x)_{x \in \boldsymbol{X}}$ is called the *multiplicity vector* of M with respect to the classifying set $\boldsymbol{X}$. Note that, by the Krull–Remak–Schmidt theorem, $m(M)$ is uniquely determined; moreover, it belongs to $\mathbb{N}^{(\boldsymbol{X})} := (\bigoplus_{x \in \boldsymbol{X}} \mathbb{Z}) \cap \mathbb{N}^{\boldsymbol{X}}$.

The problem of determining the multiplicity vectors $m(M)$ is strongly related to that of description of orbits in the variety of Λ-modules with a fixed dimension vector and to the question how to effectively decide if $M \cong M'$ for a pair M, M' of Λ-modules (see [3, 4], and also [10] which is the continuation of this paper).

In [9], a general method of handling this problem is presented. It relies on computing the sequence

$$h(M) = (h_x) \in \mathbb{N}^{\boldsymbol{X}}$$

of dimensions $h_x = \dim_k \operatorname{Hom}_\Lambda(M, X_x)$, and the so-called Auslander–Reiten matrix $T_\Lambda \in \mathbb{M}_{\boldsymbol{X} \times \boldsymbol{X}}(\mathbb{Z})$ for Λ; equivalently, the Auslander–Reiten quiver Γ_Λ for Λ. (Under a suitable assumption on the algebra Λ, it is enough to find the Cartan matrix $C(\Lambda) \in \mathbb{M}_{\boldsymbol{X} \times \boldsymbol{X}}(\mathbb{Z})$ of the Auslander category for Λ). Once we know these two data and k is an algebraically closed field, the coordinates $m(M)_x = m_x$ of the vector $m(M)$ can be computed by applying the formula

$$(*) \qquad m_x = \begin{cases} h_x + h_z - \sum_{y,\, \varepsilon(y) \in {}^-\varepsilon(x)} d_{y,x} h_y & \text{if } X_x \text{ is nonprojective,} \\ h_x - \sum_{y,\, \varepsilon(y) \in {}^-\varepsilon(x)} d_{y,x} h_y & \text{if } X_x \text{ is projective,} \end{cases}$$

where $d_{y,x}$ is the number of arrows $\varepsilon(y) \to \varepsilon(x)$ in the Auslander–Reiten translation quiver $\Gamma_\Lambda = (\Gamma_\Lambda, \tau)$, ${}^-\varepsilon(x)$ denotes the set of all direct predecessors of $\varepsilon(x)$ in Γ_Λ, and $\varepsilon(z) = [\tau X_x]_{\cong}$ (see [9, Corollary 2.3]). This method is tested in [9] on the example of canonical hereditary algebras of type $\widetilde{\mathbb{A}}_{p,q}$. In this case precise algorithmic procedures for solving the multiplicity problem are given, with pessimistic computational complexity (briefly, complexity) $\mathcal{O}(n^4)$, where $n = \dim_k M$ (see [9, Algorithms 4.5 and 5.5]). The main aim of this paper is to present similar results for the whole class of domestic canonical algebras over an algebraically closed field k.

In constructing the algorithms for domestic canonical algebras, and to improve the efficiency of computations, we use general classical results on the structure of the relevant module categories and information on roots of the associated quadratic Euler form. However, a crucial role in our approach is played by the following three main results.

The first, Theorem 2.2, states that there exist algorithms computing the restricted multiplicity vector $m(M)$ for any individual component of Γ_Λ, with the same complexity as in the $\widetilde{\mathbb{A}}_{p,q}$ case, where the classifying set $\boldsymbol{X}$ consists of the postprojective roots, preinjective roots and the data called tubular coordinates, encoding the indecomposable regular modules from the 1-parameter family of stable tubes (see 1.6 and 2.1). The problem for regular components is reduced to an analogous one for algebras of type $\widetilde{\mathbb{A}}_{p,q}$, already solved in [9]. The reduction uses a certain functorial technique developed in Section 3 (see Proposition 3.1 and Lemma 3.3). As a "side effect" we also obtain a description of canonical forms for indecomposable regular modules (see Remark 3.3(i) and Corollary 3.3, cf. [18]).

To handle the problem of computing the restricted multiplicity vector for the postprojective (and preinjective) component we prove the second result, Theorem 2.3, which yields precise formulas for the coordinates $h(M)_x$ of the vector $h(M)$ for postprojective roots $x \in \boldsymbol{X}$. The result refers to the specific structure of the set of all postprojective roots (see Lemma 2.3). In the proof we apply, among other things, the description of the canonical forms for indecomposable postprojective modules over domestic canonical algebras, obtained recently in [18, 15].

The third result, Theorem 2.4, collects all necessary information on interrelations between various combinatorial structures on the postprojective component. In particular, it yields an alternative method (in comparison to "knitting") of computing consecutive dimension vectors in the postprojective component, which together with formula $(*)$ and Proposition 5.7 forms a basis for computing the multiplicities $m(M)_x$ for postprojective roots $x \in \boldsymbol{X}$.

Section 6, containing Algorithms 6.1, 6.2 and 6.3, is in some sense the most significant part of the paper, as it recapitulates all previous considerations. There the algorithms are precisely formulated in an integrated pseudocode form. The presentation of the final part of the paper is intended to create a complete ("up to" [9]) and self-contained "computer algebra project", which is just ready for implementation. Therefore, in Section 7 we provide tables containing initial parts of the postprojective components and the inverses of the Coxeter matrices, for all domestic canonical algebras Λ. (The algorithms use the data from the theorems and from the tables.) We also comment on the efficiency and memory management aspects. In particular, we show how to decrease the complexity of Algorithm 6.2 and to achieve the announced one, $\mathcal{O}(n^4)$. To this end we apply a detailed analysis of some computational linear algebra problems, strongly related with very specific shapes of matrices which appear in the formulation of Theorem 2.3 (see 6.4 and 6.5).

The paper is organized as follows. In Section 1 we recall basic definitions and fix the notation used throughout. There we introduce, in particular, the concept of tubular coordinates (1.3). We recall the definition of domestic canonical algebras (1.4), and the classical theorems on the structure of module categories and classification of indecomposables modules for this class of algebras (see Theorems 1.5 and 1.6). In Section 2 we specify the classifying set $\boldsymbol{X}$ (2.1) and formulate our main results: Theorems 2.2, 2.3 and 2.4. Section 3 is devoted to determining the restricted multiplicity vector for regular components. We prove the results on functorial reduction (Proposition 3.1, Lemma 3.3) and Theorem 2.2(a+b). Section 4 is devoted to the proof of Theorem 2.3, preceded by several technical facts. In Section 5, the proof of Theorem 2.4 is given. Section 6 contains the pseudo-code descriptions of Algorithms 6.1, 6.2 and 6.3, a result that allows us to decrease complexity of Algorithm 6.2 (Lemma 6.4), and the proof of Theorem 2.2(c) (see 6.5). Section 7 consists of the tables containing the data for domestic canonical algebras, mentioned above.

1. PRELIMINARIES AND NOTATION

The definitions and notation we use are standard. Nevertheless, for the benefit of the reader, we briefly recall some of them. We also collect some facts describing the module categories for domestic canonical algebras. For basic information and notation concerning representation theory of algebras (respectively, canonical algebras, categories, linear algebra, algorithm theory) we refer to [2] (respectively, [22, 23], [1], [16], [6]).

1.1. For any positive $n \in \mathbb{N} = \{0, 1, \ldots\}$, we set $[n] = \{1, \ldots, n\}$ and $\mathbb{Z}_n = \{0, \ldots, n-1\}$; by $\mathbb{Z}_n = (\mathbb{Z}_n, \oplus_n)$ we always mean the group of remainders modulo n. For $m \in \mathbb{Z}$, the integral quotient and remainder of m modulo n are denoted by $\mathrm{quo}_n(m)$ and $\mathrm{rem}_n(m)$, respectively. Given a set S, we write $|S|$ for the cardinality of S. If G is a group and $g \in G$, we denote by

(g) the cyclic subgroup of G generated by g, and by $|g|$ $(=|(g)|)$ the order of g.

Throughout the paper, k always denotes a field, usually algebraically closed. For any $m, n \in \mathbb{N}$, we denote by $\mathbb{M}_{m\times n}(k)$ the set of all $m \times n$-matrices with coefficients in k. The identity matrix in $\mathbb{M}_{n\times n}(k)$ is denoted by I_n.

Given $M \in \mathbb{M}_{m\times n}(k)$, we denote by $\mathrm{r}(M)$ the rank of M and by $\mathrm{cor}(M) = m - \mathrm{r}(M)$ its corank. For any $1 \le i \le m$ (resp. $1 \le j \le n$), $M_{|i}$ (resp. $M^{|j}$) is the matrix in $\mathbb{M}_{i\times n}(k)$ (resp. $\mathbb{M}_{m\times j}(k)$) consisting of the first i rows (resp. first j columns) of M. We denote by $\widehat{M}$ the echelon upper triangular matrix obtained by deleting all zero rows from the echelon matrix resulting from the standard Gaussian row elimination procedure applied to M (see [16]).

1.2. By a k-algebra we always mean a finite-dimensional associative connected basic unitary algebra over k. For a k-algebra Λ (respectively, locally bounded category Λ, see [5]), we denote by $\mathrm{mod}\,\Lambda$ the category of all finite-dimensional Λ-modules, by $J = J(\Lambda)$ the Jacobson radical of Λ, and by $\mathrm{rad}_\Lambda = \mathrm{rad}(\mathrm{mod}\,\Lambda)$ the Jacobson radical of the category $\mathrm{mod}\,\Lambda$. If (Q, I) is a bound quiver (see [2]) and the algebra (resp. locally bounded category) Λ has the form $\Lambda = kQ/I$, then we always identify $\mathrm{mod}\,\Lambda$ with the category of all finite-dimensional representations of the quiver $Q = (Q_0, Q_1)$, satisfying the relations from the ideal I. For the definition of the path algebra kQ, we refer to [2]. For any $v \in Q_0$, we denote by $S(v)$ (resp. $P(v)$, $Q(v)$) the simple (resp. indecomposable projective, injective) module corresponding to the vertex v.

Let $\mathrm{K}_0(\Lambda) = \mathrm{K}_0(\mathrm{mod}\,\Lambda)$ denote the Grothendieck group of Λ, or more precisely, of the category $\mathrm{mod}\,\Lambda$. The class of a finite-dimensional Λ-module X in the Grothendieck group $\mathrm{K}_0(\Lambda)$ is denoted by $[X]$. In case $\Lambda = kQ/I$, where (Q, I) is a bound quiver, we use the standard identification $\mathrm{K}_0(\Lambda) \cong \mathbb{Z}^{Q_0}$ induced by associating to X the dimension vector $\underline{\dim}\, X$.

For any pair X, Y of modules in $\mathrm{mod}\,\Lambda$, we set

$$[X, Y] = \dim_k \mathrm{Hom}_\Lambda(X, Y),$$

and we denote by $m(Y)_X$ the maximal integer $n \in \mathbb{N}$ such that X^n is isomorphic to a direct summand of Y.

Throughout the paper $D : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda^{\mathrm{op}}$ means the standard duality $D(-) = \mathrm{Hom}_k(-, k)$.

Given a class $\mathcal{C}$ of objects in $\mathrm{mod}\,\Lambda$ we denote by $\mathrm{add}\,\mathcal{C}$ the additive closure of $\mathcal{C}$ in $\mathrm{mod}\,\Lambda$.

Let $\mathcal{U}$ be an abelian category. Recall that $\mathcal{U}$ is *serial* if it is a length category and each of its indecomposable objects is uniquely determined, up to isomorphism, by its length and socle. In contrast to the category

$\operatorname{mod} \Lambda$, the length (resp. socle) of an object X from $\mathcal{U}$ will be called its $\mathcal{U}$-length (resp. $\mathcal{U}$-socle), and denoted by $\ell_{\mathcal{U}}(X)$ (resp. $\operatorname{soc}_{\mathcal{U}}(X)$). (This is especially important if $\mathcal{U}$ is a full proper exact subcategory of $\operatorname{mod} \Lambda$). We say that a serial category $\mathcal{U}$ is of *type* (n, ∞) if there exist exactly n pairwise nonisomorphic simple objects in $\mathcal{U}$, and for each pair consisting of a simple object X_0 in $\mathcal{U}$ and a positive integer $l \in \mathbb{N}$, there exists an indecomposable object X in $\mathcal{U}$ such that $\operatorname{soc}_{\mathcal{U}}(X) \cong X_0$ and $\ell_{\mathcal{U}}(X) = l$ (cf. [12]).

By the *Auslander–Reiten quiver* Γ_Λ of Λ (A-R quiver, for short), we always mean the translation quiver

$$\Gamma = (\Gamma_0, \Gamma_1, \tau)$$

defined in a standard way (the set of vertices Γ_0 consists of the isoclasses $[X]_{\cong}$ of indecomposable objects X in $\operatorname{mod} \Lambda$, the sets $\Gamma_1([X]_{\cong}, [Y]_{\cong})$ of all arrows from $[Y]_{\cong}$ to $[X]_{\cong}$ consist of $\dim_k(\operatorname{rad}_\Lambda(X,Y)/\operatorname{rad}^2_\Lambda(X,Y))$ elements, and $\tau[X]_{\cong} = [\tau X]_{\cong}$, where τ is the Auslander–Reiten translate).

For any $[X]_{\cong} \in \Gamma_0$, we denote by ${}^-[X]_{\cong}$ (resp. $[X]^+_{\cong}$) the set of all immediate predecessors (resp. successors) of $[X]_{\cong}$ in Γ_Λ, i.e. the set of all vertices $[Y]_{\cong} \in \Gamma_0$ such that there exists an arrow $[Y]_{\cong} \to [X]_{\cong}$ (resp. $[X]_{\cong} \to [Y]_{\cong}$) in Γ_Λ. Similar notation is used for an arbitrary translation quiver $\Gamma = (\Gamma, \tau)$.

Let $\mathcal{C}$ be a connected component in Γ_Λ. Then the additive closure $\operatorname{add}(\bigcup_{[X]_{\cong} \in \mathcal{C}_0} [X]_{\cong})$ is denoted for simplicity by $\operatorname{add} \mathcal{C}$. For a Λ-module X the phrase "X belongs to $\mathcal{C}$" means "$[X]_{\cong}$ belongs to $\mathcal{C}_0$".

Following [2], a connected component $\mathcal{C}$ of Γ_Λ is called *postprojective* if it is acyclic and for any indecomposable Λ-module M in $\mathcal{C}$ there exists $t \in \mathbb{N}$ and an indecomposable projective module P such that $M \cong \tau^{-n} P$.

Finally, a connected convex acyclic full subquiver Σ of the connected translation quiver $\Gamma = (\Gamma, \tau)$ is called a *sectional subquiver* (briefly, a *section*) in Γ if for each $x \in \Gamma_0$ there exists a unique $n \in \mathbb{Z}$ such that $\tau^{-n} x \in \Sigma_0$.

1.3. Recall that a *stable tube* $\mathcal{T}(n)$ of rank $n \geq 1$ is a quiver $\mathbb{Z}\mathbb{A}_\infty/(\tau^n)$ with the translation τ induced from that in the translation quiver $\mathbb{Z}\mathbb{A}_\infty$ (see [22, 23]). Stable tubes of rank 1 are called *homogeneous*. We fix a standard notation of vertices in $\mathcal{T}(n)$ by setting $\mathcal{T}(n)_0 = \{(s,l) : s \in \mathbb{Z}_n,\ l \geq 1\}$. Then $\mathcal{T}(n)$ has the following shape:

$$\begin{tikzcd}[column sep=small]
 & (s\oplus_n 1,1) \arrow[dr] & & (s,1) \arrow[dr] & & (s\ominus_n 1,1) & \\
\cdots & & (s\oplus_n 1,2) \arrow[ur] \arrow[dr] & & (s,2) \arrow[ur] \arrow[dr] & & \cdots \\
 & (s\oplus_n 2,3) \arrow[ur] & \vdots & (s\oplus_n 1,3) \arrow[ur] & \vdots & (s,3) &
\end{tikzcd}$$

Let $\mathcal{T}$ be a connected component of the A-R quiver Γ_Λ of an algebra Λ, which is a stable standard tube of rank n. Then the category $\operatorname{add}\mathcal{T}$ is an abelian serial category of type (n,∞) and each indecomposable module X from $\operatorname{add}\mathcal{T}$ is uniquely determined by its $\mathcal{T}$-socle and $\mathcal{T}$-length, defined by

$$\operatorname{soc}_{\mathcal{T}}(X) = \operatorname{soc}_{\operatorname{add}\mathcal{T}}(X) \quad \text{and} \quad \ell_{\mathcal{T}}(X) = \ell_{\operatorname{add}\mathcal{T}}(X).$$

The $\mathcal{T}$-simple modules (i.e. the simple objects in $\operatorname{add}\mathcal{T}$) are exactly those lying in the mouth of the tube $\mathcal{T}$. They correspond to the vertices $(s,1) \in \mathcal{T}(n)_0$, $s \in \mathbb{Z}_n$. Moreover, the $\mathcal{T}$-socle of the module corresponding to the vertex $(s,l) \in \mathcal{T}(n)_0$ is a $\mathcal{T}$-simple module corresponding to $(s,1) \in \mathcal{T}(n)_0$, and its $\mathcal{T}$-length is l. This yields a precise encoding of indecomposable modules in $\mathcal{T}$. It suffices to write down the precise forms of the consecutive modules from the mouth of $\mathcal{T}$ and to choose arbitrarily one of them to correspond to the vertex $(0,1) \in \mathcal{T}(n)_0$. We denote it by $X(\mathcal{T},0,1)$. In practice, one has to describe only one of them, the remaining can be obtained as its τ-shifts. Then the isoclass of an idecomposable module X from $\mathcal{T}$ is uniquely encoded in the form $X \cong X(\mathcal{T},s,l)$; this means that X is a module such that $\ell_{\mathcal{T}}(X) = l$ and $\operatorname{soc}_{\mathcal{T}}(X) \cong X(\mathcal{T},s,1) = \tau^s X(\mathcal{T},0,1)$. It is clear that in the above notation, the almost split sequences in $\mathcal{T}$ (more precisely, in the subcategory $\operatorname{add}\mathcal{T}$ of $\operatorname{mod}\Lambda$) have the following shape:

$$\begin{aligned} 0 \to X(\mathcal{T},s,l) \to X(\mathcal{T},s\ominus_n 1,l-1) \oplus X(\mathcal{T},s,l+1)& \\ \to (\mathcal{T},s\ominus_n 1,l) \to 0& \end{aligned}$$

for any $s \in \mathbb{Z}_n$, $l \geq 1$ (we assume that $X(\mathcal{T},s,0)$ is a zero-module).

This encoding of indecomposable objects in $\operatorname{add}\mathcal{T}$ is called the *system of tubular coordinates*.

1.4. Consider a subclass of canonical algebras (see [22] for the definition) consisting of the finite-dimensional k-algebras of the form $\Lambda_{p,q,r} = kQ_{p,q,r}/I_{p,q,r}$, $p,q,r \geq 1$, where $Q_{p,q,r}$ is a quiver

$$\begin{tikzcd}
 & a_1 \arrow[r, "\alpha_2"] & \cdots \arrow[r, "\alpha_{p-1}"] & a_{p-1} \arrow[dr, "\alpha_p"] & \\
0 \arrow[ur, "\alpha_1"] \arrow[r, "\beta_1"] \arrow[dr, "\gamma_1"'] & b_1 \arrow[r, "\beta_2"] & \cdots \arrow[r, "\beta_{q-1}"] & b_{q-1} \arrow[r, "\beta_q"] & \omega \\
 & c_1 \arrow[r, "\gamma_2"] & \cdots \arrow[r, "\gamma_{r-1}"] & c_{r-1} \arrow[ur, "\gamma_r"'] &
\end{tikzcd}$$

and $I_{p,q,r}$ is the ideal generated by $\alpha+\beta-\gamma$, $\alpha = \alpha_1\cdots\alpha_p$, $\beta = \beta_1\cdots\beta_q$ and $\gamma = \gamma_1\cdots\gamma_r$. (Later on, the composition $\alpha_i\alpha_{i+1}\cdots\alpha_j$ for $i \leq j$ will be denoted by $\alpha_{i,j}$ and analogously for β and γ). Note that $\Lambda_{p,q,1}$ is isomorphic to the hereditary algebra $\Lambda_{p,q}$ of type $\mathbb{A}_{p,q}$, given by the quiver

$$Q_{p,q}: \quad \begin{tikzcd}[column sep=small] & a_1 \arrow[r, "\alpha_2"] & \cdots \arrow[r, "\alpha_{p-1}"] & a_{p-1} \arrow[dr, "\alpha_p"] & \\ 0 \arrow[ur, "\alpha_1"] \arrow[r, "\beta_1"] & b_1 \arrow[r, "\beta_2"] & \cdots \arrow[r, "\beta_{q-1}"] & b_{q-1} \arrow[r, "\beta_q"] & \omega \end{tikzcd}$$

We often treat $Q_{p,q}$ as a full subquiver of $Q_{p,q,r}$ via the embedding $(Q_{p,q})_0 \hookrightarrow (Q_{p,q,r})_0$.

Let $\Lambda = \Lambda_{p,q,r}$ for some triple (p,q,r). Then the finite-dimensional Λ-modules can be interpreted as linear representations

$$M = (\{M_v\}_{v\in(Q_{p,q,r})_0}, \{M_\delta\}_{\delta\in(Q_{p,q,r})_1})$$

of the quiver $Q_{p,q,r}$, with dimension vector $\underline{\dim}\, M = (\dim_k M_v) \in \mathbb{N}^{(Q_{p,q,r})_0}$, satisfying the relation $\alpha + \beta = \gamma$. For obvious reasons we will restrict attention to matrix representations of the algebra Λ. More precisely, we consider only those finite-dimensional Λ-modules M, with $\underline{\dim}\, M = \underline{n} = (n_v) \in \mathbb{N}^{(Q_{p,q,r})_0}$, for which the spaces M_v over the vertices $0, a_1, \dots, a_{p-1}$, $b_1, \dots, b_{q-1}$, $c_1, \dots, c_{r-1}, \omega$ are resp. $k^{n_0}, k^{n_{a_1}}, \dots, k^{n_{a_{p-1}}}$, $k^{n_{b_1}}, \dots,$ $k^{n_{b_{q-1}}}$, $k^{n_{c_1}} \dots, k^{n_{c_{r-1}}}, k^{n_\omega}$, and the maps M_δ corresponding to the arrows $\alpha_1, \dots, \alpha_p$, $\beta_1, \dots, \beta_q$, $\gamma_1, \dots, \gamma_r$ are left multiplications by some matrices $A_1, \dots, A_p$, $B_1, \dots, B_q$, $C_1, \dots, C_r$ of appropriate dimensions. We allow matrices with zero columns or rows. In this situation, we simply say that a module M is given by the triple (A, B, C), where $A = (A_i)_{i\in[p]}$, $B = (B_i)_{i\in[q]}$, $C = (C_i)_{i\in[r]}$. Sometimes we identify M with (A, B, C).

Analogously, we consider only $\Lambda_{p,q}$-modules that are pairs (A, B), where A, B are as above. In both cases, $\Lambda = \Lambda_{p,q,r}$ and $\Lambda = \Lambda_{p,q}$, we will also use the notation $A_{s,t} = A_s A_{s-1} \dots A_t$ for $t \le s$, and $\overline{A} = A_{p,1}$ (and similarly for B).

Following [23–25], a canonical algebra Λ is called *domestic* if $\Lambda \cong \Lambda_{p,q,r}$, where

$$(p,q,r) \in \mathcal{D} := \{(p,q,1), p,q \ge 1;\ (p,2,2), p \ge 2;\ (3,3,2);\ (4,3,2);\ (5,3,2)\}.$$

1.5. Let $\Lambda = \Lambda_{p,q,2}$ be a domestic canonical algebra. The well-known results of Ringel [22] yield a description of the category $\operatorname{mod}\Lambda$, and the classification of indecomposable Λ-modules, by use of the concept of *rank* (see also [23, 13]).

Recall that the rank function

$$\operatorname{rk} : \mathrm{K}_0(\Lambda) \to \mathbb{Z}$$

on the Grothendieck group $\mathrm{K}_0(\Lambda)$ is given by the formula

$$\operatorname{rk}(d) = d_\omega - d_0$$

for $d \in \mathbb{Z}^{(Q_{p,q,2})_0}$, under the standard identification $\mathrm{K}_0(\Lambda) = \mathbb{Z}^{(Q_{p,q,2})_0}$. We also consider the so-called *growth vector* $\mathrm{gr}(d) \in \mathbb{Z}^{(Q_{p,q,2})_1}$, defined by three sequences

$$\begin{aligned} r_\alpha &= (r_{\alpha_1}, \dots, r_{\alpha_p}) = (d_{a_1} - d_0, \dots, d_\infty - d_{a_{p-1}}), \\ r_\beta &= (r_{\beta_1}, \dots, r_{\beta_q}) = (d_{b_1} - d_0, \dots, d_\omega - d_{b_{q-1}}), \\ r_\gamma &= (r_{\gamma_1}, r_{\gamma_2}) = (d_{c_1} - d_0, d_\omega - d_{c_1}). \end{aligned}$$

The class $\mathrm{ob}(\mathrm{ind}\,\Lambda)$ of all indecomposable Λ-modules splits naturally into a disjoint union of three subclasses $\mathcal{P} = \mathcal{P}(\Lambda)$, $\mathcal{Q} = \mathcal{Q}(\Lambda)$ and $\mathcal{R} = \mathcal{R}(\Lambda)$, consisting of all M such that $\mathrm{rk}(\underline{\dim}\, M) > 0$, $\mathrm{rk}(\underline{\dim}\, M) < 0$ and $\mathrm{rk}(\underline{\dim}\, M) = 0$, respectively. For reasons to be explained below, the modules from these classes are called *postprojective, preinjective* and *regular*, respectively (see the theorem).

It is proved in [22] that there is another description of the classes $\mathcal{P}$ and $\mathcal{Q}$, common for all canonical algebras. Namely, $\mathcal{P}$ (resp. $\mathcal{Q}$) consists of all indecomposable Λ-modules M such that all maps M_δ, $\delta \in (Q_{p,q,r})_1$, are monomorphisms and $\mathrm{gr}(\underline{\dim}\, M) \neq 0$ (resp. epimorphisms and $\mathrm{gr}(-\underline{\dim}\, M) \neq 0$).

The following result furnishes important information on the structure of the category $\mathrm{mod}\,\Lambda$.

THEOREM ([22, 23]). *Let Λ be a domestic canonical algebra.*

(a) *The isomorphism classes of all modules from $\mathcal{P}$ (resp. $\mathcal{Q}$) form a connected postprojective (resp. preinjective) component in the quiver Γ_Λ containing the isoclasses of all indecomposable projective (resp. injective) Λ-modules.*

(b) $\mathrm{add}\,\mathcal{R}$ *is an abelian serial category closed under extensions*, *and*

$$\mathrm{add}\,\mathcal{R} \simeq \coprod_{\lambda \in k \cup \{\infty\}} \mathrm{add}\,\mathcal{T}_\lambda$$

where $\mathcal{T} = \{\mathcal{T}_\lambda\}_{\lambda \in k \cup \{\infty\}}$ is a 1-parameter family of stable standard tubes of (tubular) type $(p, q, 2)$, and $\mathrm{add}\,\mathcal{T}_\lambda$ is an abelian subcategory of $\mathrm{add}\,\mathcal{R}$.

(c) $\mathrm{Hom}_\Lambda(\mathcal{Q}, \mathcal{P}) = 0$; $\mathrm{Hom}_\Lambda(\mathcal{R}, \mathcal{P}) = 0$; $\mathrm{Hom}_\Lambda(\mathcal{Q}, \mathcal{R}) = 0$.

From now on, the notation $\mathcal{P}$ and $\mathcal{Q}$ is used for the components of Γ_Λ rather than for the classes of all postprojective and preinjective indecomposable Λ-modules, respectively.

1.6. We recall that $\mathrm{gl.dim}\,\Lambda = 2$ and a crucial role in the precise classification of indecomposable modules over domestic canonical algebras Λ is played by the Euler quadratic form

$$q = q_\Lambda : \mathrm{K}_0(\Lambda) \to \mathbb{Z}$$

associated to the $\mathbb{Z}$-bilinear

$$\langle -,-\rangle : \mathrm{K}_0(\Lambda)\times \mathrm{K}_0(\Lambda)\to \mathbb{Z}$$

given by the formula

$$\begin{aligned}\langle \underline{\dim}\, M, \underline{\dim}\, N\rangle = {} & \dim_k \mathrm{Hom}_\Lambda(M,N)\\ & - \dim_k \mathrm{Ext}^1_\Lambda(M,N) + \dim_k \mathrm{Ext}^2_\Lambda(M,N)\end{aligned}$$

for M and N in $\operatorname{mod}\Lambda$. The quadratic form q is also defined in terms of the Cartan matrix $C_\Lambda \in \mathbb{M}_{s\times s}(k)$ of the algebra $\Lambda = \Lambda_{p,q,2}$, by the formula

$$q(x) = x^t (C_\Lambda^t)^{-1} x$$

for $x\in\mathbb{Z}^s$, under the identification $\mathbb{Z}^{(Q_{p,q,2})_0} = \mathbb{Z}^s$, where $s = |(Q_{p,q,2})_0|$.

Let $\operatorname{rad} q = \{x\in \mathbb{Z}^{(Q_{p,q,2})_0} : q(x) = 0\}$ denote the radical of the form q. Since Λ is a concealed algebra of Euclidean type, it follows that $\operatorname{rad} q$ is a subgroup of $\mathbb{Z}^{(Q_{p,q,2})_0}$, q is positive semidefinite of corank 1 and

$$(*) \qquad \operatorname{rad} q = \mathbb{Z}\cdot \mathbb{1},$$

where $\mathbb{1}\in \mathbb{Z}^{(Q_{p,q,2})_0}$ is the all-one vector.

According to Ringel's classification [22], $\operatorname{mod}\Lambda$ is controlled by the form q_Λ. In more detail and in a slightly modified version, taking into account $(*)$ and Theorem 1.5(b), this can be phrased as follows:

THEOREM ([22, 23]). *Let Λ be a domestic canonical algebra.*

(a) *For an indecomposable module X in* $\operatorname{add}\mathcal{T}_\lambda$, $\lambda\in k\cup\{\infty\}$, *we have*

$$\underline{\dim}\, X = m\cdot\mathbb{1} \quad \textit{if and only if} \quad \ell_{\mathcal{T}_\lambda}(X) = mn_\lambda,$$

for $m\ge 1$, where n_λ is the rank of $\mathcal{T}_\lambda$.

(b) *The function $\underline{\dim}$ yields bijections of the vertex sets $\mathcal{P}_0$ and $\mathcal{Q}_0$ with the sets*

$$\boldsymbol{P} := \{x\in\mathbb{N}^{(Q_{p,q,2})_0} : q(x) = 1,\ \mathrm{rk}(x) > 0\},$$
$$\boldsymbol{Q} := \{x\in\mathbb{N}^{(Q_{p,q,2})_0} : q(x) = 1,\ \mathrm{rk}(x) < 0\}$$

respectively. Moreover, the set

$$\{x\in\mathbb{N}^{(Q_{p,q,2})_0} : q(x) = 1,\ \mathrm{rk}(x) = 0\}$$

corresponds bijectively via $\underline{\dim}$ to the set of isoclasses of all indecomposable modules X in $\operatorname{add}\mathcal{T}_\lambda$, $\lambda\in k\cup\{\infty\}$, *such that $n_\lambda \nmid \ell_{\mathcal{T}_\lambda}(X)$ $(n_\lambda\ge 2)$.*

We call $\boldsymbol{P}$ (resp. $\boldsymbol{Q}$) the set of all *postprojective* (resp. *preinjective*) *positive roots* of the quadratic Euler form $q = q_\Lambda$.

REMARK. From (b) and the description of postprojective (resp. preinjective) modules over canonical algebras in terms of the growth vector $\mathrm{gr}(x)$, it follows that for any $x\in\mathbb{N}^{(Q_{p,q,2})_0}$ such that $q(x) = 1$, we have $\mathrm{rk}(x) > 0$

(resp. $\mathrm{rk}(x) < 0$) if and only if $\mathrm{gr}(x) \in \mathbb{N}^{(Q_{p,q,2})_1} \setminus \{0\}$ (resp. $\mathrm{gr}(-x) \in \mathbb{N}^{(Q_{p,q,2})_1} \setminus \{0\}$).

Corollary. *The sets* $\boldsymbol{P}$ *and* $\boldsymbol{Q}$ *are classifying sets of invariants for indecomposable modules from the components* $\mathcal{P}$ *and* $\mathcal{Q}$*.*

To define a full classifying set for the whole class of indecomposable Λ-modules one has to specify tubular coordinates for the subcategories $\operatorname{add} \mathcal{T}_\lambda$, $\lambda \in k \cup \{\infty\}$. To this end we have in fact to fix those $\lambda \in k \cup \{\infty\}$ for which n_λ takes the values 2, p and q, respectively, and next to give a precise description of one selected module in the mouth, for each tube $\mathcal{T}_\lambda$.

2. THE MAIN RESULTS

Before we formulate our main results we need to establish some extra notation and first of all to complete the process of precise encoding for indecomposable modules over domestic canonical algebras, i.e. to specify the classifying set $\boldsymbol{X}$.

2.1. To fix the encoding for regular indecomposable modules by tubular coordinates, we apply the tubular structure of the category $\operatorname{add} \mathcal{R}$ (see Theorem 1.5(b)).

Let $\Lambda = \Lambda_{p,q,2}$ be a domestic canonical algebra. As already stated in 1.5, the regular Λ-modules form a 1-parameter family $\mathcal{T} = \{\mathcal{T}_\lambda\}_{\lambda \in k \cup \{\infty\}}$ of stable tubes of type $(p,q,2)$ and each of the categories $\operatorname{add} \mathcal{T}_\lambda$, $\lambda \in k \cup \{\infty\}$, is serial of type (n_λ, ∞), where n_λ denotes the rank of $\mathcal{T}_\lambda$. Additionally, one can assume that $\mathcal{T}_\lambda = \mathcal{T}_\lambda^{p,q,2}$, where $\mathcal{T}^{p,q,2} = \{\mathcal{T}_\lambda^{p,q,2}\}_{\lambda \in k \cup \{\infty\}}$ is a 1-parameter family of tubular type $(p,q,2)$ such that $\mathcal{T}_0^{p,q,2}$, $\mathcal{T}_1^{p,q,2}$, $\mathcal{T}_\infty^{p,q,2}$, $\mathcal{T}_\lambda^{p,q,2}$, $\lambda \in k \setminus \{0,1\}$, are stable tubes of rank p, 2, q and 1, respectively. Moreover, indecomposable modules from the exceptional tubes can be encoded, according to 1.3, as described below (see also [18, 23]).

We can set:

$$X(\mathcal{T}_0, 0, 1) \quad = \quad \begin{tikzcd} & 0 \arrow[r, "0"] & \cdots \arrow[r, "0"] & 0 \arrow[dr, "0"] & \\ k \arrow[ur, "0"] \arrow[r, "1"] \arrow[drr, "1"'] & k \arrow[r, "1"] & \cdots \arrow[r, "1"] & k \arrow[r, "1"] & k \\ & & k \arrow[urr, "1"'] & & \end{tikzcd}$$

and $X(\mathcal{T}_0, s, 1) = S(a_s)$ for $s \in \mathbb{Z}_p \setminus \{0\}$, where $X(\mathcal{T}_0, s', l)$ is the module in the tube $\mathcal{T}_0$ corresponding to the vertex $(s', l) \in \mathcal{T}(p)_0$, for all $s' \in \mathbb{Z}_p$ and $l \geq 1$;

$$X(\mathcal{T}_1, 0, 1) \quad = \quad \begin{tikzcd} & k \arrow[r, "1"] & \cdots \arrow[r, "1"] & k \arrow[dr, "1"] & \\ k \arrow[ur, "1"] \arrow[r, "-1"] \arrow[drr, "0"'] & k \arrow[r, "1"] & \cdots \arrow[r, "1"] & k \arrow[r, "1"] & k \\ & & 0 \arrow[urr, "0"'] & & \end{tikzcd}$$

and $X(\mathcal{T}_1, 1, c_1) = S(c_1)$, where $X(\mathcal{T}_1, s', l)$ is the module in the tube $\mathcal{T}_1$ corresponding to the vertex $(s', l) \in \mathcal{T}(2)_0$, for all $s' \in \mathbb{Z}_2$ and $l \geq 1$;

$$X(\mathcal{T}_\infty, 0, 1) \quad = \quad \begin{tikzcd}[row sep=small]
 & k \arrow[r,"1"] & \cdots \arrow[r,"1"] & k \arrow[dr,"1"] & \\
k \arrow[ur,"1"] \arrow[r,"0"] \arrow[drr,"1"'] & 0 \arrow[r,"0"] & \cdots \arrow[r,"0"] & 0 \arrow[r,"0"] & k \\
 & & k \arrow[urr,"1"'] & &
\end{tikzcd}$$

and $X(\mathcal{T}_\infty, s, 1) = S(b_s)$ for $s \in \mathbb{Z}_q \setminus \{0\}$, where $X(\mathcal{T}_\infty, s', l)$ is the module in the tube $\mathcal{T}_\infty$ corresponding to the vertex $(s', l) \in \mathcal{T}(q)_0$, for all $s' \in \mathbb{Z}_q$ and $l \geq 1$.

To establish the encoding for indecomposable regular modules from the tubes $\mathcal{T}_\lambda$, $\lambda \in k \setminus \{0, 1\}$, of rank 1, it suffices to give a precise description of the unique $\mathcal{T}_\lambda$-simple module, for each λ.

For $\lambda \in k \setminus \{0, 1\}$, we can set

$$X(\mathcal{T}_\lambda, 0, 1) \quad = \quad \begin{tikzcd}[row sep=small]
 & k \arrow[r,"1"] & \cdots \arrow[r,"1"] & k \arrow[dr,"1"] & \\
k \arrow[ur,"\lambda"] \arrow[r,"-1"] \arrow[drr,"\lambda-1"'] & k \arrow[r,"1"] & \cdots \arrow[r,"1"] & k \arrow[r,"1"] & k \\
 & & k \arrow[urr,"1"'] & &
\end{tikzcd}$$

Here $X(\mathcal{T}_\lambda, 0, l)$ is the module of $\mathcal{T}_\lambda$-length l in the tube $\mathcal{T}_\lambda$ for all $l \geq 1$.

Further, for simplicity, we will use the abbreviate notation: $X(\lambda, s, l) = X(\mathcal{T}_\lambda, s, l)$ for $\lambda = 0, 1, \infty$, and $X(\lambda, l) = X(\mathcal{T}_\lambda, 0, l)$ for $\lambda \in k \setminus \{0, 1\}$.

As a consequence, indecomposable regular modules modules are precisely encoded by the following classifying set:

$$\boldsymbol{T} = \bigsqcup_{\lambda \in k \cup \{\infty\}} \boldsymbol{T}_\lambda,$$

where

$$\boldsymbol{T}_\lambda = \begin{cases} \{[0, s, l] : s \in \mathbb{Z}_p,\ l \geq 1\} & \text{for } \lambda = 0, \\ \{[1, s, l] : s \in \mathbb{Z}_2,\ l \geq 1\} & \text{for } \lambda = 1, \\ \{[\infty, s, l] : s \in \mathbb{Z}_q,\ l \geq 1\} & \text{for } \lambda = \infty, \\ \{[\lambda, l] : l \geq 1\} & \text{for } \lambda \in k \setminus \{0, 1\}. \end{cases}$$

Since postprojective and preinjective modules are fully described in terms of their dimension vectors by the sets $\boldsymbol{P}$ and $\boldsymbol{Q}$, we have the following.

PROPOSITION. *The set*

$$\boldsymbol{X} := \boldsymbol{P} \sqcup \boldsymbol{T} \sqcup \boldsymbol{Q}$$

(with the obvious map ε) is a classifying set of invariants for indecomposable Λ-modules.

2.2. Now fix integers $p \geq 1$, $n_0, n_{a_1}, \ldots n_{a_{p-1}}, n_\omega \geq 0$. Let $D \in \mathbb{M}_{n_\omega \times n_0}(k)$ and $A = (A_1, \ldots, A_p)$ be a system of matrices of size $n_{a_1} \times n_0$, $n_{a_2} \times n_{a_1}$,

$\dots,\ n_{a_{p-1}} \times n_{a_{p-2}},\ n_\omega \times n_{a_{p-1}}$, respectively. Then for any $2 \le i \le p+1$, $0 \le j \le p-1$, $n \ge -1$, we define a block matrix $\mathcal{M}^{i,j,n}(D,A) \in \mathbb{M}_{((n+1)n_\omega + n_{a_j}) \times ((n+1)n_0 + n_{a_{i-2}})}(k)$ by setting

$$\mathcal{M}^{i,j,n}(D,A) = \begin{bmatrix} \overline{A}_{p,i-1} & D & 0 & 0 & \cdots & 0 \\ 0 & -\overline{A} & D & 0 & \cdots & 0 \\ 0 & 0 & -\overline{A} & D & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -\overline{A} & D \\ 0 & 0 & \cdots & 0 & 0 & -\overline{A}_{j,1} \end{bmatrix}$$

if $n \ge 0$, and

$$\mathcal{M}^{i,j,n}(D,A) = \overline{A}_{j,i-1}$$

if $n = -1$. We set $n_{a_0} = n_0$, $\overline{A}_{0,1} = I_{n_0}$.

Moreover, for a given collection $n, n_0, n_\omega 1 \ge 0$, $\lambda \in k$ and $E, F \in \mathbb{M}_{n_\omega \times n_0}(k)$ we define a block matrix $\mathcal{M}^n_\lambda(E,F) \in \mathbb{M}_{nn_\omega \times nn_0}(k)$ by setting

$$\mathcal{M}^n_\lambda(E,F) = \begin{bmatrix} G & 0 & 0 & \cdots & 0 \\ F & G & 0 & \cdots & 0 \\ 0 & F & G & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & F & G \end{bmatrix},$$

where $G = G(\lambda) = E + \lambda F$.

For some technical reasons (explained in the proof of Theorem 2.2), we also need the indexing map

$$\mu_p : \mathbb{Z}_p \times (\mathbb{N} \setminus \{0\}) \to \{2, \dots, p+1\} \times \mathbb{Z}_p \times (\mathbb{N} \cup \{-1\})$$

defined by

$$\mu_p(s,l) = \begin{cases} (s-l+2, s, -1), & l \le s, \\ (p - \operatorname{rem}_p(l-s-1) + 1, s, \operatorname{quo}_p(l-s-1)), & l > s. \end{cases}$$

Now, using the above notation, we formulate the main theorem of this paper.

THEOREM. *Let $\Lambda = \Lambda_{p,q,2}$ be a domestic canonical algebra, $\boldsymbol{X}$ a classifying set for indecomposable Λ-modules defined above, M a finite-dimensional Λ-module, with $n = \dim_k M$ and $\underline{\dim}\, M = \underline{n}$, given by a triple (A,B,C), $A = (A_i)_{i\in[p]}$, $B = (B_i)_{i\in[q]}$, $C = (C_i)_{i\in[2]}$ (see* 1.4).

(a) *The coordinates of the restricted multiplicity vector*

$$m(M)_{|\boldsymbol{T}} = (m(M)_x)_{x \in \boldsymbol{T}}$$

of M *with respect to* $\boldsymbol{T}$ *are*:

(i) $m(M)_{[0,s,l]} = h(s,l) + h(s \ominus_p 1, l) - h(s \ominus_p 1, l-1) - h(s, l+1)$,
(ii) $m(M)_{[1,s',l]} = f(s',l) + f(s' \ominus_2 1, l) - f(s' \ominus_2 1, l-1) - f(s', l+1)$,
(iii) $m(M)_{[\infty,s'',l]} = g(s'',l) + g(s'' \ominus_q 1, l) - g(s'' \ominus_q 1, l-1) - g(s'', l+1)$,
(iv) $m(M)_{[\lambda,l]} = 2f_\lambda(l) - f_\lambda(l-1) - f_\lambda(l+1)$,

where $h(u,t) = \operatorname{cor} \mathcal{M}^{\mu_p(u,t)}(\overline{B}, A)$, $f(u',t) = \operatorname{cor} \mathcal{M}^{\mu_2(u',t)}(-\overline{B}, C)$, $g(u'',t) = \operatorname{cor} \mathcal{M}^{\mu_q(u'',t)}(\overline{A}, B)$ *and* $f_\lambda(t) = \operatorname{cor} \mathcal{M}^t_\lambda(\overline{A}, \overline{B})$, $\lambda \in k \setminus \{0,1\}$, $u \in \mathbb{Z}_p$, $u' \in \mathbb{Z}_2$, $u'' \in \mathbb{Z}_q$ *and* $t \geq 1$ (*we set* $f(*,0) = g(*,0) = h(*,0) = f_\lambda(0) = 0$). *Moreover*, *if the finite set* $\sigma(M)$ *consisting of all* $\lambda \in k \setminus \{0,1\}$ *such that* M *contains a direct summand from* $\operatorname{add} \mathcal{T}_\lambda$ *is known*, *then there exists an algorithm with pessimistic complexity* $\mathcal{O}(n^4)$, *determining* $m(M)_{|\boldsymbol{T}}$.

(b) *A scalar* $\lambda_0 \in k \setminus \{0,1\}$ *belongs to* $\sigma(M)$ *if and only if* λ_0 *is a common root of all* $(n_\omega - \operatorname{rk}_{\mathcal{P}}(M))$-*minors of the matrix* $\overline{A} + \lambda \overline{B}$, *regarded as polynomials in* $k[\lambda]$, *where* $\operatorname{rk}_{\mathcal{P}}(M)$ *denotes the rank of the maximal postprojective direct summand of* M. *Moreover*, $\operatorname{rk}_{\mathcal{P}}(M)$ *is equal to the number of all postprojective summands in a decomposition of the Kronecker module* $\overline{M} = (\overline{A}, -\overline{B})$ *into a direct sum of indecomposables in* $\operatorname{mod} \Lambda_{1,1}$. *Consequently*, *the number of summands from the tube* $\mathcal{T}_{\lambda_0}$ *in a decomposition of* M *into a direct sum of indecomposables is equal to*

$$\operatorname{cor}(\overline{A} + \lambda_0 \overline{B}) - \operatorname{rk}_{\mathcal{P}}(M)$$

and there exists an algorithm computing the integer $\operatorname{rk}_{\mathcal{P}}(M)$ *with pessimistic complexity* $\mathcal{O}(n^4)$, *which does not require* (!) *any knowledge of the vector* $m(M)_{|\boldsymbol{P}}$ (*see Remark* 3.5).

(c) *There exists an algorithm with pessimistic complexity* $\mathcal{O}(n^4)$ *which determines the vector*

$$m(M)_{|\boldsymbol{P} \sqcup \boldsymbol{Q}} = (m(M)_x)_{x \in \boldsymbol{P} \sqcup \boldsymbol{Q}}.$$

The proofs of (a) and (b) are given in Section 3. The proof of (c) needs a much deeper analysis and preparation; it will be completed at the end of Section 6. In fact, we not only prove the existence of algorithms with the required properties, but we also give a detailed description in the integrated pseudo-code form (cf. [9], see Section 6). In particular, we precisely describe an algorithm computing $m(M)_{|\boldsymbol{P}}$ (see Algorithms 6.1 and 6.2 in Section 6), but we only explain how to reduce the computation of $m(M)_{|\boldsymbol{T}}$ to the analogous problem for the Kronecker algebra $\Lambda_{1,1}$ and hereditary algebras $\Lambda_{p',q'}$ of type $\widetilde{\mathbb{A}}_{p',q'}$ (see 3.4, cf. also [9]).

2.3. The most difficult problem is to determine the restricted multiplicity vector for the postprojective component. To do this, given a Λ-module M, we give precise formulas for the positive integers $h(M)_d$ for $d \in \boldsymbol{P}$. We start by fixing some extra notation.

For any vector $d \in \mathbb{Z}^{(Q_{p,q,2})_0} = \mathrm{K}_0(\Lambda)$, we set $\overline{d} = d - d_0\mathbb{1}$. We say that d is *reduced* provided $d = \overline{d}$, equivalently $d_0 = 0$. Clearly, $\mathrm{rk}(d) = \mathrm{rk}(\overline{d})$ and $\mathrm{gr}(d) = \mathrm{gr}(\overline{d})$. We denote by $\mathrm{K}_0(\Lambda)_{\mathrm{red}}$ the subgroup of $\mathrm{K}_0(\Lambda)$ consisting of all reduced vectors. Clearly, $\mathrm{K}_0(\Lambda) = \mathrm{K}_0(\Lambda)_{\mathrm{red}} \oplus \mathbb{Z}[S(0)]$. It is easily seen that the mapping $d \mapsto (\overline{d}, d_0\mathbb{1})$ yields another decomposition

$$\mathrm{K}_0(\Lambda) = \mathrm{K}_0(\Lambda)_{\mathrm{red}} \oplus \mathbb{Z} \cdot \mathbb{1}$$

($\mathbb{1} = \sum_{v \in (Q_{p,q,2})_0}[S(v)]$). Following [18], for d as above, we denote by d' the contraction of $\overline{d}$ "along identities", i.e. the vector in $\mathbb{Z}^{(Q^{(d)}_{p,q,2})_0}$ obtained in a natural way from $\overline{d}$, where $Q^{(d)}_{p,q,2}$ is constructed from $Q_{p,q,2}$ by contracting all arrows δ with $r_\delta = 0$ (see [18]).

Let $Q' = Q'_{p,q,2}$ be the full subquiver of $Q_{p,q,2}$ formed by the set $(Q_{p,q,2})_0 \setminus \{0\}$ of vertices and let $\Lambda' = kQ'$. Then Q' is a Dynkin quiver of type $\Delta = (p, q, 2)$. It is clear that $\mathrm{K}_0(\Lambda') = \mathbb{Z}^{Q'_0}$ can be naturally identified with $\mathrm{K}_0(\Lambda)_{\mathrm{red}}$. Then $\mathrm{K}_0(\Lambda) = \mathrm{K}_0(\Lambda') \oplus \mathbb{Z}[S(0)]$ and $\mathrm{K}_0(\Lambda) = \mathrm{K}_0(\Lambda') \oplus \mathbb{Z} \cdot \mathbb{1}$. We denote by $\boldsymbol{L}$ the set of all positive roots $d \in \mathbb{N}_0^{Q'_0}$ of the quadratic form $q' = q_\Delta$, such that all components $r_{\alpha_2}, \dots, r_{\alpha_p}, r_{\beta_2}, \dots, r_{\beta_q}, r_{\gamma_2}$ of $\mathrm{gr}(d)$ are nonnegative.

Now we restrict our attention to the set $\boldsymbol{P}$. It is clear that for any $d \in \boldsymbol{P} = \boldsymbol{P}(\Lambda)$, $\Lambda = \Lambda_{p,q,2}$, we have $\mathrm{rk}(d) > 0$, $\mathrm{gr}(d) \in \mathbb{N}^{(Q_{p,q,2})_1} \setminus \{0\}$ and $\overline{d} \in \mathbb{N}^{(Q_{p,q,2})_0} \setminus \{0\}$. We set $\overline{\boldsymbol{P}} = \{d \in \boldsymbol{P} : d_0 = 0\}$ (clearly, $\overline{\boldsymbol{P}} = \boldsymbol{P} \cap \mathrm{K}_0(\Lambda)_{\mathrm{red}}$).

The following fact summarizes the most essential properties of $\boldsymbol{P}$ (cf. also [18]).

Lemma.

(a) *The mappings $d \mapsto d_{|Q'_0}$ and $d \mapsto (d_0, \overline{d})$ yield bijections* (i) $\overline{\boldsymbol{P}} \leftrightarrow \boldsymbol{L}$ *and* (ii) $\boldsymbol{P} \leftrightarrow \mathbb{N} \times \overline{\boldsymbol{P}}$ *of sets, respectively; in particular, the set $\overline{\boldsymbol{P}}$ is finite.*

(b) $\mathrm{rk}(\boldsymbol{P}(\Lambda)) := \{\mathrm{rk}(d) : d \in \boldsymbol{P}(\Lambda)\} \subset \{1, \dots, 6\}$; *moreover*, $6 \in \mathrm{rk}(\boldsymbol{P}(\Lambda))$ *if and only if $\Lambda = \Lambda_{5,3,2}$.*

(c) *The set* $\mathrm{Con} := \bigcup_{(p,q,2) \in \mathcal{D}} \{d' : d \in \boldsymbol{P}(\Lambda_{p,q,2})\}$ *has* $|\mathrm{Con}| = 18$ (*see Table* 1 *below*).

Proof. Assertion (a) follows from the facts that $q_{|\mathbb{Z}^{(Q'_{p,q,2})_0}} = q'$ and that $\mathbb{1} \in \mathrm{rad}\, q$, where $q = q_\Lambda$, $\Lambda = \Lambda_{p,q,2}$. Assertions (b) and (c) are consequences of the respective properties of root sets for Dynkin diagrams. ∎

From now on we will identify the sets $\boldsymbol{L}$ and $\overline{\boldsymbol{P}}$ using the bijection (i). Note, in particular, that the bijection (ii) equips the set $\boldsymbol{P}$ with some extra "coordinate system".

To formulate our result we need some technical notation. Given $N \in \mathbb{M}_{m\times n}(k)$, for positive $i \in \mathbb{N}$, we form the $m \times in$-matrix

$$N^{(i)} = [N \,|\, -N \,|\, N \,|\, -N \,|\, \ldots] \in \mathbb{M}_{m\times in}(k).$$

Then, for any $i \in \mathbb{N}$, we set

$$N^{(\infty_{|i})} = (N^{(j)})^{|i} \in \mathbb{M}_{m\times i}(k),$$

where $j \in \mathbb{N}$ is an arbitrary positive integer such that $jn \geq i$.

Analogously, for any $i \in \mathbb{N}$, we define $N_{(i)} \in \mathbb{M}_{im\times n}(k)$ and $N_{(\infty_{|i})} \in \mathbb{M}_{i\times n}(k)$, by setting

$$N_{(i)} = ((N^t)^{(i)})^t \quad \text{and} \quad N_{(\infty_{|i})} = (N_{(j)})_{|i},$$

where $j \in \mathbb{N}$ is such that $jm \geq i$. Clearly, we have $N_{(\infty_{|i})} = ((N^t)^{(\infty_{|i})})^t$.

Let $N \in \mathbb{M}_{m\times m}(k)$. For any $i \in \mathbb{N}$, we denote by $i * N$ the block diagonal matrix

$$i * N = \begin{bmatrix} N & 0 & \ldots & 0 \\ 0 & N & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & N \end{bmatrix} \in \mathbb{M}_{im\times im}(k).$$

Let $P = [p_{i,j}] \in \mathbb{M}_{m_1\times n_1}(k)$ and $Q \in \mathbb{M}_{m_2\times n_2}(k)$. Then we denote by $P \otimes Q$ the matrix in $\mathbb{M}_{m_1 m_2\times n_1 n_2}(k)$ that, under the standard identification

$$\mathbb{M}_{m_1 m_2\times n_1 n_2}(k) \cong \mathbb{M}_{m_1\times n_1}(\mathbb{M}_{m_2\times n_2}(k)),$$

has the form

$$P \otimes Q = [p_{i,j} \cdot Q]_{1\leq i\leq m_1,\, 1\leq j\leq n_1}.$$

$P \otimes Q$ can be interpreted as the matrix of the tensor product $(P\cdot) \otimes (Q\cdot) : k^{n_1} \otimes k^{n_2} \to k^{m_1} \otimes k^{m_2}$ of the linear maps $P\cdot : k^{n_1} \to k^{m_1}$ and $Q\cdot : k^{n_2} \to k^{m_2}$, with respect to the standard bases of $k^{n_1} \otimes k^{n_2}$ and $k^{m_1} \otimes k^{m_2}$, respectively, ordered lexicographically.

Let M be a $\Lambda_{p,q}$-module with $\underline{\dim}\, M = \underline{n}$, defined by the pair (A, B), $A = (A_i)_{i\in[p]}$, $B = (B_j)_{j\in[q]}$, and $d \in \mathbb{N}^{(Q_{p,q})_0}$ be a vector such that its growth vector $\mathrm{gr}(d) \in \mathbb{Z}^{(Q_{p,q})_1}$, which is given by sequences r_α and r_β as above, belongs to $\mathbb{N}^{(Q_{p,q})_1}$. For the pair (M, d) as above, we denote by $W(A, B, d)$ the matrix $W = [W_1 \mid W_2 \mid W_3] \in \mathbb{M}_{d_\omega n_\omega \times c}(k)$ of the form

$$\left[\begin{array}{ccc|ccccc|ccc}
r_{\beta_q} * B_{q,q} & & & A_{p,1} & & & & & & & \\
 & \ddots & & & \ddots & & & & & & \\
 & & r_{\beta_1} * B_{q,1} & & & \ddots & & & & & \\
 & & & -B_{q,1} & & & \ddots & & & & \\
 & & & & \ddots & & & A_{p,1} & & & \\
 & & & & & \ddots & & & r_{\alpha_1} * A_{p,1} & & \\
 & & & & & & \ddots & & & \ddots & \\
 & & & & & & & -B_{q,1} & & & r_{\alpha_p} * A_{p,p},
\end{array}\right],$$

where all entries lying outside the two block diagonals are zero and $c = (r_{\beta_q} n_{b_{q-1}} + \cdots + r_{\beta_1} n_0) + d_0 n_0 + (r_{\alpha_1} n_0 + \cdots + r_{\alpha_p} n_{a_{p-1}})$.

Now we can formulate the announced result.

THEOREM. *Let $\Lambda = \Lambda_{p,q,2}$ be a domestic canonical algebra and M a Λ-module with $\underline{\dim}\, M = \underline{n}$, given by the triple (A, B, C), where $A = (A_i)_{i \in [p]}$, $B = (B_i)_{j \in [q]}$ and $C = (C_1, C_2)$ are sequences of matrices defining the structure maps in M corresponding to arrows $\{\alpha_i\}_{i \in [p]}$, $\{\beta_i\}_{j \in [q]}$ and $\{\gamma_1, \gamma_2\}$, respectively. Then for any $d \in \boldsymbol{P}(\Lambda)$ we have*

$$h(M)_d = \operatorname{cor} \mathcal{M}(M, d)$$

where

$$\mathcal{M}(M, d) = \left[W(A, B, d_{|(p,q)}) \,\middle|\, \left(\begin{bmatrix} -I_{r_{\gamma_2}} \\ U(d) \end{bmatrix}_{(\infty_{|\,d_\omega})} \right) \otimes C_2 \right],$$

$d_{|(p,q)} = d_{|(Q_{p,q})_0}$, *and $U(d) \in \mathbb{M}_{\bar{d}_{c_1} \times r_{\gamma_2}}(k)$ is uniquely determined by d. Moreover*:

(a) *If $\operatorname{rk}(d) = 1$, equivalently $d' = [0, 1]$, then $U(d)$ depends only on $\overline{d}_{c_1}$, $\overline{d}_{c_1} = 0$ or 1, and it is a trivial matrix in $\mathbb{M}_{0 \times 1}(k)$ or in $\mathbb{M}_{0 \times 0}(k)$, respectively.*

(b) *If $\operatorname{char}(k) \neq 2$ and $2 \leq \operatorname{rk}(d)$, or $\operatorname{char}(k) = 2$ and $2 \leq \operatorname{rk}(d) \leq 5$, then $U(d)$ depends only on d', and $U(d) = U(d')$ belongs to the 17-element, in fact 13-element, list consisting of all matrices $U(e)$, $e \in \mathrm{Con}$ (see Table 1).*

(c) *If $\operatorname{char}(k) = 2$ and $\operatorname{rk}(d) = 6$ (consequently, $\Lambda = \Lambda_{5,3,2}$), then $U(d)$ depends only on the pair $(d', \operatorname{rem}_6(d_0))$, and $U(d) = U(d', \operatorname{rem}_6(d_0))$ belongs to the 30-element list consisting of all matrices $U(e, i)$, $(e, i) \in \{f \in \mathrm{Con} : \operatorname{rk}(f) = 6\} \times \mathbb{Z}_6$ (see Table 2).*

Table 1. The shapes of the matrices $U(d')$

d'	$U(d')$	d'	$U(d')$
$0\ 1$	–	$\begin{smallmatrix} & 1 & \\ 0 & 1 & 2 \\ & 1 & \end{smallmatrix}$	$[1]$
$\begin{smallmatrix} & 1 & 2 & \\ 0 & 1 & 2 & 3 \\ & 1 & & \end{smallmatrix}$	$[1\ 1]$	$\begin{smallmatrix} & 1 & 2 & \\ 0 & 1 & 2 & 3 \\ & 2 & & \end{smallmatrix}$	$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$
$\begin{smallmatrix} & 1 & 2 & 3 & \\ 0 & 2 & & 3 & 4 \\ & & 2 & & \end{smallmatrix}$	$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$	$\begin{smallmatrix} & 1 & 2 & 3 & \\ 0 & 1 & & 3 & 4 \\ & & 2 & & \end{smallmatrix}$	$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$
$\begin{smallmatrix} & 1 & 2 & 3 & \\ 0 & 1 & & 2 & 4 \\ & & 2 & & \end{smallmatrix}$	$\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$		
$\begin{smallmatrix} & 1 & 2 & 3 & 4 & \\ 0 & & 1 & 3 & & 5 \\ & & 2 & & & \end{smallmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$	$\begin{smallmatrix} & 1 & 2 & 3 & 4 & \\ 0 & & 2 & 3 & & 5 \\ & & 2 & & & \end{smallmatrix}$	$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$
$\begin{smallmatrix} & 1 & 2 & 3 & 4 & \\ 0 & & 1 & 3 & & 5 \\ & & 3 & & & \end{smallmatrix}$	$\begin{bmatrix} 1 & 1 \\ 0 & -1 \\ 0 & 1 \end{bmatrix}$	$\begin{smallmatrix} & 1 & 2 & 3 & 4 & \\ 0 & & 2 & 4 & & 5 \\ & & 3 & & & \end{smallmatrix}$	$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$
$\begin{smallmatrix} & 1 & 2 & 3 & 4 & \\ 0 & & 2 & 3 & & 5 \\ & & 3 & & & \end{smallmatrix}$	$\begin{bmatrix} 0 & -1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}$	$\begin{smallmatrix} & 1 & 2 & 3 & 4 & \\ 0 & & 2 & 4 & & 5 \\ & & 2 & & & \end{smallmatrix}$	$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$
$\begin{smallmatrix} & 2 & 3 & 4 & 5 & \\ 0 & & 2 & 4 & & 6 \\ & & 3 & & & \end{smallmatrix}$	$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$	$\begin{smallmatrix} & 1 & 3 & 4 & 5 & \\ 0 & & 2 & 4 & & 6 \\ & & 3 & & & \end{smallmatrix}$	$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$
$\begin{smallmatrix} & 1 & 2 & 4 & 5 & \\ 0 & & 2 & 4 & & 6 \\ & & 3 & & & \end{smallmatrix}$	$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$	$\begin{smallmatrix} & 1 & 2 & 3 & 5 & \\ 0 & & 2 & 4 & & 6 \\ & & 3 & & & \end{smallmatrix}$	$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$
$\begin{smallmatrix} & 1 & 2 & 3 & 4 & \\ 0 & & 2 & 4 & & 6 \\ & & 3 & & & \end{smallmatrix}$	$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{bmatrix}$		

Table 2. The shapes of the matrices $U(d', \mathrm{rem}_6(d_0))$

	$d' = d^{(1)}$	$d' = d^{(2)}$	$d' = d^{(3)}$	$d' = d^{(4)}$	$d' = d^{(5)}$
$\mathrm{rem}_6(d_0) = 0$	U_1	U_2	U_1	U_4	U_5
$\mathrm{rem}_6(d_0) = 1$	U_2	U_1	U_2	U_3	U_2
$\mathrm{rem}_6(d_0) = 2$	U_1	U_2	U_1	U_2	U_1
$\mathrm{rem}_6(d_0) = 3$	U_2	U_1	U_2	U_4	U_5
$\mathrm{rem}_6(d_0) = 4$	U_1	U_2	U_1	U_2	U_1
$\mathrm{rem}_6(d_0) = 5$	U_2	U_1	U_2	U_3	U_2

The matrices $U_1,\dots,U_5$ are defined as follows:

$$U_1=\begin{bmatrix}0&1&1\\1&0&1\\1&1&0\end{bmatrix},\ U_2=\begin{bmatrix}1&1&1\\0&1&1\\1&0&1\end{bmatrix},\ U_3=\begin{bmatrix}1&1&0\\1&0&1\\1&1&1\end{bmatrix},\ U_4=\begin{bmatrix}1&0&1\\1&1&1\\0&1&0\end{bmatrix},\ U_5=\begin{bmatrix}1&1&1\\0&1&0\\1&1&1\end{bmatrix};$$

whereas $d^{(1)},\dots,d^{(5)}$ denote the last five vectors d' in Table 1, which as vectors from $\mathrm{K}_0(\Lambda_{5,3,2})$ are distinguished by the conditions $\mathrm{rk}(d^{(i)})=6$ and $r_{\alpha_i}(d^{(i)})=2$, $i=1,\dots,5$.

The full proof of the theorem is given in Section 4.

2.4. Now we consider the problems essential for computing the vector $m(M)_{|\boldsymbol{P}}$ for a Λ-module M. We formulate a long theorem collecting very specific, detailed properties of the set of all positive postprojective roots. These properties are mainly connected with the shape of the component $\mathcal{P}$ and with the various structures $\boldsymbol{P}$ is equipped with (cf. Lemma 2.3(a) and the considerations below). In particular, we give formulas controlling the "changes of coordinates" resulting from individual structures. The theorem determines the nature and scheme of the algorithms, discusses the stop problem for them and indicates how to improve their efficiency.

Let $\Lambda=\Lambda_{p,q,2}$ be a domestic canonical algebra. Then, as already stated, the component $\mathcal{P}$ in Γ_Λ containing all indecomposable projective Λ-modules is postprojective in the sense of 1.2. It is also infinite, since Λ is a concealed algebra of Euclidean type (see [23]). In particular, $\mathcal{P}$ admits sections and all of them are Euclidean quivers of the same type. For each section Σ in $\mathcal{P}$ we have $|\Sigma_0|=|(Q_{p,q,2})_0|$, and $\mathcal{P}$ is isomorphic, as a translation quiver, to the full subquiver of $\mathbb{Z}\Sigma$, formed by all vertices $(n,x)\in(\mathbb{Z}\Sigma)_0=\mathbb{Z}\times\Sigma_0$ such that $\tau^n x$ is defined in $\mathcal{P}$. Moreover, under the identification $\tau^n x\mapsto(n,x)$, each choice of a section Σ yields a disjoint splitting $\mathcal{P}_0=(\mathcal{P}^0)_0\cup(\mathcal{P}')_0$, where $\mathcal{P}^0=\mathcal{P}^0(\Sigma)$ and $\mathcal{P}'=\mathcal{P}'(\Sigma)$ are full subquivers of $\mathbb{Z}\Sigma$ such that $\mathcal{P}^0$ is the finite full translation subquiver of $(\mathbb{N}\setminus\{0\})\Sigma$ and $\mathcal{P}'=-\mathbb{N}\Sigma$, since there are no injective modules in $\mathcal{P}$ (see [23]).

We know that $\underline{\dim}$ yields a bijection

$$\mathcal{P}_0\leftrightarrow\boldsymbol{P}$$

(see Theorem 1.6(b)). Consequently, $\boldsymbol{P}$ is endowed with the canonical structure of translation quiver, transported from $\mathcal{P}$ along $\underline{\dim}$. (The translation in $\boldsymbol{P}$ is denoted by the same letter τ.) We assume that all notions and notations introduced above for $\mathcal{P}$ are automatically transported to $\boldsymbol{P}$.

Let

$$\phi=\phi_\Lambda:\mathrm{K}_0(\Lambda)\to\mathrm{K}_0(\Lambda)$$

be the Coxeter transformation for Λ. Recall that ϕ is a $\mathbb{Z}$-linear map, which can be interpreted as the map

$$\phi = (-C^t C^{-1})\cdot : \mathbb{Z}^s \to \mathbb{Z}^s,$$

under the identification $\mathrm{K}_0(\Lambda) = \mathbb{Z}^s$, where $C = C_\Lambda \in \mathbb{M}_{s\times s}(k)$ is the Cartan matrix of Λ and $s = |(Q_{p,q,2})_0|$. Note that ϕ is an isomorphism, since gl.dim Λ is finite and C is nonsingular (see [2, Section III.3] and [22]).

We set $\overline{\mathrm{K}_0(\Lambda)} = \mathrm{K}_0(\Lambda)/\mathrm{rad}\, q_\Lambda$, where $\mathrm{rad}\, q_\Lambda$ is the radical of q_Λ. Since $\phi(\mathbb{1}) = \mathbb{1}$ (and $\mathrm{rad}\, q_\Lambda = \mathbb{Z}\cdot\mathbb{1}$), ϕ induces the so-called *reduced Coxeter transformation*, which is a $\mathbb{Z}$-isomorphism

$$\overline{\phi} : \overline{\mathrm{K}_0(\Lambda)} \to \overline{\mathrm{K}_0(\Lambda)}$$

defined by the formula

$$\overline{\phi}(x + \mathrm{rad}\, q_\Lambda) = \phi(x) + \mathrm{rad}\, q_\Lambda$$

for $x \in \mathrm{K}_0(\Lambda)$ (see [8] and [23, Section XI.1]). Observe that $\pi_| : \mathrm{K}_0(\Lambda)_{\mathrm{red}} \to \overline{\mathrm{K}_0(\Lambda)}$ is an isomorphism, where $\pi : \mathrm{K}_0(\Lambda) \to \overline{\mathrm{K}_0(\Lambda)}$ denotes the canonical projection; the inverse of $\pi_|$ is induced by the epimorphism $^- : \mathrm{K}_0(\Lambda) \to \mathrm{K}_0(\Lambda)_{\mathrm{red}}$, $x \mapsto \overline{x} = x - x_0\mathbb{1}$. We often use the identifications $\mathrm{K}_0(\Lambda') = \mathrm{K}_0(\Lambda)_{\mathrm{red}} = \overline{\mathrm{K}}_0(\Lambda)$ (see also 2.3). In this way we view $\overline{\phi}$ as a map $\mathrm{K}_0(\Lambda)_{\mathrm{red}} \to \mathrm{K}_0(\Lambda)_{\mathrm{red}}$ given by the formula

$$\overline{\phi}(\overline{x}) = \overline{\phi(x)}$$

for $x \in \mathrm{K}_0(\Lambda)$ (similarly for $\mathrm{K}_0(\Lambda')$).

It turns out that $\overline{\phi}$ furnishes some important extra structure on the set $\boldsymbol{L} = \overline{\boldsymbol{P}}$, and consequently, on the set $\boldsymbol{P}$ (cf. Lemma 2.3(a)).

THEOREM. *Let $\Lambda = \Lambda_{p,q,2}$ be a domestic canonical algebra, $\phi = \phi_\Lambda : \mathrm{K}_0(\Lambda) \to \mathrm{K}_0(\Lambda)$ the Coxeter transformation for Λ, $\overline{\phi} : \overline{\mathrm{K}_0(\Lambda)} \to \overline{\mathrm{K}_0(\Lambda)}$ the reduced Coxeter transformation for Λ, and let the subsets $\boldsymbol{L}$ $(= \overline{\boldsymbol{P}})$, $\boldsymbol{P} \subset \mathrm{K}_0(\Lambda)$ be as before. Then*:

(a) *The set $\boldsymbol{L}$ is $\overline{\phi}$-invariant and, for any fixed section Σ in $\boldsymbol{P}$,*

$$\boldsymbol{L} = \mathcal{O}(\overline{x(1)}) \cup \cdots \cup \mathcal{O}(\overline{x(s)}),$$

where $\Sigma_0 = \{x(1), \ldots, x(s)\}$ and $\mathcal{O}(\overline{x(i)})$ is the orbit of the reduced vector $\overline{x(i)}$ under the action of the finite cyclic group $G = (\overline{\phi})$ on $\overline{\mathrm{K}_0(\Lambda)}$.

(b) *Given a section Σ as above, fix a sequence $i_1, \ldots, i_r$ such that $\boldsymbol{L} = \mathcal{O}(\overline{x(i_1)}) \cup \cdots \cup \mathcal{O}(\overline{x(i_r)})$ is a disjoint union. Set $y_{l,j} = (\overline{\phi})^{-l}(\overline{x(i_j)})$ for any pair $(l, j) \in \{0, \ldots, \nu_j\} \times [r]$, and $\kappa_j = u_{1,j} + \cdots + u_{\nu_j,j}$ for any $j \in [r]$, where $\nu_j = |\mathcal{O}(\overline{x(i_j)})|$ and the integers $u_{l,j}$ are defined by the equalities $\phi^{-1}(y_{l-1,j}) = y_{l,j} + u_{l,j}\mathbb{1}$. Then*:

- $\mathcal{O}(\overline{x(i_j)}) = \{y_{0,j}, \ldots, y_{\nu_j-1,j}\}$ *and* $y_{\nu_j,j} = y_{0,j}$ *for every* $j \in [r]$; *hence the mapping* $(l,j) \mapsto y_{l,j}$ *yields a bijection*

$$\boldsymbol{L} \leftrightarrow \{(l,j) : j \in [r],\, l \in \mathbb{Z}_{\nu_j}\}.$$

Moreover, for any $i \in [s]$, *there exist unique* $j = j(i) \in [r]$ *such that* $\mathcal{O}(\overline{x(i)}) = \mathcal{O}(\overline{x(i_j)})$ *and* $l = l(i) \in \mathbb{Z}_{\nu_j}$ *such that* $\overline{x(i)} = y_{l,j}$.
- $\nu_\Sigma = \nu = \mathrm{lcm}\{\nu_j : j \in [r]\}$, *where* $\nu = |\overline{\phi}|$, $\nu_\Sigma = |\overline{\phi}_\Sigma|$ *and* $\overline{\phi}_\Sigma$ *is the reduced Coxeter transformation for the Euclidean type quiver* Σ.
- $\kappa_j > 0$ *and* $\kappa_j = |[s]_j|$ *for every* $j \in [r]$, *where* $[s]_j = \{i \in [s] : j(i) = j\}$; *consequently*, $\sum_{j=1}^{r} \kappa_j = s$. *Moreover, set* $\overline{\varrho}_j(l,i) = \mathrm{rem}_{\kappa_j}(\varrho_j(l,i))$ *for any* $j = 1, \ldots, r$ *and* $(l,i) \in \mathbb{Z}_{\nu_j} \times [s]_j$, *where*

$$\varrho_j(l,i) = \begin{cases} x(i)_0 & \text{if } l = l(i), \\ x(i)_0 + u_{l(i)+1,j} + \cdots + u_{l,j} & \text{if } l > l(i), \\ x(i)_0 + u_{l(i)+1,j} + \cdots + u_{\nu_j,j} + u_{1,j} + \cdots + u_{l,j} & \text{if } l < l(i). \end{cases}$$

Then

$$\{\overline{\varrho}_j(l,i) : i \in [s]_j\} = \mathbb{Z}_{\kappa_j} \tag{$*$}$$

for every $l \in \mathbb{Z}_{\nu_j}$.

(c) *Set* $x(n,i) = \tau^{-n}(x(i))$ *for any pair* $(n,i) \in \mathbb{N} \times [s]$. *Then*

$$x(n,i) = \begin{cases} y_{n \oplus l(i), j(i)} + \varrho_{j(i)}(n \oplus l(i), i)\mathbb{1} & \text{if } n < \nu_{j(i)}, \\ x(\mathrm{rem}_{\nu_{j(i)}}(n), i) + \mathrm{quo}_{\nu_{j(i)}}(n)\kappa_{j(i)}\mathbb{1} & \text{if } n \geq \nu_{j(i)}, \end{cases}$$

where $\oplus$ *denotes addition in* $\mathbb{Z}_{\nu_{j(i)}}$.

(d) *Let* $x \in \mathbb{N}^{Q_0}$ *be a vector from* $\boldsymbol{P}$, *and* $(j,l,i) \in [r] \times \mathbb{Z}_{\nu_j} \times [s]_j$ *the triple uniquely determined by the equalities* $\overline{x} = y_{l,j}$ *and* $\overline{\varrho}_j(l,i) = \mathrm{rem}_{\kappa_j}(x_0)$, *where* $Q = Q_{p,q,2}$. *Then*:
- $x \in \boldsymbol{P}'$ *if and only if* $x_0 \geq \varrho_j(l,i)$.
- *If* $x \in \boldsymbol{P}'$ *then*

$$x = \begin{cases} x(l - l(i) + (x_0 - \varrho_j(l,i))\nu_j/\kappa_j, i) & \text{if } l \geq l(i), \\ x(l - l(i) + \nu_j + (x_0 - \varrho_j(l,i))\nu_j/\kappa_j, i) & \text{if } l < l(i). \end{cases}$$

(e) *For any* $m, m' \in \mathbb{N}$, *the inequality*

$$\sum_{v \in Q_0} x(n,i)_v > m \qquad (\textit{resp. } x(n,i)_v > m',\, v \in Q_0),$$

holds for all $i \in \Sigma_0$ *and* $n \geq m/s\eta + \nu$ (*resp.* $n \geq m'/\eta + \nu$), *where* $\eta = \min\{\kappa_j/\nu_j : j \in [r]\}$.

(f) *There exists a section Σ in $\boldsymbol{P}$ with the property that for any pair $x = x(n,i)$, $y = x(n',i')$ in $\boldsymbol{P}' = \boldsymbol{P}'(\Sigma)$ such that $\overline{x} = \overline{y}$ the following hold*:

- *The inequalities $n \leq n'$ and $x_0 \leq y_0$ are equivalent.*
- *If $n < n'$ and $\overline{z} \neq \overline{x}$ for all $z = x(n'',i'')$ with $n < n'' < n'$, then $y_0 = x_0 + 1$.*

A complete proof of Theorem 2.4 is given in Section 5.

3. THE RESTRICTED MULTIPLICITY VECTOR FOR REGULAR COMPONENTS

The first part of this section (Subsections 3.1–3.3) is devoted to preparations for the proof of Theorem 2.2(a+b). Subsections 3.4 and 3.5 contain the proofs of assertions (a) and (b), respectively.

3.1. We start by proving a useful general fact.

PROPOSITION. *Let k be a field, R, S two finite-dimensional k-algebras, $\mathcal{C}$ a connected component of the quiver Γ_R, and let*

$$\operatorname{mod} R \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \operatorname{mod} S$$

be a pair of k-linear functors such that Ψ is left adjoint for Φ. Assume that Φ is exact and the restricted functor $\Phi_{|\operatorname{add}\mathcal{C}} : \operatorname{add}\mathcal{C} \to \operatorname{mod} S$ "preserves the Auslander–Reiten structure", i.e.:

(a) *$\Phi(X)$ is indecomposable for any indecomposable X in $\operatorname{add}\mathcal{C}$,*
(b) *for any indecomposable X in $\operatorname{add}\mathcal{C}$, $\Phi(f)$ is a right (resp. left) minimal almost split homomorphism in $\operatorname{mod} S$ provided that so is $f : Y \to X$ (resp. $f : X \to Y$) in $\operatorname{mod} R$,*
(c) *for any indecomposable X in $\operatorname{add}\mathcal{C}$, $\Phi(X)$ is a simple projective in $\operatorname{mod} S$ provided that so is X in $\operatorname{mod} R$.*

Then

$$m(M)_{\Phi(X)} = m(\Psi(M))_X$$

for any M in $\operatorname{mod} S$ and any indecomposable X in $\operatorname{add}\mathcal{C}$.

Proof. For any nonprojective X in $\mathcal{C}$ there exists an almost split sequence

$$0 \to \tau X \to \bigoplus_{Z \in {}^{-}X} Z^{d'_{Z,X}} \to X \to 0$$

in $\operatorname{mod} R$. Since $\Phi_{|\operatorname{add}\mathcal{C}}$ is exact and satisfies (a) and (b), the sequence

$$0 \to \Phi(\tau X) \to \bigoplus_{Z \in {}^{-}X} \Phi(Z)^{d'_{Z,X}} \to \Phi(X) \to 0$$

is almost split in $\operatorname{mod} S$. Therefore, for any S-module M,

$$m(\Psi(M))_X = [\Psi(M), X] + [\Psi(M), \tau X] - \sum_{Z \in {}^-X} d'_{Z,X}[\Psi(M), Z],$$

$$m(M)_{\Phi(X)} = [M, \Phi(X)] + [M, \Phi(\tau X)] - \sum_{Z \in {}^-X} d'_{Z,X}[M, \Phi(Z)]$$

(see formula $(*)$ in the Introduction). Since (Ψ, Φ) is a pair of adjoint functors, we get $m(M)_{\Phi(X)} = m(\Psi(M))_X$.

It remains to prove the assertion for X projective in $\mathcal{C}$. In this case there exists a right minimal almost split homomorphism

$$\bigoplus_{Z \in {}^-X} Z^{d'_{Z,X}} \cong JX \hookrightarrow X$$

in $\operatorname{mod} R$. By similar arguments to those above, applying (a)–(c), we get

$$\begin{aligned} m(\Psi(M))_X &= [\Psi(M), X] - \sum_{Z \in {}^-X} d'_{Z,X}[\Psi(M), Z] \\ &= [M, \Phi(X)] - \sum_{Z \in {}^-X} d'_{Z,X}[M, \Phi(Z)] = m(M)_{\Phi(X)} \end{aligned}$$

for any S-module M, and the proof is complete. ■

REMARK.

(i) Let (Ψ, Φ) be as in Proposition 3.1. Then there exists a unique connected component $\mathcal{C}'$ in Γ_S such that $\Phi(X) \in \operatorname{add} \mathcal{C}'$, for any X in $\mathcal{C}$, and the induced functor $\Phi_{|\operatorname{add} \mathcal{C}} : \operatorname{add} \mathcal{C} \to \operatorname{add} \mathcal{C}'$ is dense. Moreover, the problem of determining the restricted multiplicity vector of an S-module M for the subcategory $\operatorname{add} \mathcal{C}' \subset \operatorname{mod} S$ can be reduced to the analogous one for the module $\Psi(M)$ and the subcategory $\operatorname{add} \mathcal{C} \subset \operatorname{mod} R$.

(ii) Assume that there are no projective (resp. injective) modules in $\mathcal{C}$ and, in addition, the functor $\Phi_{|\operatorname{add} \mathcal{C}}$ is full and faithful. Then the assertion of the proposition remains valid if instead of the assumption on preserving the Auslander–Reiten structure, we require that $\Phi(\tau_R X) \cong \tau_S \Phi(X)$ (resp. $\Phi(\tau_R^{-1} X) \cong \tau_S^{-1} \Phi(X)$) for any indecomposable module X in $\mathcal{C}$. (This follows by the properties of almost split sequences, in particular from [2, Corollary 3.2(a)].)

3.2. Now we introduce four pairs of special functors which satisfy the assumptions of Proposition 3.1. Given a module M over a domestic canonical algebra Λ, we use them to reduce the problem of determining the restricted multiplicity vectors $m(M)_{|\mathcal{C}}$ for all regular components $\mathcal{C}$ in Γ_Λ to the analogous one for algebras of type $\widetilde{\mathbb{A}}_{p,q}$ for $p, q \geq 2$, in some cases even $\widetilde{\mathbb{A}}_{1,1}$ (the Kronecker algebra). For this class of algebras, the problem is already solved in [9].

Let $\Lambda = \Lambda_{p,q,2}$ be a domestic canonical algebra with $p, q \geq 2$. We define the functors

$$\operatorname{mod} \Lambda_{p,q} \underset{\Psi_0}{\overset{\Phi_0}{\rightleftarrows}} \operatorname{mod} \Lambda_{p,q,2}, \qquad \operatorname{mod} \Lambda_{2,q} \underset{\Psi_1}{\overset{\Phi_1}{\rightleftarrows}} \operatorname{mod} \Lambda_{p,q,2},$$
$$\operatorname{mod} \Lambda_{q,p} \underset{\Psi_\infty}{\overset{\Phi_\infty}{\rightleftarrows}} \operatorname{mod} \Lambda_{p,q,2}, \qquad \operatorname{mod} \Lambda_{1,1} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \operatorname{mod} \Lambda_{p,q,2}.$$

Without loss of generality, we can restrict our attention to matrix representations (see 1.4 for the precise definition).

For a module M given by a triple (A, B, C), with $A = (A_i)_{i\in[p]}$, $B = (B_i)_{i\in[q]}$, and $C = (C_i)_{i\in[2]}$, we set

$$\Psi_0(M) = (A, B), \quad \Psi_\infty(M) = (B, A),$$
$$\Psi_1(M) = (C, B'), \qquad \Psi(M) = (\overline{A}, -\overline{B}),$$

where $B' = (B'_i)_{i\in[q]}$ with $B'_1 = -B_1$ and $B'_i = B_i$ for $i \geq 2$.

To define the remaining four functors we need some extra notation. For any $D \in \mathbb{M}_{v\times w}(k)$ and integer $i \geq 2$, we set $\mathrm{I}^{(i)}(D) = (D_1, \ldots, D_i)$, where $D_1 = D$ and $D_j = I_v$ for all $j = 2, \ldots, i$. If the value of i is obvious then we omit the upper index and write simply $\mathrm{I}(D)$.

Now, for a Λ'-module N given by the pair (A, B), where Λ' is equal to $\Lambda_{p,q}$, $\Lambda_{2,q}$, $\Lambda_{q,p}$ and $\Lambda_{1,1}$, respectively, we set

$$\Phi_0(N) = (A, B, \mathrm{I}(\overline{A} + \overline{B})), \quad \Phi_1(N) = (\mathrm{I}(\overline{A} + \overline{B}), B', A),$$
$$\Phi_\infty(N) = (B, A, \mathrm{I}(\overline{A} + \overline{B})), \quad \Phi(N) = (\mathrm{I}(A_1), \mathrm{I}(-B_1), \mathrm{I}(A_1 - B_1)),$$

where B' is as above.

The eight mappings introduced above can be extended to k-linear functors by defining their values on morphisms in an obvious way. These functors have the following properties.

LEMMA.

(a) *The functors $\Phi_1, \Phi_0, \Phi_\infty, \Phi$ are full, faithful and exact.*
(b) (Ψ_0, Φ_0), (Ψ_1, Φ_1), $(\Psi_\infty, \Phi_\infty)$, (Ψ, Φ) *are pairs of adjoint k-linear functors.*

Proof. An easy check on the definitions. ■

3.3. By Remark 3.1(ii), to apply Proposition 3.1 for regular components, it suffices to show that the functors $\Phi_0, \Phi_1, \Phi_\infty, \Phi$, restricted to appropriate subcategories, commute with the Auslander–Reiten translate. We show this by proving that $\Phi_{0|\operatorname{add}\mathcal{T}_0^{p,q}}$, $\Phi_{1|\operatorname{add}\mathcal{T}_0^{2,q}}$, $\Phi_{\infty|\operatorname{add}\mathcal{T}_0^{q,p}}$, $\Phi_{|\operatorname{add}\mathcal{T}_\lambda^{1,1}}$, $\lambda\in k\setminus\{0,1\}$, yield, respectively, the equivalences

$$(*)\qquad \begin{aligned} &\operatorname{add}\mathcal{T}_0^{p,q}\simeq\operatorname{add}\mathcal{T}_0^{p,q,2}, \quad \operatorname{add}\mathcal{T}_0^{2,q}\simeq\operatorname{add}\mathcal{T}_1^{p,q,2},\\ &\operatorname{add}\mathcal{T}_0^{q,p}\simeq\operatorname{add}\mathcal{T}_\infty^{p,q,2}, \quad \operatorname{add}\mathcal{T}_\lambda^{1,1}\simeq\operatorname{add}\mathcal{T}_\lambda^{p,q,2}, \quad \lambda\in k\setminus\{0,1\},\end{aligned}$$

of serial categories, where $\mathcal{T}'=\{\mathcal{T}_\lambda^{p',q'}\}_{\lambda\in k\cup\{\infty\}}$, for $p',q'\geq 1$, denotes the 1-parameter family of stable tubes of type (p',q'), containing all regular indecomposable modules over the hereditary algebra $\Lambda'=\Lambda_{p',q'}$ of type $\widetilde{\mathbb{A}}_{p',q'}$.

The following fact plays a crucial role in the proof of $(*)$.

Lemma. *Let $\Upsilon:\operatorname{mod}R\to\operatorname{mod}S$ be a full faithfull exact functor and $\mathcal{U}$ (resp. $\mathcal{U}'$) a full subcategory of $\operatorname{mod}R$ (resp. $\operatorname{mod}S$) closed under isomorphisms, which as an exact subcategory is a serial (and abelian) category of type (n,∞) for some $n\geq 1$. Assume that:*

(a) *$\mathcal{U}'$ is closed under extensions,*

(b) *for any simple object X in $\mathcal{U}$, $\Upsilon(X)$ is a simple object in $\mathcal{U}'$.*

Then $\Upsilon_{|\mathcal{U}}$ yields an equivalence $\mathcal{U}\simeq\mathcal{U}'$ of abelian categories. In particular, if an object X in $\mathcal{U}$ with $\mathcal{U}$-socle X_1 has $\mathcal{U}$-length l then $\Upsilon(X)$ has $\mathcal{U}'$-length l and its $\mathcal{U}'$-socle is isomorphic to $\Upsilon(X_1)$.

Proof. We first prove that $\Upsilon(X)\in\mathcal{U}'$ for any X in $\mathcal{U}$. We apply induction on $l=\ell_{\mathcal{U}}(X)$. If $l=1$ then the claim holds by (b). Assume that $l\geq 2$ and the claim holds for all X' in $\mathcal{U}$ with $\ell_{\mathcal{U}}(X')<l$. For any fixed X with $\ell_{\mathcal{U}}(X)=l$, there exists an exact sequence

$$0\to X'\to X\to X''\to 0$$

in $\mathcal{U}$ such that $\ell_{\mathcal{U}}(X'),\ell_{\mathcal{U}}(X'')<l$. Then the sequence

$$0\to\Upsilon(X')\to\Upsilon(X)\to\Upsilon(X'')\to 0$$

is exact in $\mathcal{U}'$, since Υ is an exact functor. The objects $\Upsilon(X')$ and $\Upsilon(X'')$ belong to $\mathcal{U}'$ by the inductive assumption. Hence, by (a), so does $\Upsilon(X)$, and the proof of the claim is complete. Consequently, Υ induces a functor $\Upsilon_{|\mathcal{U}}:\mathcal{U}\to\mathcal{U}'$. We have to show that $\Upsilon_{|\mathcal{U}}$ is dense.

Denote by $X_0,\ldots,X_{n-1}$ all (up to isomorphism) pairwise nonisomorphic simple objects in $\mathcal{U}$. We can assume that their numbering is such that all pairwise nonisomorphic objects $X(s,l)$, $s\in\mathbb{Z}_n$, of $\mathcal{U}$-length $l\geq 1$ in $\mathcal{U}$ are uniquely determined by composition series of the form $(X_s,\ldots,X_{s-l+1})$, where $X_i=X_{\operatorname{rem}_n(i)}$ for $i\geq n$. Then applying (b) and the fact that Υ is full and faithful, we infer that the objects $Y_s:=\Upsilon(X_s)$, $s\in\mathbb{Z}_n$, are all nonisomorphic simple objects in $\mathcal{U}'$. Next, we show by induction on l that the pairwise nonisomorphic indecomposable objects $Y(s,l):=\Upsilon(X(s,l))$, $s\in\mathbb{Z}_n$, in $\mathcal{U}'$ have $\mathcal{U}'$-length l and are determined by composition series of the form $(Y_s, \dots, Y_{s-l+1})$, $s \in \mathbb{Z}_n$, where $Y_i = Y_{\mathrm{rem}_n(i)}$ for $i \geq n$. This follows, by exactness of Υ, from the existence of exact sequences

$$0 \to X(s,1) \to X(s,l) \to X(s \ominus_n 1, l-1) \to 0,$$

$s \in \mathbb{Z}_n$, for any $l \geq 2$. Consequently, $\Upsilon_{|\mathcal{U}} : \mathcal{U} \to \mathcal{U}'$ is dense and yields the required equivalence of abelian categories. Now the final assertion is straightforward. ∎

Let $\Lambda' = \Lambda_{p',q'}$ be a hereditary algebra of type $\widetilde{\mathbb{A}}_{p',q'}$, where $p', q' \geq 1$. As already mentioned, the regular Λ-modules form a 1-parameter family $\mathcal{T}' = \{\mathcal{T}_\lambda^{p',q'}\}_{\lambda \in k \cup \{\infty\}}$ of stable tubes of type (p', q') and each of the categories $\operatorname{add} \mathcal{T}_\lambda^{p',q'}$, $\lambda \in k \cup \{\infty\}$, is serial of type (n_λ, ∞), where n_λ is the rank of $\mathcal{T}_\lambda^{p',q'}$. Assume that $\mathcal{T}_0^{p',q'}$, $\mathcal{T}_\infty^{p',q'}$ and $\mathcal{T}_\lambda^{p',q'}$, $\lambda \in k \setminus \{0\}$, are stable tubes of rank p, q and 1, respectively. Below we list all, consecutive with respect to the "cyclic order", regular simple modules from the mouth of each tube $\mathcal{T}_\lambda^{p',q'}$, according to the convention of 1.3. The list yields the encodings of all indecomposable regular Λ'-modules given by tubular coordinates.

We set:

$$X(\mathcal{T}_0^{p',q'}, 0, 1) \;=\; \begin{tikzcd}[row sep=small] & 0 \arrow[r, "0"] & \cdots \arrow[r, "0"] & 0 \arrow[dr, "0"] & \\ k \arrow[ur, "0"] \arrow[r, "1"'] & k \arrow[r, "1"'] & \cdots \arrow[r, "1"'] & k \arrow[r, "1"'] & k \end{tikzcd}$$

and $X(\mathcal{T}_0^{p',q'}, s, 1) = S(a_s)$ for $s \in \mathbb{Z}_{p'} \setminus \{0\}$, where $X(\mathcal{T}_0^{p',q'}, s', l)$ is the module in the tube $\mathcal{T}_0^{p',q'}$ corresponding to the vertex $(s', l) \in \mathcal{T}(p')_0$ for all $s' \in \mathbb{Z}_{p'}$ and $l \geq 1$;

$$X(\mathcal{T}_\infty^{p',q'}, 0, 1) \;=\; \begin{tikzcd}[row sep=small] & k \arrow[r, "1"] & \cdots \arrow[r, "1"] & k \arrow[dr, "1"] & \\ k \arrow[ur, "1"] \arrow[r, "0"'] & 0 \arrow[r, "0"'] & \cdots \arrow[r, "0"'] & 0 \arrow[r, "0"'] & k \end{tikzcd}$$

and $X(\mathcal{T}_\infty^{p',q'}, s, 1) = S(b_s)$ for $s \in \mathbb{Z}_{q'} \setminus \{0\}$, where $X(\mathcal{T}_\infty^{p',q'}, s', l)$ is the module in the tube $\mathcal{T}_\infty^{p',q'}$ corresponding to the vertex $(s', l) \in \mathcal{T}(q')_0$ for all $s' \in \mathbb{Z}_{q'}$ and $l \geq 1$.

To fix a precise list of regular simple Λ'-modules lying in homogeneous tubes, we simply give a description of all indecomposable regular modules from tubes $\mathcal{T}_\lambda^{p',q'}$, $\lambda \in k \setminus \{0\}$, of rank 1. We set

$$X(\mathcal{T}_\lambda, 0, l) \;=\; \begin{tikzcd}[row sep=small] & k^l \arrow[r, "I_l"] & \cdots \arrow[r, "I_l"] & k^l \arrow[dr, "I_l"] & \\ k^l \arrow[ur, "J_l(\lambda)"] \arrow[r, "I_l"'] & k^l \arrow[r, "I_l"'] & \cdots \arrow[r, "I_l"'] & k^l \arrow[r, "I_l"'] & k^l \end{tikzcd}$$

where $X(\mathcal{T}_\lambda^{p',q'}, 0, l)$ is the module of $\operatorname{add} \mathcal{T}_\lambda^{p',q'}$-length l in the tube $\mathcal{T}_\lambda^{p',q'}$ for all $l \geq 1$.

Further on, for $\lambda \in k\backslash\{0\}$ we use the abbreviated notation $X^{p',q'}(\lambda,s,l) = X(\mathcal{T}_\lambda^{p',q'},s,l)$ for $\lambda = 0,\infty$, and $X^{p',q'}(\lambda,l) = X(\mathcal{T}_\lambda^{p',q'},0,l)$ for $\lambda \in k\setminus\{0\}$.

COROLLARY.

(a) *The functors* $\Phi_{0|\operatorname{add}\mathcal{T}_0^{p,q}}$, $\Phi_{1|\operatorname{add}\mathcal{T}_0^{2,q}}$, $\Phi_{\infty|\operatorname{add}\mathcal{T}_0^{q,p}}$, $\Phi_{|\operatorname{add}\mathcal{T}_\lambda^{1,1}}$ *yield the equivalences* $(*)$.

(b) *We have the isomorphisms*

(i) $\Phi_0(X^{p,q}(0,s,l)) \cong X(0,s,l)$,

(ii) $\Phi_1(X^{2,q}(0,s',l)) \cong X(1,s',l)$,

(iii) $\Phi_\infty(X^{q,p}(0,s'',l)) \cong X(\infty,s'',l)$,

(iv) $\Phi(X^{1,1}(\lambda,l)) \cong X(\lambda,l)$

for all $s \in \mathbb{Z}_p$, $s' \in \mathbb{Z}_2$, $s'' \in \mathbb{Z}_q$, $l \geq 1$ *and* $\lambda \in k\setminus\{0,1\}$.

Proof. The functors $\Phi_0,\Phi_1,\Phi_\infty,\Phi$ and the pairs $(\operatorname{add}\mathcal{T}_0^{p,q},\operatorname{add}\mathcal{T}_0^{p,q,2})$, $(\operatorname{add}\mathcal{T}_0^{2,q},\operatorname{add}\mathcal{T}_1^{p,q,2})$, $(\operatorname{add}\mathcal{T}_0^{q,p},\operatorname{add}\mathcal{T}_\infty^{p,q,2})$, $(\operatorname{add}\mathcal{T}_\lambda^{1,1},\operatorname{add}\mathcal{T}_\lambda^{p,q,2})_{\lambda\in k\setminus\{0,1\}}$ of serial subcategories of the respective module categories satisfy the first assumptions and condition (a) of Lemma 3.3 (see Lemma 3.2(a) and [22, 23]). It is easy to check that for $l = 1$, the isomorphisms (i)–(iv) hold trivially (they are in fact equalities), so (b) is also satisfied for each of the four functors. Consequently, (a) holds automatically by Lemma 3.3. Assertion (b) follows immediately from the final assertion of Lemma 3.3 and the definition of tubular coordinates. ∎

REMARK.

(i) By the definition of the functor Φ, the $\Lambda_{p,q,2}$-modules $X(\lambda,l)$ in the homogeneous tubes $\mathcal{T}_\lambda^{p,q,2}$, $\lambda \in k\setminus\{0,1\}$, have the form

$$X(\lambda,l) \quad = \quad \begin{tikzcd}[column sep=small] & k^l \arrow[r, "I_l"] & \cdots \arrow[r, "I_l"] & k^l \arrow[dr, "I_l"] & \\ k^l \arrow[ur, "J_l(\lambda)"] \arrow[r, "-I_l"] \arrow[drr, "J_l(\lambda-1)"'] & k^l \arrow[r, "I_l"] & \cdots \arrow[r, "I_l"] & k^l \arrow[r, "I_l"] & k^l \\ & & k^l \arrow[urr, "I_l"'] & & \end{tikzcd}$$

(see also [18]). The formulas for the remaining indecomposable regular modules from the tubes $\mathcal{T}_0^{p,q,2}$, $\mathcal{T}_1^{p,q,2}$ and $\mathcal{T}_\infty^{p,q,2}$ do not have such regular shape but of course can be reconstructed, by applying the functors Φ_0, Φ_1 and Φ_∞ and the description of regular nonhomogeneous modules over hereditary algebras Λ' of type $\widetilde{\mathbb{A}}_{p',q'}$ in terms of walks in the quiver $Q_{p'q'}$ (see for example [9]).

(ii) The functor Φ_∞ induces a homomorphism

$$\varphi_\infty : K(\Lambda_{q,p}) \to K(\Lambda_{p,q,2})$$

of Grothendieck groups, given by $[M] \mapsto [\Phi_\infty(M)]$ for M in $\operatorname{mod}\Lambda_{q,p}$. Applying only additivity of the dimension vector on exact sequences, exactness of Φ_∞ and the isomorphisms (i)–(iv) for $l = 1$, one can easily obtain the formula $\varphi_\infty([X^{q,p}(s,l)]) = [X(\infty,s,l)]$ for all $s \in \mathbb{Z}_q$ and $l \geq 1$. Consequently,

$$\Phi_\infty(X^{q,p}(0,s,l)) \cong X(\infty,s,l)$$

for all $s \in \mathbb{Z}_q$ and l such that $q \nmid l$, since the modules $X(\infty,s,l)$ are uniquely determined by their dimension vectors in this case (see [23]). It remains to define an analogous isomorphism in case $q \mid l$. The situation for the functors $\Phi_{0|\mathrm{add}\,\mathcal{T}_0^{p,q}}$ and $\Phi_{1|\mathrm{add}\,\mathcal{T}_0^{2,q}}$ is analogous.

3.4. *Proof of Theorem 2.2(a).* The pairs of functors (Ψ_0,Φ_0),(Ψ_1,Φ_1), $(\Psi_\infty,\Phi_\infty)$, (Ψ,Φ) satisfy the assumptions of Proposition 3.1 (see Lemma 3.2, Corollary 3.3 and Remark 3.1(ii)). Thus, the following formulas hold:

$$(*)\qquad \begin{aligned} m(M)_{[0,s,l]} &= m(\Psi_0(M))_{X^{p,q}(0,s,l)},\\ m(M)_{[1,s',l]} &= m(\Psi_1(M))_{X^{2,q}(0,s',l)},\\ m(M)_{[\infty,s'',l]} &= m(\Psi_\infty(M))_{X^{q,p}(0,s'',l)},\\ m(M)_{[\lambda,l]} &= m(\Psi(M))_{X^{1,1}(\lambda,l)}, \end{aligned}$$

for all $s \in \mathbb{Z}_p$, $s' \in \mathbb{Z}_2$, $s'' \in \mathbb{Z}_q$, $l \geq 1$ and $\lambda \in k \setminus \{0,1\}$.

Following the notation introduced in [9], for any $p',q' \geq 1$, $s \in \mathbb{Z}_{p'}$ and $l \geq 1$, there exist $i \in \{2,\ldots,p'+1\}$, $j \in \mathbb{Z}_{p'}$ and $n \geq 0$ such that an indecomposable module $X^{p',q'}(0,s,l)$ is given by the walk $w(i,j,-1) = \alpha_{i,j}$ or $w(i,j,n) = \alpha_{i,p'}(\beta^{-1}\alpha)^n\beta^{-1}\alpha_{1,j}$ in the quiver $Q_{p',q'}$, where $\alpha_{p'+1,p'} = (\infty)$, $\alpha_{i+1,i} = (a_i)$, $\alpha_{1,0} = (0)$ are trivial walks in $Q_{p',q'}$ (see [9] for details). Applying simple induction, one can show the equality

$$(i,j,n) = \mu_{p'}(s,l),$$

where $\mu_{p'}$ is the indexing map defined in 2.2. Now, given a $\Lambda_{p',q'}$-module M defined by the pair (A,B), $A = (A_i)_{i\in[p']}$, $B = (B_i)_{i\in[q']}$, and integers $s \in \mathbb{Z}_{p'}$, $m \geq 1$, we have

$$(**)\qquad [M, X^{p',q'}(0,s,l)] = \operatorname{cor}\mathcal{M}^{i,j,n}(\bar{B},A),$$

where $(i,j,n) = \mu_{p'}(s,l)$ (see [9, Lemma 5.6]). Moreover, for any $s \in \mathbb{Z}_{p'}$ and $l \geq 1$ we have the formula

$$(***)\qquad \begin{aligned} &m(M)_{X^{p',q'}(0,s,l)}\\ &\quad= [M, X^{p',q'}(0,s,l)] - [M, X^{p',q'}(0,s\ominus_{p'}1,l-1)]\\ &\qquad - [M, X^{p',q'}(0,s,l+1)] + [M, X^{p',q'}(0,s\ominus_{p'}1,l)], \end{aligned}$$

where $X^{p',q'}(0,s,0) = 0$ (see [9, Corollary 5.3]).

Now we can complete the proof. Combining formulas $(***)$ and $(**)$, for (p',q') equal to (p,q), $(2,q)$ and (q,p), respectively, with $(*)$, we obtain (i), (ii), (iii) of Theorem 2.2(a). Formula (iv) holds by analogous arguments and the equality

$$[M, X^{1,1}(\lambda, l)] = \operatorname{cor} \mathcal{M}_\lambda^l(A_1, -B_1)$$

for any $\lambda \in k \setminus \{0,1\}$, $l \geq 0$ (see [9, Lemma 4.6]).

Assume now that for a Λ-module M the set $\sigma(M)$ is known. Observe that the existence of the algorithm with the required properties follows from the formulas $(*)$. They reduce the problem of determining the restricted multiplicity vectors $m(M)_{|\mathcal{C}}$, for all regular components $\mathcal{C}$ in Γ_Λ, to the analogous one for concrete four modules, $\Psi_0(M)$, $\Psi_1(M)$, $\Psi_\infty(M)$ and $\Psi(M)$, over four algebras Λ' of type $\widetilde{\mathbb{A}}_{p',q'}$, and a finite number of already determined regular connected components $\mathcal{C}'$ in $\Gamma_{\Lambda'}$, for each of these algebras. Following [9], for a Λ'-module M' with $\dim_k M' = n'$ and a regular component $\mathcal{C}'$, there exists an algorithm of pessimistic complexity $\mathcal{O}(n'^4)$ which computes $m(M')_{|\mathcal{C}'}$. Since $n' \leq \dim_k M$ for $M' = \Psi_0(M), \Psi_1(M), \Psi_\infty(M)$, or $\Psi(M)$, the proof of the existence of the algorithm, and hence Theorem 2.2(a), is complete. ∎

Corollary. *Let* $\boldsymbol{T}_{\{0,1,\infty\}} = \bigsqcup_{\lambda \in k\setminus\{0,1\}} \boldsymbol{T}_\lambda$. *Then* $m(M)_{|\boldsymbol{T}_{\{0,1,\infty\}}} = m(\Psi(M))_{|\mathcal{R}'_{\{0,1,\infty\}}}$ *and the problem of algorithmic computing of the vector* $m(M)_{|\boldsymbol{T}_{\{0,1,\infty\}}}$, *in particular determining the set* $\sigma(M)$, *is fully reduced to the analogous problems for the Kronecker algebra*, $\Lambda_{1,1,}$, *for the restricted vector* $m(\Psi(M))_{|\mathcal{R}'_{\{0,1,\infty\}}}$ *and the set* $\sigma(\psi(M))$, *where* $\mathcal{R}'_{\{0,1,\infty\}} = \bigsqcup_{\lambda \in k\setminus\{0,1\}} (\mathcal{T}_\lambda^{1,1})_0$ *and* $\sigma(\psi(M))$ *consists of all* $\lambda \in k$ *such that* $\Psi(M)$ *contains a direct summand from* $\operatorname{add} \mathcal{T}_\lambda^{1,1}$ (cf. [9, Proposition 4.4 and Algorithm 4.4(3)]).

Remark. The algorithmic determining of the vector $m(M)_{|\boldsymbol{T}_\lambda}$, for a fixed $\lambda \in k \cup \{\infty\}$, relies on an appropriate reduction and is described in the final part of the proof above. To determine the integer $m(M)_x$, for a fixed single $x \in \boldsymbol{T}$, we can apply directly formulas (i)–(iv) from Theorem 2.2(a).

3.5. *Proof of Theorem 2.2(b).* Let $\Lambda' = \Lambda_{1,1} = k(0 \rightrightarrows \omega)$ be the Kronecker algebra and $\Psi : \operatorname{mod} \Lambda \to \operatorname{mod} \Lambda'$ the functor defined in 3.2. First we prove that $\operatorname{rk}_{\mathcal{P}}(M)$ is equal to the number of postprojective summands in a decomposition of the module $\Psi(M) = (\overline{A}, -\overline{B})$ into a direct sum of indecomposable Λ'-modules, where M is given by a triple (A, B, C).

Denote by $\operatorname{add} \mathcal{P}'$, $\operatorname{add} \mathcal{R}'$ and $\operatorname{add} \mathcal{Q}'$ the subcategories of all postprojective, regular and preinjective modules in $\operatorname{mod} \Lambda'$, respectively. Recall that the dimension vector $\underline{\dim}\, P'$ of an indecomposable module P' in $\operatorname{add} \mathcal{P}'$ has the form $\underline{\dim}\, P' = [m, m+1]$ for some $m \geq 0$. Denote by $\operatorname{res} : \operatorname{mod} \Lambda \to \operatorname{mod} \Lambda'$ the standard restriction functor, given by $\operatorname{res}(M) = (\overline{A}, \overline{B})$ for M as above.

Let M be a fixed Λ-module given by a triple (A, B, C), and

$$M \cong P \oplus R \oplus Q$$

a decomposition of M with P in $\operatorname{add}\mathcal{P}$, R in $\operatorname{add}\mathcal{R}$ and Q in $\operatorname{add}\mathcal{Q}$. Then, by [23, Chapter 12], the modules $\operatorname{res}(P)$, $\operatorname{res}(R)$ and $\operatorname{res}(Q)$ belong to the subcategories $\operatorname{add}\mathcal{P}'$, $\operatorname{add}\mathcal{R}'$ and $\operatorname{add}\mathcal{Q}'$, respectively. Observe that Ψ can be presented as a composite functor

$$\operatorname{mod}\Lambda \xrightarrow{\mathrm{res}} \operatorname{mod}\Lambda' \xrightarrow{\Theta} \operatorname{mod}\Lambda',$$

where Θ is the autoequivalence defined by the formula $\Theta(N) = (A', -B')$ for a Λ'-module N given by the pair (A', B') of appropriate matrices. The equivalence Θ preserves the subcategories $\operatorname{add}\mathcal{P}'$, $\operatorname{add}\mathcal{R}'$ and $\operatorname{add}\mathcal{Q}'$, since it preserves the dimension vectors. Hence, $\Psi(P)$, $\Psi(R)$ and $\Psi(Q)$ belong to $\operatorname{add}\mathcal{P}'$, $\operatorname{add}\mathcal{R}'$ and $\operatorname{add}\mathcal{Q}'$, respectively. Thus, $\Psi(P)$ is a maximal postprojective direct summand of $\Psi(M)$.

Let

$$\Psi(P) \cong \bigoplus_{i=1}^{t} P'_i$$

be a decomposition of $\Psi(P)$ into a direct sum of postprojective indecomposable Λ'-modules. Then

$$\underline{\dim}\,\Psi(P) = \sum_{i=1}^{t} [s_i, s_i + 1],$$

where $\underline{\dim}\,P'_i = [s_i, s_i + 1]$ for $i = 1, \dots, t$; on the other hand,

$$\underline{\dim}\,\Psi(P) = [s, s + r],$$

where $s = \dim_k P_0$ and $r = \operatorname{rk}(P) = \operatorname{rk}_{\mathcal{P}}(M)$. Consequently, $t = \operatorname{rk}_{\mathcal{P}}(M)$ and our claim is proved.

Now we prove the remaining assertions of Theorem 2.2(b).

Fix $\lambda_0 \in k \setminus \{0, 1\}$. Then λ_0 belongs to $\sigma(M)$ if and only if $m(M)_{[\lambda_0,l]} = m(\Psi(M))_{X^{1,1}(\lambda_0,l)} \neq 0$ for some $l \geq 1$ (see 3.4($*$)). By [9, Proposition 4.4] and the equality $t = \operatorname{rk}_{\mathcal{P}}(M)$, this is equivalent to λ_0 being a common root of all $(n_\omega - \operatorname{rk}_{\mathcal{P}}(M))$-minors of the matrix $\overline{A} + \lambda\overline{B}$, regarded as polynomials in $k[\lambda]$, and we are done.

The formula for the number of indecomposable direct summands of M from one tube $\mathcal{T}_{\lambda_0}$ follows immediately, by the equality $t = \operatorname{rk}_{\mathcal{P}}(M)$, from [9, Corollary 4.4].

Finally, the required algorithm computing $\operatorname{rk}_{\mathcal{P}}(M)$ with low complexity can be obtained by applying the algorithm computing the vector $m(N)_{|\mathcal{P}'}$ for modules N over the Kronecker algebra Λ', to the module $N = \Psi(M)$ (see [9]). In this way the proof of Theorem 2.2(b) is complete. ■

Remark. Suppose we want to determine the vector $m(M)_{|\boldsymbol{T}_{\{0,1,\infty\}}}$. Then we apply the method described above. More precisely, we execute [9, Algorithm 4.5(1)] to compute $m(\Psi(M))_{|\mathcal{P}'}$, and hence the integer t, which is necessary to determine $\sigma(\Psi(M))$. Next, applying [9, Algorithm 4.5(3)], we compute $m(\Psi(M))_{|\mathcal{R}_{\{0,1,\infty\}}}$ and $m(M)_{|\boldsymbol{T}_{\{0,1,\infty\}}}$. In case the integer $\mathrm{rk}_{\mathcal{P}}(M) = t$ is already known (in particular, if $m(M)_{|\boldsymbol{P}}$ as a solution of a partial task in determining $m(M)$ is already computed), we can clearly omit computing $m(\Psi(M))_{|\mathcal{P}'}$ and pass at once to further steps of the procedure (cf. Remark 6.2(c)). One should stress that the algorithm computing $m(M)_{|\boldsymbol{P}}$, constructed in Section 6, has the same complexity, $\mathcal{O}(n^4)$, as [9, Algorithm 4.5(1)], but it is much more complicated and uses rather deep knowledge of postprojective indecomposable modules over domestic canonical algebras.

4. COMPUTING THE INTEGERS $h(M)_d$, $d \in \boldsymbol{P}$

This section is devoted to the proof of Theorem 2.3.

4.1. We start with some general observation.

Let M, N be modules over some locally bounded category R (i.e. k-linear functors from R to $\operatorname{mod} k$). Assume that there exists a full subcategory R' of R such that for every $x \in \operatorname{ob} R \setminus \operatorname{ob} R'$ there exists a morphism $\alpha \in R(x, y)$, for some $y \in \operatorname{ob} R'$, such that $N(\alpha) : N(x) \to N(y)$ is a monomorphism. Then the linear map

$$\iota = \iota^{R'}_{M,N}\ \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_{R'}(M_{|R'}, N_{|R'})$$

induced by the standard restriction functor res : $\operatorname{mod} R \to \operatorname{mod} R'$ is a monomorphism.

We precisely describe the image of ι in some particular situations. For this we need some extra notation.

Following [18], for any $r \geq s$ we consider the block matrices

$$X_{r,s} = \left[\frac{I_s}{0}\right] \quad \text{and} \quad Y_{r,s} = \left[\frac{0}{I_s}\right]$$

in $\mathbb{M}_{r\times s}(k)$, where 0 denotes the zero matrix in $\mathbb{M}_{(r-s)\times s}(k)$.

Lemma.

(a) *Let $\Lambda = \Lambda_{1,1,2}$ and M, M' be finite-dimensional Λ-modules of dimension vectors $\underline{n}, \underline{n}'$, which are given by the triples (A, B, C) and (A', B', C'), respectively. Assume that $C'_2 : k^{n'_{c_1}} \to k^{n'_\omega}$ is a monomorphism and $D_2 \in \mathbb{M}_{n'_{c_1}\times n'_{c_1}}(k)$ is invertible, where $C'_2 = \left[\frac{D_1}{D_2}\right] \in \mathbb{M}_{n'_\omega\times n'_{c_1}}(k)$, $D_1 \in \mathbb{M}_{(n'_\omega - n'_{c_1})\times n'_{c_1}}(k)$. Then $\iota^{\Lambda'}_{M,M'}$, for $\Lambda' = \Lambda_{1,1}$, yields a k-isomorphism*

$$\mathrm{Hom}_\Lambda(M, M') \cong \{(y, x) \in \mathrm{Hom}_{\Lambda'}(M_{|\Lambda'}, M'_{|\Lambda'}) : x_{(1)}C_2 = D_1D_2^{-1}x_{(2)}C_2\},$$

where $(y,x) \in \mathbb{M}_{n'_0 \times n_0}(k) \times \mathbb{M}_{n'_\omega \times n_\omega}(k)$ *and* $x = \left[\frac{x_{(1)}}{x_{(2)}}\right]$ *with* $x_1 \in \mathbb{M}_{(n'_\omega - n'_{c_1}) \times n_\omega}(k)$ *and* $x_2 \in \mathbb{M}_{n'_{c_1} \times n_\omega}(k)$.

(b) *Let* $\Lambda = \Lambda_{p,q}$ *and* M, M' *be finite-dimensional* Λ*-modules of dimension vectors* $\underline{n}$, $\underline{n}'$, *which are given by the pairs* (A,B) *and* (A',B'), *respectively. Assume that* $\mathrm{gr}(\underline{n}') \in \mathbb{N}^{(Q_{p,q})_1}$, *all the matrices* A'_i *are of the form* $X_{*,*}$ *and all the matrices* B'_j *are of the form* $Y_{*,*}$. *Then* $\iota^{\Lambda'}_{M,M'}$, *for* $\Lambda' = k_\omega (= k)$, *yields a* k*-isomorphism*

$$\mathrm{Hom}_\Lambda(M,M') \cong \left\{ x = \begin{bmatrix} x_1 \\ \vdots \\ x_{n'_\omega} \end{bmatrix} \in \mathbb{M}_{n'_\omega \times n_\omega}(k) : [x_1 | \dots | x_{n'_\omega}] \cdot W = 0 \right\},$$

where $W = W(A,B,\underline{n}')$ (*see* 2.3).

Proof. (a) We start by proving that a triple $(y,x,u) \in \mathbb{M}_{n'_0 \times n_0}(k) \times \mathbb{M}_{n'_\omega \times n_\omega}(k) \times \mathbb{M}_{n'_{c_1} \times n_{c_1}}(k)$ belongs to $\mathrm{Hom}_\Lambda(M,M')$ if and only if it satisfies the system of three matrix equations:

$$\begin{cases} \text{(i) } xA_1 = A'_1 y, \\ \text{(ii) } xB_1 = B'_1 y, \\ \text{(iii) } xC_2 = C'_2 u. \end{cases}$$

Fix (y,x,u) satisfying (i)–(iii). We have to show that for (y,x,u) also

(iv) $$uC_1 = C'_1 y.$$

To this end we use another form of (iii), namely,

$$\begin{cases} \text{(iii)}_1 \ x_{(1)}C_2 = D_1 u, \\ \text{(iii)}_2 \ x_{(2)}C_2 = D_2 u, \end{cases}$$

obtained from (iii) by using the block matrix presentations of x and C'_2. By the assumptions, (iii)_2 is equivalent to

$\text{(iii)}'_2$ $$D_2^{-1} x_{(2)} C_2 = u.$$

The relations in Λ and (i), (ii), $\text{(iii)}'_2$ yield

$$\begin{aligned} uC_1 &= D_2^{-1} x_{(2)} C_2 C_1 = D_2^{-1} x_{(2)} (A_1 + B_1) = D_2^{-1} \pi_2 x (A_1 + B_1) \\ &= D_2^{-1} \pi_2 (A'_1 + B'_1) y = (D_2^{-1} \pi_2 C'_2) C'_1 y = C'_1 y, \end{aligned}$$

where $\pi_2 = [0 \,|\, I_{n'_{c_1}}] \in \mathbb{M}_{n'_{c_1} \times n'_\omega}(k)$. Consequently, the claim is proved.

Now we show (a). By the above we have

$$\mathrm{Hom}_\Lambda(M,M') = \{(y,x,u) : \text{(i)}, \text{(ii)}, \text{(iii)}_1, \text{(iii)}'_2\}.$$

Observe that subtracting from (iii)_1 equation $\text{(iii)}'_2$ multiplied by D_1 from the left, we obtain a new system $(\text{(iii)}'_1, \text{(iii)}'_2)$, equivalent to $(\text{(iii)}_1, \text{(iii)}'_2)$, where

$$\text{(iii)}'_1 \qquad D_1 D_2^{-1} x_{(2)} C_2 = x_{(1)} C_2.$$

Moreover, by the shape of $\text{(iii)}'_2$, the projection $(y, x, u) \mapsto (y, x)$ yields a k-isomorphism

$$\mathrm{Hom}_\Lambda(M, M') \cong \{(y, x) \in \mathbb{M}_{n'_0 \times n_0}(k) \times \mathbb{M}_{n'_\omega \times n_\omega}(k) : \text{(i)}, \text{(ii)}, \text{(iii)}'_1\}.$$

In this way the proof of (a) is complete.

(b) To show the required isomorphism, we interpret $\mathrm{Hom}_\Lambda(M, M')$ as the set of all tuples

$$\varphi = (y, v^{(1)}, \dots, v^{(p-1)}, w^{(1)}, \dots, w^{(q-1)}, x)$$

in

$$\mathbb{M}_{n'_0 \times n_0}(k) \times \prod_{i=1}^{p-1} \mathbb{M}_{n'_{a_i} \times n_{a_i}}(k) \times \prod_{i=1}^{q-1} \mathbb{M}_{n'_{b_j} \times n_{b_j}}(k) \times \mathbb{M}_{n'_\omega \times n_\omega}(k)$$

satisfying a system $\text{(v)} = (\text{(v)}_{\alpha_1}, \dots, \text{(v)}_{\alpha_p}; \text{(v)}_{\beta_1}, \dots, \text{(v)}_{\beta_q})$ of $p+q$ matrix equations, given by the commutativity of structure maps in M and M' corresponding to all arrows of $Q_{p,q}$, with the components of φ corresponding to the appropriate vertices of $Q_{p,q}$,. To better understand the system (v), we present each coordinate of φ in block matrix form given by rows, in the following way:

$$y = \begin{bmatrix} y_1 \\ \vdots \\ y_{n'_0} \end{bmatrix}, \quad v^{(i)} = \begin{bmatrix} v^{(i)}_1 \\ \vdots \\ v^{(i)}_{n'_{a_i}} \end{bmatrix}, \quad w^{(j)} = \begin{bmatrix} w^{(j)}_1 \\ \vdots \\ w^{(j)}_{n'_{b_j}} \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_{n'_\omega} \end{bmatrix},$$

where $i = 1, \dots, p-1$ and $j = 1, \dots, q-1$. Then, by applying the formulas defining the matrices A'_i and B'_j, the system (v) has the form

$$\text{(v)}'_{\alpha_1} : \begin{bmatrix} v^{(1)}_1 \\ \vdots \\ v^{(1)}_{n'_{a_1}} \end{bmatrix} A_1 = \left[\begin{array}{c} y_1 \\ \vdots \\ y_{n'_0} \\ \hline 0 \\ \vdots \\ 0 \end{array}\right], \qquad \text{(v)}'_{\beta_1} : \begin{bmatrix} w^{(1)}_1 \\ \vdots \\ w^{(1)}_{n'_{b_1}} \end{bmatrix} B_1 = \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline y_1 \\ \vdots \\ y_{n'_0} \end{array}\right],$$

$$\text{(v)}'_{\alpha_2} : \begin{bmatrix} v^{(2)}_1 \\ \vdots \\ v^{(2)}_{n'_{a_2}} \end{bmatrix} A_2 = \left[\begin{array}{c} v^{(1)}_1 \\ \vdots \\ v^{(1)}_{n'_{a_1}} \\ \hline 0 \\ \vdots \\ 0 \end{array}\right], \qquad \text{(v)}'_{\beta_2} : \begin{bmatrix} w^{(2)}_1 \\ \vdots \\ w^{(2)}_{n'_{b_2}} \end{bmatrix} B_2 = \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline w^{(1)}_1 \\ \vdots \\ w^{(1)}_{n'_{b_1}} \end{array}\right],$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$(\mathrm{v})'_{\alpha_{p-1}}:\ \begin{bmatrix} v_1^{(p-1)} \\ \vdots \\ v_{n'_{a_{p-1}}}^{(p-1)} \end{bmatrix} A_{p-1} = \begin{bmatrix} v_1^{(p-2)} \\ \vdots \\ v_{n'_{a_{p-2}}}^{(p-2)} \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad (\mathrm{v})'_{\beta_{q-1}}:\ \begin{bmatrix} w_1^{(q-1)} \\ \vdots \\ w_{n'_{b_{q-1}}}^{(q-1)} \end{bmatrix} B_{q-1} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \hline w_1^{(q-2)} \\ \vdots \\ w_{n'_{b_{q-2}}}^{(q-2)} \end{bmatrix},$$

$$(\mathrm{v})'_{\alpha_p}:\ \begin{bmatrix} x_1 \\ \vdots \\ x_{n'_\omega} \end{bmatrix} A_p = \begin{bmatrix} v_1^{(p-1)} \\ \vdots \\ v_{n'_{a_{p-1}}}^{(p-1)} \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad (\mathrm{v})'_{\beta_q}:\ \begin{bmatrix} x_1 \\ \vdots \\ x_{n'_\omega} \end{bmatrix} B_q = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \hline w_1^{(q-1)} \\ \vdots \\ w_{n'_{b_{q-1}}}^{(q-1)} \end{bmatrix},$$

Each of the above equations can be written in the form $\begin{bmatrix}(\mathrm{v})'_{\alpha_{i,1}} \\ (\mathrm{v})'_{\alpha_{i,2}}\end{bmatrix}$ (resp. $\begin{bmatrix}(\mathrm{v})'_{\beta_{j,1}} \\ (\mathrm{v})'_{\beta_{j,2}}\end{bmatrix}$), where the divisions are indicated by horizontal lines in the vectors on the right hand sides. Now we inductively transform the systems $(\mathrm{v})'_\alpha = ((\mathrm{v})'_{\alpha_1}, \ldots, (\mathrm{v})'_{\alpha_p})$ and $(\mathrm{v})'_\beta = ((\mathrm{v})'_{\beta_1}, \ldots, (\mathrm{v})'_{\beta_q})$, separately, to equivalent systems $(\mathrm{v})''_\alpha = ((\mathrm{v})''_{\alpha_1}, \ldots, (\mathrm{v})''_{\alpha_p})$ and $(\mathrm{v})''_\beta = ((\mathrm{v})''_{\beta_1}, \ldots, (\mathrm{v})''_{\beta_q})$ of a simpler form, in the following way.

We start with $(\mathrm{v})'_\alpha$ and set $(\mathrm{v})''_{\alpha_p} = (\mathrm{v})'_{\alpha_p}$. Assume that $(\mathrm{v})''_{\alpha_i}$ for $1 < i \le p$ is already constructed. Then we define $(\mathrm{v})''_{\alpha_{i-1}}$ as the sum of $(\mathrm{v})'_{\alpha_{i-1}}$ and $(\mathrm{v})''_{\alpha_{i,1}}$ multiplied from the right by the matrix A_i. The resulting system $(\mathrm{v})''_\alpha$, consisting of $n'_{a_1} + \cdots + n'_{a_{p-1}} + n'_\omega$ equations, looks as follows:

$$(\mathrm{v})''_{\alpha_1}:\ \begin{cases} x_1 A_{p,1} = y_1, \\ \cdots\cdots\cdots\cdots \\ x_{n'_0} A_{p,1} = y_{n'_0}, \\ x_{n'_0+1} A_{p,1} = 0, \\ \cdots\cdots\cdots\cdots \\ x_{n'_{a_1}} A_{p,1} = 0, \end{cases} \qquad (\mathrm{v})''_{\alpha_2}:\ \begin{cases} x_1 A_{p,2} = v_1^{(1)}, \\ \cdots\cdots\cdots\cdots \\ x_{n'_{a_1}} A_{p,2} = v_{n'_{a_1}}^{(1)}, \\ x_{n'_{a_1}+1} A_{p,2} = 0, \\ \cdots\cdots\cdots\cdots \\ x_{n'_{a_2}} A_{p,2} = 0, \end{cases}$$

$$\vdots$$

$$(\mathrm{v})''_{\alpha_{p-1}}:\ \begin{cases} x_1 A_{p,p-1} = v_1^{(p-2)}, \\ \cdots\cdots\cdots\cdots \\ x_{n'_{a_{p-2}}} A_{p,p-1} = v_{n'_{a_{p-2}}}^{(p-2)}, \\ x_{n'_{a_{p-2}+1}} A_{p,p-1} = 0, \\ \cdots\cdots\cdots\cdots \\ x_{n'_{a_{p-1}}} A_{p,p-1} = 0, \end{cases} \qquad (\mathrm{v})''_{\alpha_p}:\ \begin{cases} x_1 A_p = v_1^{(p-1)}, \\ \cdots\cdots\cdots\cdots \\ x_{n'_{a_{p-1}}} A_p = v_{n'_{a_{p-1}}}^{(p-1)}, \\ x_{n'_{a_{p-1}+1}} A_p = 0, \\ \cdots\cdots\cdots\cdots \\ x_{n'_\omega} A_p = 0. \end{cases}$$

Similarly we proceed with $(\mathrm{v})'_\beta$. We set $(\mathrm{v})''_{\beta_q} = (\mathrm{v})'_{\beta_q}$. Assume that $(\mathrm{v})''_{\beta_j}$, for $1 < j \le q$, is already constructed. Then we define $(\mathrm{v})''_{\beta_{j-1}}$ to be the sum of

$(\mathrm{v})'_{\beta_{j-1}}$ and $(\mathrm{v})''_{\beta_{j,2}}$ multiplied from the right by the matrix B_j. The resulting system $(\mathrm{v})''_{\beta}$, consisting of $n'_{b_1}+\ldots+n'_{b_{q-1}}+n'_{\omega}$ equations, looks as follows:

$$(\mathrm{v})''_{\beta_1}:\ \begin{cases} x_{n'_\omega-n'_{b_1}+1}B_{q,1}=0,\\ \cdots\cdots\cdots\cdots\\ x_{n'_\omega-n'_0}B_{q,1}=0,\\ x_{n'_\omega-n'_0+1}B_{q,1}=y_1,\\ \cdots\cdots\cdots\cdots\\ x_{n'_\omega}B_{q,1}=y_{n'_0}, \end{cases}\qquad (\mathrm{v})''_{\beta_2}:\ \begin{cases} x_{n'_\omega-n'_{b_2}+1}B_{q,2}=0,\\ \cdots\cdots\cdots\cdots\\ x_{n'_\omega-n'_{b_1}}B_{q,2}=0,\\ x_{n'_\omega-n'_{b_1}+1}B_{q,2}=w_1^{(1)},\\ \cdots\cdots\cdots\cdots\\ x_{n'_\omega}B_{q,2}=w^{(1)}_{n'_{b_1}}, \end{cases}$$

$$\vdots$$

$$(\mathrm{v})''_{\beta_{q-1}}:\ \begin{cases} x_{n'_{b_\omega}-n'_{b_{q-1}}+1}B_{q,q-1}=0,\\ \cdots\cdots\cdots\cdots\\ x_{n'_{b_\omega}-n'_{b_{q-2}}}B_{q,q-1}=0,\\ x_{n'_{b_\omega}-n'_{b_{q-2}}+1}B_{q,q-1}=w_1^{(q-2)},\\ \cdots\cdots\cdots\cdots\\ x_{n'_\omega}B_{q,q-1}=w^{(q-2)}_{n'_{b_{q-2}}}, \end{cases}\qquad (\mathrm{v})''_{\beta_q}:\ \begin{cases} x_1B_q=0,\\ \cdots\cdots\cdots\cdots\\ x_{n'_\omega-n'_{b_{q-1}}}B_q=0,\\ x_{n'_\omega-n'_{b_{q-1}}+1}B_q=w_1^{(q-1)},\\ \cdots\cdots\cdots\cdots\\ x_{n'_\omega}B_q=w^{(q-1)}_{n'_{b_{q-1}}}. \end{cases}$$

Now we complete the proof. It is easily seen that the projection

$$(y,v^{(1)},\ldots,v^{(p-1)},w^{(1)},\ldots,w^{(q-1)},x)\mapsto x=\begin{bmatrix} x_1\\ \vdots\\ x_{n'_\omega}\end{bmatrix}$$

yields a k-isomorphism

$$\mathrm{Hom}_\Lambda(M,M')\cong\left\{x=\begin{bmatrix} x_1\\ \vdots\\ x_{n'_\omega}\end{bmatrix}\in\mathbb{M}_{n'_\omega\times n_\omega}(k):(*)\right\},$$

where $(*)$ denotes system

$$((\mathrm{v})''_{\alpha_1,2},\ldots,(\mathrm{v})''_{\alpha_p,2};\ (\mathrm{v})''_{\beta_1,1},\ldots,(\mathrm{v})''_{\beta_q,1};\ (\mathrm{v})''_{\alpha_1,1}-(\mathrm{v})''_{\beta_1,2}).$$

Note that the matrix of $(*)$ is W, once we interpret $x\in\mathbb{M}_{n'_\omega\times n_\omega}(k)$ as a row vector $[x_1\,|\,\ldots\,|\,x_{n'_\omega}]\in\mathbb{M}_{1\times n_\omega n'_\omega}(k)$. In this way the proof of assertion (b), and of the whole lemma, is complete. ∎

4.2. Now we prove an important fact concerning special homomorphism spaces for modules over domestic canonical algebras.

PROPOSITION. *Let $\Lambda=\Lambda_{p,q,2}$, $p,q\ge 2$, and M,M' be finite-dimensional Λ-modules of dimension vectors $\underline{n},\underline{n}'$, which are given by the triples (A,B,C) and (A',B',C'), respectively. Assume that C'_2 and (A',B') satisfy the assumptions of* (a) *and* (b) *of Lemma* 4.1, *respectively. Then*

$$[M,M']=\mathrm{cor}[W\,|\,W'],$$

where $W=W(A,B,\underline{n}'_{|(p,q)})$ and $W'=\left[\dfrac{-I_{n'_\omega-n'_{c_1}}}{(D_1D_2^{-1})^t}\right]\otimes C_2$.

Proof. Consider the commutative diagram

$$\begin{array}{ccccc}
H & \xrightarrow{\iota_2'} & H_2 & & \\
\downarrow{\scriptstyle \iota_1'} & & \downarrow{\scriptstyle \iota_2} & & \\
H_1 & \xrightarrow{\iota_1} & H_3 & \xrightarrow{\iota_3} & H_4 \cong \mathbb{M}_{n'_\omega \times n_\omega}(k)
\end{array}$$

of k-vector spaces, where $H = \mathrm{Hom}_\Lambda(M, M')$, H_1, H_2, H_3, H_4 stand for the homomorphism spaces $\mathrm{Hom}_{\Lambda'}(M_{|\Lambda'}, M'_{|\Lambda'})$ for $\Lambda' = \Lambda_{p,q}, \Lambda_{1,1,2}, \Lambda_{1,1}, k_\omega$, respectively, and $\iota_1', \iota_2', \iota_1, \iota_2, \iota_3$ denote the maps given by the respective restrictions. Observe that by the introductory remark in 4.1 and the assumptions on M' all five homomorphisms in the diagram are monomorphisms. Moreover, it is easily seen that the pair (ι_1', ι_2') induces a k-isomorphism

$$H \cong H_1 \sqcap_{H_3} H_2,$$

where $H_1 \sqcap_{H_3} H_2$ is the fibre product of H_1 and H_2 along the pair (ι_1, ι_2) of homomorphisms. Consequently, the monomorphisms $\iota_1 \circ \iota_1'$ $(= \iota_2 \circ \iota_2')$ and ι_3 yield a k-isomorphism

$$H \cong \mathrm{Im}\,\iota_1 \cap \mathrm{Im}\,\iota_2 \cong \mathrm{Im}(\iota_3 \circ \iota_1) \cap \iota_3(\mathrm{Im}\,\iota_2).$$

Hence, by Lemma 4.1,

$$H \cong \{x \in \mathbb{M}_{n'_\omega \times n_\omega}(k) : [x_1 \mid \ldots \mid x_{n'_\omega}] \cdot W = 0;\ D_1 D_2^{-1} x_{(2)} C_2 = x_{(1)} C_2\}.$$

Note that the second equation, as a matrix equation in $\mathbb{M}_{(n'_\omega - n'_{c_1}) \times n_{c_1}}(k)$, looks as follows:

$$[-I_{n'_\omega - n'_{c_1}} \mid D_1 D_2^{-1}] \cdot \left[\frac{x_{(1)}}{x_{(2)}}\right] \cdot C_2 = 0.$$

By the lemma below, it is equivalent to the equation

$$[x_1 | \ldots | x_{n'_\omega}] \cdot \left(\left[\frac{-I_{n'_\omega - n'_{c_1}}}{(D_1 D_2^{-1})^t} \right] \otimes C_2 \right) = 0$$

in $\mathbb{M}_{1 \times (n'_\omega - n'_{c_1})\, n_{c_1}}(k)$. This finishes the proof. ∎

LEMMA. *For any $P \in \mathbb{M}_{m_1 \times m}(k)$, $x \in \mathbb{M}_{m \times n}(k)$ and $Q \in \mathbb{M}_{n \times n_2}(k)$, $PxQ = 0$ in $\mathbb{M}_{m_1 \times n_2}(k)$ if and only if $[x_1 | \ldots | x_m] \cdot (P^t \otimes Q) = 0$ in $\mathbb{M}_{1 \times m_1 n_2}(k)$, where $x_1, \ldots, x_m \in \mathbb{M}_{1 \times n}(k)$ are the rows of x.*

Proof. An easy check on definitions. ∎

4.3. Let $Z \in \mathbb{M}_{r \times s}(k)$. Following [18], for any $n \in \mathbb{N}$, we denote by $Z[n]$ the nth *enlargement* of Z. Recall that $Z[n] \in \mathbb{M}_{(r+n) \times (s+n)}(k)$ is given by

$$Z[n] = \left[\begin{array}{c|ccc} & 1 & & \\ Z & & \ddots & \\ \hline & 1 & & 1 \\ 0 & & \ddots & \\ & & & 1 \end{array}\right],$$

where all entries off the two diagonals of length n (consisting of ones) are zeros. We set $Z(0) = Z$.

Assume now that $r \geq s$. Then Z can be written in the form

$$Z = \left[\begin{array}{c} Z_1 \\ \hline Z_2 \end{array}\right],$$

where $Z_1 \in \mathbb{M}_{(r-s)\times s}(k)$ and $Z_2 \in \mathbb{M}_{s\times s}(k)$. Analogously, for any $n \in \mathbb{N}$,

$$Z[n] = \left[\begin{array}{c} Z[n]_1 \\ \hline Z[n]_2 \end{array}\right],$$

where $Z[n]_1 \in \mathbb{M}_{(r-s)\times(n+s)}(k)$ and $Z[n]_2 \in \mathbb{M}_{(n+s)\times(n+s)}(k)$. Observe that $Z[n]_2$ is a block upper triangular matrix, so $Z[n]_2$ is invertible if and only if Z_2 is.

LEMMA. *Let $Z \in \mathbb{M}_{r\times s}(k)$, $r \geq s$, be such that Z_2 is invertible. Then for any $n \in \mathbb{N}$, we have the equality*

$$[-I_{r-s} \,|\, Z[n]_1 Z[n]_2^{-1}] = [-I_{r-s} \,|\, U]^{(\infty|n+r)}$$

of matrices in $\mathbb{M}_{(r-s)\times(n+r)}(k)$, where $U = Z_1 Z_2^{-1} \in \mathbb{M}_{(r-s)\times s}(k)$.

Proof. Recall that the matrix $[-I_{r-s} \,|\, U]^{(\infty|n+r)} \in \mathbb{M}_{(r-s)\times(n+r)}(k)$ is by definition given by the formula

$$[-I_{r-s} \,|\, U]^{(\infty|n+r)} = ([-I_{r-s} \,|\, U]^{(i)})^{|n+r},$$

where $(i-1)r \geq n$, and $[-I_{r-s} \,|\, U]^{(i)} \in \mathbb{M}_{(r-s)\times ir}(k)$ has the form

$$[-I_{r-s} \,|\, U]^{(i)} = [I_{r-s} \,|\, U \,|\, I_{r-s} \,|\, -U \,|\, -I_{r-s} \,|\, U \,|\, \ldots].$$

Fix $n \geq 1$ and set, for simplicity, $\widetilde{Z}_1 = Z[n]_1$ and $\widetilde{Z}_2 = Z[n]_2$. To show the assertion we have to compute the matrix $\widetilde{Z}_1 \widetilde{Z}_2^{-1}$. To this end, we write $\widetilde{Z}_1 \in \mathbb{M}_{(r-s)\times(n+s)}(k)$ and $\widetilde{Z}_2 = \mathbb{M}_{(n+s)\times(n+s)}(k)$ in block matrix form

$$\widetilde{Z}_1 = [Z_1 \,|\, I_{r-s} \,|\, 0] \quad \text{and} \quad \widetilde{Z}_2 = \left[\begin{array}{c|c} Z_2 & Z_2' \\ \hline 0 & Z_2'' \end{array}\right],$$

where $Z_2' \in \mathbb{M}_{s\times n}(k)$ and $Z_2'' \in \mathbb{M}_{n\times n}(k)$. Here

$$Z_2' = [0_{r-s} \,|\, I_s \,|\, 0] \quad \text{and} \quad Z_2'' = I_n + N^r,$$

where $0_{r-s} \in \mathbb{M}_{s\times(r-s)}(k)$ is the zero matrix and $N = J_n(0) \in \mathbb{M}_{n\times n}(k)$ is the (nilpotent) upper triangular Jordan block with eigenvalue 0.

The matrix Z_2'' is clearly invertible. We claim that its inverse is

$$Z_2''^{-1} = \sum_{i=0}^{\infty} (-1)^i N^{ir}.$$

Since the mapping $\overline{T} \mapsto N$ yields an algebra isomorphism between the truncated polynomial algebra $k[T]/(T^n)$ and the subalgebra $k[N] \subseteq \mathbb{M}_{n\times n}(k)$, where $\overline{T} = T + (T^n)$, the claim follows from the fact that the inverse of the invertible element $1 + \overline{T}^r \in k[T]/(T^n)$ is equal to $\sum_{i=0}^{\infty} (-1)^i\, \overline{T}^{ir}$.

Next observe that $\widetilde{Z}_2$ is a block upper triangular matrix, so the inverse of $\widetilde{Z}_2$ has the form

$$\widetilde{Z}_2^{-1} = \left[\begin{array}{c|c} Z_2^{-1} & -Z_2^{-1} Z_2'\, Z_2''^{-1} \\ \hline 0 & Z_2''^{-1} \end{array}\right].$$

Consequently,

$$\widetilde{Z}_1 \widetilde{Z}_2^{-1} = [Z_1 Z_2^{-1} \,|\, -Z_1 Z_2^{-1} Z_2'\, Z_2''^{-1} + [I_{r-s} \,|\, 0] Z_2''^{-1}],$$

Moreover, applying the formula for $Z_2''^{-1}$, we have

$$\begin{aligned} -Z_1 Z_2^{-1} Z_2'\, Z_2''^{-1} &= -Z_1 Z_2^{-1}([0_{r-s} \,|\, I_s \,|\, 0] Z_2''^{-1}) \\ &= -Z_1 Z_2^{-1}[0_{r-s} \,|\, I_s \,|\, -0_{r-s} \,|\, -I_s \,|\, \ldots] \\ &= [0_{r-s} \,|\, -Z_1 Z_2^{-1} \,|\, -0_{r-s} \,|\, Z_1 Z_2^{-1} \,|\, \ldots] \end{aligned}$$

and

$$[I_{r-s} \,|\, 0] Z_2''^{-1} = [I_{r-s} \,|\, 0_s \,|\, -I_{r-s} \,|\, 0_s \,|\, \ldots],$$

where $0_s \in \mathbb{M}_{(r-s)\times s}(k)$ is the zero matrix. Now inserting these two final formulas into that for $\widetilde{Z}_1 \widetilde{Z}_2^{-1}$, we immediately obtain the assertion. ∎

Let $\Lambda = \Lambda_{p,q,2}$, $Z \in \mathbb{M}_{r\times s}(k)$, $r > s$, and let $d \in \mathbb{N}^{(Q_{p,q,2})_0}$ be a vector such that $d_0 = 0$, $d_{c_1} = s$, $d_\omega = r$, $\mathrm{gr}(d) \in \mathbb{N}^{(Q_{p,q,2})_1}$. Then for any $l \in \mathbb{N}$, we denote by $d[l]$ the vector

$$d[l] = d + l \cdot \mathbb{1}$$

and by $N[l]$ the Λ-module

$$N[l] = N(Z, d, l)$$

of dimension vector $d[l]$, given by the triple (A', B', C'), where $A_i' = X_{*,*}$ for $i = 1, \ldots, q$, $B_j' = Y_{*,*}$ for $j = 1, \ldots, q$, $C_1' = Y_{d_{c_1}+l,l}$ and $C_2' = Z[l]$. (Note that $N[l]$ is really a Λ-module).

REMARK. If $\Lambda = \Lambda_{p,q,2}$ is a domestic canonical algebra then the mapping $(l,d) \mapsto d[l]$ yields the inverse of the map which defines the bijection (ii) in Lemma 2.3(a).

As an immediate consequence of Proposition 4.2 and Lemma 4.3 we obtain the following.

COROLLARY. *Let Z and d be as above. Assume that $Z_2 \in \mathbb{M}_{s\times s}(k)$ is invertible, where $Z = \left[\begin{smallmatrix} Z_1 \\ Z_2 \end{smallmatrix}\right]$. Then for any integer $l \in \mathbb{N}$ and Λ-module M of dimension vector n, given by the triple (A,B,C), we have*

$$[M, N[l]] = \operatorname{cor}\left[W(A,B,d[l]_{|(p,q)}) \,\middle|\, \left[\begin{smallmatrix} -I_{r-s} \\ U \end{smallmatrix}\right]_{(\infty_{|l+r})} \otimes C_2\right],$$

where $U = (Z_1 Z_2^{-1})^t$.

4.4. *Proof of Theorem 2.3.* Let $d \in \boldsymbol{P}$. Then clearly $d = \overline{d}[d_0]$ and $\operatorname{gr}(\overline{d}) \in \mathbb{N}^{(Q_{p,q,2})_1}$. Applying the results of [18, 15], we know that, if $r_{\gamma_2}(d) \geq 1$ (which holds always if $\operatorname{rk}(d) \geq 2$), then the unique indecomposable postprojective Λ-module P_d, with dimension vector $d = \overline{d}[d_0]$, can be represented in the form $P_d = N[d_0]$, $N[d_0] = N(Z(d), \overline{d}, l)$, where $Z(d) \in \mathbb{M}_{\bar{d}_\omega \times \bar{d}_{c_1}}(k)$ is uniquely determined by d. Moreover, the set

$$\mathcal{Z} = \{Z : \exists_{e\in \boldsymbol{P},\, r_{\gamma_2}(e)\geq 1}\ Z = Z(e)\}$$

is finite and it is described by two tables; the first from [18] for the case as in Theorem 2.3(b), and the second from [15] for the case as in (c) (for e with $\operatorname{rk}(e) = 1$ and $r_{\gamma_2}(e) = 1$, $Z(e)$ is a trivial matrix in $\mathbb{M}_{1\times 0}(k)$).

One can easily check, by inspection, that each $Z \in \mathcal{Z}$ has the property that the matrix $Z_2 \in \mathbb{M}_{\bar{d}_{c_1}\times \bar{d}_{c_1}}(k)$ is invertible, where $Z = Z(d)$ and $Z = \left[\begin{smallmatrix} Z_1 \\ Z_2 \end{smallmatrix}\right]$ with $Z_1 \in \mathbb{M}_{(\bar{d}_\omega - \bar{d}_{c_1})\times \bar{d}_{c_1}}(k)$.

Now we complete the proof. For any $d \in \boldsymbol{P}$ such that $r_{\gamma_2}(d) \geq 1$, we set $U(d) = (Z_1 Z_2^{-1})^t$, where $Z \in \mathcal{Z}$ is such that $Z = Z(d)$. Then the first assertion of Theorem 2.3, for $d \in \boldsymbol{P}$ as above, follows immediately from Corollary 4.3. We still have to discuss the case of $d \in \boldsymbol{P}$ such that $r_{\gamma_2}(d) = 0$ (and then $\operatorname{rk}(d) = 1$). Note that by Proposition 4.2, the formula for $h(M)_d$ holds trivially in this case, since following [18] the unique indecomposable postprojective Λ-module P_d with $\underline{\dim}\, P_d = d$ is given by the triple (A', B', C') such that $A'_i = X_{*,*}$ for $i \in [p]$, $B'_j = Y_{*,*}$ for $j \in [q]$, $C'_1 = X_{d_{c_1},d_0} + Y_{d_{c_1},d_0}$ and $C'_2 = I_{d_{c_1}} \in \mathbb{M}_{d_\omega \times d_{c_1}}(k)$ (notice that $r_{\gamma_2}(d) = d_\omega - d_{c_1} = 0$).

Now the remaining assertion, in particular Tables 1 and 2, can be obtained from the two tables in [18, 15] mentioned above by computing the matrices $U(d)$ from the definition. In this way the proof of Theorem 2.3 is complete. ∎

5. THE STRUCTURE OF THE SET $\boldsymbol{P}$

This section is mainly devoted to the proof of Theorem 2.4. To do this we need some preparatory facts. The proof will be completed in 5.6. We also define a linear order relation $\prec\ \subset \boldsymbol{P}_0 \times \boldsymbol{P}_0$ to be applied in the next section.

5.1. Let $\Lambda = \Lambda_{p,q,2}$ be a domestic canonical algebra, Σ a section in the translation quiver $\boldsymbol{P} = \boldsymbol{P}(\Lambda)$, and $\boldsymbol{P}_0 = \boldsymbol{P}_0^0 \cup \boldsymbol{P}'_0$, $\boldsymbol{P}^0 = \boldsymbol{P}^0(\Sigma)$, $\boldsymbol{P}' = \boldsymbol{P}'(\Sigma) = -\mathbb{N}\Sigma$, be a splitting of $\boldsymbol{P}$ induced by Σ (see 2.4 for definitions and identifications). Denote by $\boldsymbol{P}''$ the full subquiver of $\boldsymbol{P}'$ with vertex set $\boldsymbol{P}''_0 = \{(-n, x) \in -\mathbb{N}\Sigma : n \neq 0\}$. We start by proving the following property of the Auslander–Reiten translate τ.

LEMMA. *Let $x \in \boldsymbol{P}$.*

(a) *If $x \in \boldsymbol{P}'$ then $\tau^{-1}x = \phi^{-1}(x)$.*
(b) *If $x \in \boldsymbol{P}''$ then $\tau x = \phi(x)$.*

Proof. (a) Let X be an indecomposable postprojective module with $\underline{\dim}\, X = x \in \boldsymbol{P}'_0$. From Theorem 1.5(c) we have $\mathrm{Hom}_\Lambda(D({}_\Lambda\Lambda), X) = 0$. Moreover, $\mathrm{Hom}_\Lambda(\tau^{-1}X, \Lambda) = 0$, since the component $\mathcal{P}$ is standard, and by the definition of section, there is no projective Λ-module P with $\underline{\dim}\, P \in \boldsymbol{P}''_0$ (see [22, 2]). The last equality is equivalent to the fact that inj.dim $X \le 1$, so (a) now follows easily (see [22, 2.4.1*, 2.4.4*]).

(b) Let X be an indecomposable postprojective module with $\underline{\dim}\, X = x \in \boldsymbol{P}''_0$. By similar arguments, we have the equalities $\mathrm{Hom}_\Lambda(X, \Lambda) = 0$ and $\mathrm{Hom}_\Lambda(D({}_\Lambda\Lambda), \tau X) = 0$. The last equality is equivalent to the fact that pd.dim. $X \le 1$ and thus we get (b) (see [22, 2.4.1, 2.4.4]). ∎

Let $\Sigma_0 = \{x(1), \ldots, x(s)\}$. For any $(n, i) \in \mathbb{N} \times [s]$, we set $x(n, i) = \tau^{-n}(x(i))$ (we assume $\tau^0(x(i)) = x(i)$). Note that the root $x(n, i) \in \boldsymbol{P}'_0$ corresponds to $(-n, x(i)) \in -\mathbb{N}\Sigma$. Moreover, by the lemma above we clearly have $x(n, i) = \phi^{-n}(x(i))$ and $x(n' + n, i) = \phi^{-n'}(x(n, i))$ for all $n, n' \in \mathbb{N}$, $i \in [s]$.

5.2. The following fact is crucial for the proof that the cyclic group $G = (\overline{\phi})$ is finite (cf. [8] and [23], see also [19]).

PROPOSITION. *There exists a minimal integer $\nu = \nu_{\Lambda_{p,q,2}}$ such that*

$$\phi^\nu(x) = x + \partial(x) \cdot \mathbb{1}$$

(*and consequently* $\phi^{-\nu}(x) = x - \partial(x) \cdot \mathbb{1}$) *for every* $x \in \mathrm{K}_0(\Lambda)$, *where* $\partial(x) \in \mathbb{Z}$. *The map* $\partial = \partial_\Lambda : \mathrm{K}_0(\Lambda) \to \mathbb{Z}$ (*called the* defect) *is a* $\mathbb{Z}$*-homomorphism such that* $\partial(x) < 0$ *for any* $x \in \boldsymbol{P}$. *The integer* ν *equals* $2p$ (*resp.* p) *if* $q = 2$ *and* p *is odd* (*resp. even*); *it equals* 6, 12 *or* 30 *if the pair* (p, q) *is equal to* $(3,3)$, $(4,3)$ *or* $(5,3)$, *respectively.*

Proof. As already mentioned, $\Lambda = \Lambda_{p,q,2}$ is a tilted algebra of a hereditary algebra $\Lambda'' = kQ''$ of Euclidean type, where Q'' is a quiver of type $\widetilde{\mathbb{D}}_{p+2}$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$ if the pair (p,q) is equal to $(p,2)$, $(3,3)$, $(4,3)$, $(5,3)$, respectively, with $p \geq 2$. (We can set $Q'' = \Sigma$, where Σ is any section in $\mathcal{P}$.)

It is well known [8, 23] that the radical $\operatorname{rad} q''$ of the quadratic form $q'' = q_{Q''}$ associated to Q'' has the description $\operatorname{rad} q'' = \mathbb{Z} \cdot h_{Q''}$, where $h = h_{Q''} \in \mathbb{N}^{Q''_0}$ is the vector uniquely determined by Q'', with all components positive and at least one of them equal to 1. Moreover, let $\phi'' = \phi_{\Lambda''} : \mathrm{K}_0(\Lambda'') \to \mathrm{K}_0(\Lambda'')$ be the Coxeter transformation for Λ''. Then $\phi''(h) = h$ and there exists a minimal integer $\nu'' = \nu_{Q''}$ such that $(\phi'')^{\nu''}(x) = x + \partial''(x) \cdot h$ and $(\phi'')^{-\nu''}(x) = x - \partial''(x) \cdot h$ for every $x \in \mathrm{K}_0(\Lambda'')$, where $\partial''(x) \in \mathbb{Z}$. The map $\partial'' = \partial_{\Lambda''} : \mathrm{K}_0(\Lambda'') \to \mathbb{Z}$ is a $\mathbb{Z}$-homomorphism such that $\partial''(\underline{\dim}\, P'') < 0$ for any indecomposable postprojective Λ''-module P''. The integer $\nu'' = \nu_{Q''}$ is equal to $2p$ (resp. to p) if Q' is of type $\widetilde{\mathbb{D}}_{p+2}$ and p is odd (resp. even); it is equal to 6, 12 or 30 if Q'' is of type $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$ or $\widetilde{\mathbb{E}}_8$, respectively.

To finish the proof recall that by general results of tilting theory there exists a $\mathbb{Z}$-isomorphism $f : \mathrm{K}_0(\Lambda'') \to \mathrm{K}_0(\Lambda)$ such that

$$q \cdot f = q'', \qquad \phi_\Lambda = f \cdot \phi_{\Lambda''} \cdot f^{-1}$$

(see [22, 4.1]) In particular, $f(h) = \mathbb{1}$. Moreover, in our case, the module used in the tilting procedure is postprojective. Hence, for any $x \in \boldsymbol{P}$, there exists an indecomposable postprojective Λ''-module P'' such that $x = f(\underline{\dim}\, P'')$.

Now, by applying the properties of f, the assertions of the proposition follow easily from the respective facts for Euclidean quivers, which were mentioned above. ∎

COROLLARY. *The following equalities hold*:

$$\nu = |\overline{\phi}| = \nu'' = |\overline{\phi}''|$$

where $\overline{\phi}'' : \overline{\mathrm{K}_0(\Lambda'')} \to \overline{\mathrm{K}_0(\Lambda'')}$ *is the reduced Coxeter transformation for* Λ'' *and* $\overline{\mathrm{K}_0(\Lambda'')} = \mathrm{K}_0(\Lambda'')/\operatorname{rad} q''$.

5.3. For the proof of the equality $\nu = \operatorname{lcm}\{\nu_j : j \in [r]\}$ we need the following lemma.

LEMMA. *Let* $\Lambda = \Lambda_{p,q,2}$ *be a domestic canonical algebra*, $S(\boldsymbol{L})$ *the group of all permutations of the set* $\boldsymbol{L}$, *and* $H = \{\psi \in \operatorname{Aut}(\mathrm{K}_0(\Lambda)_{\mathrm{red}}) : \psi(\boldsymbol{L}) \subset \boldsymbol{L}\}$. *Then* H *is a subgroup of* $\operatorname{Aut}(\mathrm{K}_0(\Lambda)_{\mathrm{red}})$ *and the group homomorphism* $R : H \to S(\boldsymbol{L})$, $\psi \mapsto \psi_{|\boldsymbol{L}}$, *is injective.*

Proof. Note first that for any $\psi \in \operatorname{Aut}(\mathrm{K}_0(\Lambda)_{\mathrm{red}})$ such that $\psi(\boldsymbol{L}) \subset \boldsymbol{L}$, we have $\psi(\boldsymbol{L}) = \boldsymbol{L}$, so $\psi^{-1}(\boldsymbol{L}) \subset \boldsymbol{L}$, since $\boldsymbol{L}$ is finite and $\psi_{|\boldsymbol{L}}$ is an injection. Consequently, H is a subgroup of $\operatorname{Aut}(\mathrm{K}_0(\Lambda)_{\mathrm{red}})$. To prove that $\ker R = \{\mathrm{id}_{\boldsymbol{L}}\}$, it suffices to know that the subset $\overline{\Sigma}_0 = \{\overline{x(1)}, \dots, \overline{x(s)}\}$ of $\boldsymbol{L}$ generates $\mathrm{K}_0(\Lambda)_{\mathrm{red}} = \overline{\mathrm{K}_0(\Lambda)}$. This follows from the fact that Σ_0 forms a $\mathbb{Z}$-basis of $\mathrm{K}_0(\Lambda)$, since the unique postprojective module $T = \bigoplus_{i=1}^{s} T(i)$ such that $\dim T(i) = x(i)$ is tilting (see [22]). ∎

5.4. In the proof of Theorem 2.4(e) we apply the results below.

LEMMA. *Let $t, n \in \mathbb{N}$. Then $x(n,i)_0 \geq t\widetilde{\eta}$ provided $n \geq t\nu$, where $\widetilde{\eta} = \min\{-\partial(x(i)) : i \in [s]\}$.*

Proof. Fix $n \in \mathbb{N}$. We argue by induction on $t \in \mathbb{N}$. The case $t = 0$ is obvious. Assume that, given $t > 0$, the assertion holds for all $t' < t$. Suppose that $n \geq t\nu$. Then clearly $n - \nu \geq (t-1)\nu \geq 0$ and applying definitions, basic properties of the defect and the inductive assumption we have

$$\begin{aligned} x(n,i) &= \phi^{-\nu}(\phi^{-(n-\nu)}(x(i))) = \phi^{-(n-\nu)}(x(i)) - \partial(x(i))\mathbb{1} \\ &= x(n-\nu, i) - \partial(x(i))\mathbb{1} \end{aligned}$$

and

$$x(n,i)_0 = x(n-\nu,i)_0 - \partial(x(i)) \geq (t-1)\widetilde{\eta} + \widetilde{\eta} = t\widetilde{\eta}. \ ∎$$

COROLLARY. *Let $m, n \in \mathbb{N}$. Then $x(n,i)_0 > m$ provided $n \geq m\nu/\widetilde{\eta} + \nu$.*

Proof. We have

$$n \geq m\nu/\widetilde{\eta} + \nu = (m/\widetilde{\eta} + 1)\nu \geq (\theta + 1)\nu,$$

where $\theta = \mathrm{quo}_{\widetilde{\eta}}(m)$. Then, from the lemma and the properties of remainders, we infer $x(n,i)_0 \geq (\theta+1)\widetilde{\eta} > m$. ∎

5.5. To prove assertion (f) of Theorem 2.4 we show the following more general fact.

LEMMA. *Let Σ be a section in $\boldsymbol{P}$ and $\boldsymbol{P} = \boldsymbol{P}^0 \cup \boldsymbol{P}'$ be the splitting of $\boldsymbol{P}$ induced by Σ, where $\boldsymbol{P}^0 = \boldsymbol{P}^0(\Sigma)$ and $\boldsymbol{P}' = \boldsymbol{P}'(\Sigma)$. If*

$(*)$ $\quad y_0 < x_0 \quad$ *for all $x \in \Sigma_0 \cup \boldsymbol{P}^0_0$, $y \in \boldsymbol{P}'_0$ such that $\overline{x} = \overline{y}$,*

then Σ has the properties as in assertion (f) *of Theorem* 2.4.

Proof. Let Σ be a section satisfying $(*)$. Note that then $\overline{x(i)} \neq \overline{x(j)}$ for any $1 \leq i, j \leq s$, $i \neq j$. Since $\overline{\phi}$ is an isomorphism, by 5.1 we have

(i) $\overline{x(n,i)} \neq \overline{x(n,j)}$ for any $i, j \in [s]$, $i \neq j$, and $n \in \mathbb{N}$.

Property $(*)$ also implies that

(ii) if $\overline{x(i)} = \overline{x(n,j)}$ then $x(i)_0 < x(n,j)_0$ for any $i, j \in [s]$ and $n > 0$.

To prove our assertion, for any $d \in \overline{\boldsymbol{P}}$ we construct inductively a sequence $\xi(d) = \{\xi_t\}_{t \in \mathbb{N}}$ of nonnegative integers, and show that $\xi(d) = \mathrm{id}_{\mathbb{N}}$.

Fix $d \in \overline{\boldsymbol{P}}$. Set $\xi_0 = x_0^{(0)}, \ldots, \xi_{t_0} = x_0^{(t_0)}$, $t_0 \geq -1$, where $x_0^{(0)}, \ldots, x_0^{(t_0)}$ are all vectors $x \in \boldsymbol{P}_0^0$ such that $\overline{x} = d$. We can assume that $\xi_0 < \xi_1 < \cdots < \xi_{t_0}$. To define ξ_{t_0+1}, let $n \in \mathbb{N}$ be minimal such that $\overline{x(n,i)} = d$ for some $i \in [s]$. Note that by (i), the index i is uniquely determined. We set $\xi_{t_0+1} = x(n,i)_0$. Assume that for $t > t_0$ the integer ξ_t is already defined and $\xi_t = x(n',i')_0$ for some $n' \in \mathbb{N}$, $i' \in [s]$. Then we set $\xi_{t+1} = x(n'',i'')_0$, where $n'' > n'$ is minimal such that $\overline{x(n'',i'')} = d$ for some $i'' \in [s]$. (Note again that i'' is uniquely determined.)

Observe that by Lemma 2.3(a), the constructed sequence $\xi(d)$ is surjective as a function $\mathbb{N} \to \mathbb{N}$. This follows directly from the construction, since for each $x \in \boldsymbol{P}$ with $\overline{x} = d$, there exists $t \geq 0$ such that $\xi_t = x_0$.

Next we prove that the sequence $\xi(d)$ is increasing. Observe first that by $(*)$ we have $\xi_{t_0} < \xi_{t_0+1}$ if $t_0 \geq 0$. We now show that $\xi_{t+1} > \xi_t$ for $t > t_0$. Referring to the definition above, this inequality has the shape $x_0 < y_0$, where $x = x(n',i')$, $y = x(n'',i'')_0$, and $n' < n''$. Applying the obvious equalities $x = \overline{x} + x_0\mathbb{1}$, $y = \overline{y} + y_0\mathbb{1}$ and $\phi^{n'}(x) = x(i')$, $\phi^{n'}(y) = x(n''-n',i'')$, we infer that $\overline{x(i')} + x(i')_0\mathbb{1} = \overline{\phi}^{n'}(\overline{x}) + (t+x_0)\mathbb{1}$ and $\overline{x(n''-n',i'')} + x(n''-n',i'')_0\mathbb{1} = \overline{\phi}^{n'}(\overline{y}) + (t+y_0)\mathbb{1}$ for some $t \in \mathbb{Z}$, since $\overline{\phi^{n'}(\overline{x}) = \phi^{n'}}(\overline{y})$. Then $x_0 = x(i')_0 - t$, $y_0 = x(n''-n',i'')_0 - t$ and $\overline{x(i')} = \overline{x(n''-n',i'')}$, since $\overline{\phi}^{n'}(\overline{x}) = \overline{\phi}^{n'}(\overline{y})$. Now the required inequality $x_0 < y_0$ follows immediately from (ii).

To complete the proof, note that $\xi(d) = \mathrm{id}_{\mathbb{N}}$ for every $d \in \overline{\boldsymbol{P}}$, since $\xi(d)$ is increasing and surjective. Now it is easily seen that for any pair $x = x(n,i)$, $y = x(n',i')$ of vectors in $\boldsymbol{P}'$, the two conditions from assertion (f) of Theorem 2.4 are satisfied provided $\overline{x} = \overline{y} = d$. ∎

5.6. *Proof of Theorem 2.4.* (a) We prove that $\boldsymbol{L}$ is G-invariant, where $G = (\overline{\phi}\,)$. Fix $x \in \boldsymbol{L}$. Then by the shape of the bijection in Lemma 2.3(a)(ii) and the finiteness of the subquiver $\boldsymbol{P}^0$, there exists $t \geq 0$ such that $y = x[t] \in \boldsymbol{P}_0''$ (see Remark 4.3). To show the first assertion of (a) observe that $\phi(y) \in \boldsymbol{P}$, since $\phi(y) = \tau(y)$ from Lemma 5.1(b). Then

$$\overline{\phi}(x) = \overline{\phi}(\overline{y}) = \overline{\phi(y)} = \overline{\tau(y)}$$

and by Lemma 2.3(a), $\overline{\phi}(x) \in \overline{\boldsymbol{P}} = \boldsymbol{L}$. Note that by Proposition 5.2 the group G is cyclic of order ν, so the first assertion is shown.

Now we prove the equality $\boldsymbol{L} = \mathcal{O}(\overline{x(1)}) \cup \cdots \cup \mathcal{O}(\overline{x(s)})$. The set $\boldsymbol{P}'$ is a cofinite subset of $\boldsymbol{P}$. Therefore, for any $x \in \boldsymbol{L}$, there exists $y \in \boldsymbol{P}'$, $y = x(n,i)$, such that $\overline{y} = \overline{x}$ (see Lemma 2.3(a)(ii)). Hence, $y = \phi^{-n}(x(i))$ and $x = \overline{y} = \overline{\phi^{-n}(x(i))} = \overline{\phi}^{-n}(\overline{x(i)})$, so $x \in \mathcal{O}(\overline{x(i)})$. In this way the proof of (a) is complete.

(b) The equality $\mathcal{O}(\overline{x(i_j)}) = \{y_{0,j}, \ldots, y_{\nu_j-1,j}\}$ follows immediately from the fact that G is a finite cyclic group. The remaining statements of the first assertion of (b) are now straightforward.

To prove the second assertion, we only have to show the equality $\widetilde{\nu} = \nu$, where $\widetilde{\nu} = \mathrm{lcm}\{v_j : j \in [r]\}$. The equality $\nu = \nu_\Sigma$ follows immediately from Proposition 5.2. (Note that $\nu_j \,|\, \nu$ for every $j \in [r]$, so $\widetilde{\nu} \,|\, \nu$, since $|G| = \nu$ and $|\mathcal{O}(\overline{x(i_j)})| = \nu_j$.)

Observe first that $(\overline{\phi}^{\nu_j}) = \mathrm{Stab}(\overline{x(i_j)})$, where $\mathrm{Stab}(\overline{x(i_j)})$ denotes the stabilizer of $\overline{x(i_j)}$ under the standard action of G on $\boldsymbol{L}$. For $l \in \mathbb{N}$, $\overline{\phi}^l$ belongs to $\mathrm{Stab}(\overline{x(i_j)})$ if and only if $\nu_j \,|\, l$. Consequently, by Lemma 5.3 and the equality $\boldsymbol{L} = \mathcal{O}(\overline{x(1)}) \cup \cdots \cup \mathcal{O}(\overline{x(s)})$,

$$\overline{\phi}^l = \mathrm{id}_{\mathrm{K}_0(\Lambda)_{\mathrm{red}}} \;\Leftrightarrow\; \overline{\phi}^l_{|\boldsymbol{L}} = \mathrm{id}_{|\boldsymbol{L}} \;\Leftrightarrow\; \overline{\phi} \in \bigcap_{j\in[r]} \mathrm{Stab}(\overline{x(i_j)}) \;\Leftrightarrow\; \widetilde{\nu} \,|\, l.$$

Then the nonempty sets consisting of all $l \in \mathbb{N}$ that satisfy separately the leftmost and rightmost conditions coincide. Taking now the minimal value in these two sets we obtain the equality $\widetilde{\nu} = \nu$.

Now we show the third assertion. By Proposition 5.2, the inequality $\kappa_j > 0$ follows immediately from the formula

$$(**) \qquad \partial_j = \frac{\nu}{\nu_j}\,\kappa_j,$$

where $\partial_j = -\partial(x(i_j))$ for $j \in [r]$. To prove $(**)$, note first that $\phi^{-\nu_j}(x(i_j)) = x(i_j) + \kappa_j \mathbb{1}$, since $\phi^{-\nu_j}(y_{l,j}) = y_{l,j} + \kappa_j \mathbb{1}$. Applying this equality, we have

$$\phi^{-\nu}(x(i_j)) = (\phi^{-\nu_j})^{(\nu/\nu_j)}(x(i_j)) = x(i_j) + \frac{\nu}{\nu_j}\,\kappa_j \mathbb{1}$$

and on the other hand also

$$\phi^{-\nu}(x(i_j)) = x(i_j) - \partial(x(i_j))\mathbb{1}.$$

In this way we obtain $(**)$, hence $\kappa_j > 0$.

Next we show formula 2.4$(*)$. We start by some general observation. For any $y \in \boldsymbol{L}$ we set $\boldsymbol{P}(y) = \{x \in \boldsymbol{P} : \overline{x} = y\}$. Then by Lemma 2.3(a)(ii), for any cofinite subset $J \subset \boldsymbol{P}(y)$ and positive integer $\kappa \in \mathbb{N}$, we have $\pi(J) = \mathbb{Z}_\kappa$, where $\pi = \pi_\kappa : \mathrm{K}_0(\Lambda) \to \mathbb{Z}_\kappa$ is given by $\pi(x) = \mathrm{rem}_\kappa(x_0)$, $x \in \mathrm{K}_0(\Lambda)$.

Now we fix $j \in [r]$ and $l \in \{0, \ldots, \nu_j - 1\}$. Then for any $i \in [s]_j$, i.e. $j = j(i)$, the set

$$\boldsymbol{P}'(j,l,i) := \{x = x(n,i) \in \boldsymbol{P}' : \overline{x} = y_{l,j}\}$$

is nonempty, since $\mathcal{O}(\overline{x(i)}) = \mathcal{O}(\overline{x(i_j)}) = \{y_{0,j}, \ldots, y_{\nu_j-1,j}\}$ and $x(n,i) - x(n,i)_0\mathbb{1} = \overline{\phi^{-n}(x(i))} = \overline{\phi}^{-n}(\overline{x(i)}) = \overline{\phi}^{-n}(y_{l(i),j}) = y_{l,j}$ for a suitable $n \in \mathbb{N}$. Observe that for the vector $x = x(n,i) \in \boldsymbol{P}'(j,l,i)$ such that $n = n(j,l,i)$ is minimal, the integer x_0 is given by the formula

$$x_0 = \varrho_j(l,i),$$

where $\varrho_j(l,i)$ is as in Theorem 2.4(b). Then we have the equality

$$\{x_0 : x \in \boldsymbol{P}'(j,l,i)\} = \varrho_j(l,i) + \kappa_j\mathbb{N},$$

or equivalently,

$$\boldsymbol{P}'(j,l,i) = y_{l,j} + (\varrho_j(l,i) + \kappa_j\mathbb{N})\mathbb{1},$$

since $\phi^{-\nu_j}(y_{l,j}) = y_{l,j} + \kappa_j\mathbb{1}$, so $\phi^{-\nu_j}(x(n',i)) = x(n',i) + \kappa_j\mathbb{1}$ for all $x(n',i)$ with $\overline{x(n',i)} = y_{l,j}$. Consequently, applying the introductory general observation, we have

$$\mathbb{Z}_{\kappa_j} = \bigcup_{i\in[s]_j} \pi_{\kappa_j}(\boldsymbol{P}'(j,l,i)) = \{\overline{\varrho}_j(l,i) : i \in [s]_j\},$$

since $\bigcup_{i\in[s]_j} \boldsymbol{P}'(j,l,i)$ is a cofinite subset of $\boldsymbol{P}(y_{l,j})$, and in this way 2.4(∗) is proved.

In particular, 2.4(∗) implies immediately $\kappa_j \le |[s]_j|$. We now show the opposite inequality. Suppose that $\overline{\varrho}_j(l,i) = \overline{\varrho}_j(l,i')$ for some $i,i' \in [s]_j$. Then the sets $\varrho_j(l,i) + \kappa_j\mathbb{N}$ and $\varrho_j(l,i') + \kappa_j\mathbb{N}$ intersect nontrivially. Hence, there exist $n,n' \in \mathbb{N}$ such that $x(n,i)_0 = x(n',i')_0$ and $\overline{x(n,i)} = \overline{x(n',i')}$. Consequently, $x(n,i) = x(n,i')$, so $i = i'$. Thus $\kappa_j = |[s]_j|$ and the proof of (b) is complete.

(c) For $n < \nu_j$ the required formula follows from the equality $x(n,i) = \overline{x(n,i)} + x(n,i)_0\mathbb{1}$, since $\overline{x(n,i)} = y_{n\oplus l(i),j(i)}$ and $x(n,i)_0 = \varrho_{j(i)}(n \oplus l(i), i)$ (see the interpretation of the integers $\varrho_j(l,i)$ in the proof of (b)). The formula for $n \ge \nu_j$ is an immediate consequence of the equality $x(n'+\nu_j,i) = x(n',i) + \kappa_{j(i)}\mathbb{1}$.

(d) Let x and (j,l,i) be as in the assumptions of (d). Clearly, we have $x = y_{l,j} + x_0\mathbb{1} \in \boldsymbol{P}(y_{l,j})$. Then, by 2.4(∗), $x \in \boldsymbol{P}'$ if and only if $x \in \boldsymbol{P}'(j,l,i) = y_{l,j} + (\varrho_j(l,i) + \kappa_j\mathbb{N})\mathbb{1}$ (see the proof of (b) for the definitions). Consequently, $x \in \boldsymbol{P}'$ is equivalent to the inequality $x_0 \ge \varrho_j(l,i)$.

Now we prove the formula for the coordinates of $x \in \boldsymbol{P}'$ in the presentation $x = x(n,i)$. Note first that for (j,l,i) as above, we have

$$n(j,l,i) = \begin{cases} l - l(i) & \text{if } l \ge l(i), \\ l - l(i) + \nu_j & \text{if } l < l(i), \end{cases}$$

where $\varrho_j(l,i) = x(n(j,l,i),i)_0$ (see proof of (b)). On the other hand, both $x = x(n,i) = y_{l,j} + x_0\mathbb{1}$ and $z = x(n(j,l,i),i) = y_{l,i} + \varrho_j(l,i)\mathbb{1}$ belong to $\boldsymbol{P}'(j,l,i)$, so $x_0 - \varrho_j(l,i) = \zeta\kappa_j$ for some $\zeta \in \mathbb{N}$. Then, by applying the formula $\phi^{-\nu_j}(z) = z + \kappa_j\mathbb{1}$, we have

$$x = z + \zeta\kappa_j\mathbb{1} = \phi^{-\zeta\nu_j}(z) = x(n(j,l,i) + \zeta\nu_j, i).$$

Now, the required formula for n follows from those for $n(j,l,i)$ and from the equality $\zeta = (x_0 - \varrho_j(l,i))\kappa_j{}^{-1}$.

(e) Note first that $-\partial(x(i)) = \partial_{j(i)}$ for any $i \in [s]$. Consequently, by $(**)$, we have $\widetilde{\eta} = \eta\nu$. Now the assertions follow easily from Corollary 5.4, properties of remainders and the fact that $x(n,i)_v \geq x(n,i)_0$ for every $v \in Q_0$.

(f) In Tables 7.1 below, for any domestic canonical algebra $\Lambda_{p,q,2}$, we provide one selected section Σ together with the subquiver $\boldsymbol{P}^0 = \boldsymbol{P}^0(\Sigma)$. It is easily seen that all these sections satisfy the assumptions of Lemma 5.5, since $x_0 = 0$ for all but one $x \in \boldsymbol{P}_0^0 \cup \Sigma$, $y = \underline{\dim}\, P(0)$ belongs to Σ_0 and $\overline{y} = \underline{\dim}\, S(\omega)$ belongs to $\boldsymbol{P}_0^0$. Consequently, (f) holds by the lemma quoted above, and the proof of Theorem 2.4 is complete. ∎

5.7. Keeping the notation of 2.4 and 5.1, we introduce the announced relation $\prec \subset \boldsymbol{P}_0 \times \boldsymbol{P}_0$.

Given an enumeration $x(1), \ldots, x(s)$ of the vertices in Σ, we define the relation

$$\prec' = \prec'_{(x(1),\ldots,x(s))} \subset \boldsymbol{P}'_0 \times \boldsymbol{P}'_0$$

by setting

$$x(n',i') \prec' x(n,i) \text{ if and only if either } n' < n, \text{ or } n' = n \text{ and } i' < i.$$

It is clear that $\prec'$ yields a lexicographic order on the set $\mathbb{N} \times \Sigma$, so also in $\boldsymbol{P}'$. This relation has the following simple property.

LEMMA. *If $(x(1), \ldots, x(s))$ is a full admissible sequence of sources (in the sense of [2]) in the section Σ, then $x(n',i') \prec' x(n,i)$ for any $x(n',i') \in$ $\in {}^{-}x(n,i)$, $n \geq 1$.*

Proof. Assume $x(n',i') \in {}^{-}x(n,i)$ for $n \geq 1$. Clearly, $n' \leq n$. If $n' = n$ then there exists an arrow $x(i') \to x(i)$ in Σ and, by the assumption on $(x(1), \ldots, x(s))$, we have $i' < i$; Consequently, $x(n',i') \prec' x(n,i)$. The case $n' < n$ is trivial. ∎

Let Σ, with $\Sigma_0 = \{x(1), \ldots, x(s)\}$, be the selected section, and $\boldsymbol{P}_0^0 = \{z(1), \ldots, z(t)\}$, $t = |\boldsymbol{P}_0^0|$, be the enumeration of the vertices in $\boldsymbol{P}^0$ established for Λ in 7.1. The collection of these two data sets for an individual domestic canonical algebra Λ is denoted further by 7.1.$I(\boldsymbol{P})_\Lambda$. We extend $\prec' = \prec'_{x(1),\ldots,x(s)}$ to a relation

$$\prec = \prec_{(x(1),\ldots,x(s);z(1),\ldots,z(t))} \subset \boldsymbol{P}_0 \times \boldsymbol{P}_0.$$

For $x, y \in \boldsymbol{P}_0$, we set $x \prec y$ if and only if one of the following, pairwise exclusive, conditions holds:

- $x = z(i)$, $y = z(j)$ and $i < j$,
- $x \in \boldsymbol{P}^0$ and $y \in \boldsymbol{P}'$,
- $x, y \in \boldsymbol{P}'$ and $x \prec' y$.

Proposition. *The relation $\prec$ defines the structure of a partially ordered set on $\boldsymbol{P}_0$ such that*

(a) $(\boldsymbol{P}_0, \prec) \simeq (\mathbb{N}, <)$,
(b) $\tau x \prec x$ *and* $y \prec x$ *for any* $x \in \boldsymbol{P}_0$ *and* $y \in {}^{-}x$.

Proof. Assertion (a) is an immediate consequence of the definitions. To prove (b), note that the sequence $(x(1), \ldots, x(2))$ from $7.1.I(\boldsymbol{P})_\Lambda$ is an admissible sequence of sources in Σ for any domestic canonical algebra Λ. Then, by Lemma 5.7, we have $y \prec x$ for $x = x(n,i) \in \boldsymbol{P}_0'$ and $y \in {}^{-}x$, provided $n \geq 1$. Moreover, $\tau x = x(n-1, i)$ in this case. In the remaining case, $x \in \boldsymbol{P}_0^0 \cup \Sigma_0$ and the assertion follows easily by inspection (see $7.1.I(\boldsymbol{P})_\Lambda$). ■

6. ALGORITHMS AND OPTIMIZATION

In this section, using a pseudo-code, we describe the consecutive steps of the announced algorithms. The most important one computes directly the restricted multiplicity vector $m(M)_{|\boldsymbol{P}}$ for modules M over a fixed domestic canonical algebra Λ. We also discuss some optimization of the algorithms and complete the proof of Theorem 2.2.

We apply the results of the previous sections and the tables of Section 7. The semantics of the pseudo-code is clear from the context (see also [6]). The only nonstandard instruction we use is "`read` y `from` Y". It means that the data y, which is "situated in the element Y" of the paper, is further available in the code as a value of the variable (or variables) named y. ("An element" is usually a table or a theorem.)

6.1. We start with a preparatory algorithm.

Algorithm (computing the initial parameters for a domestic canonical algebra $\Lambda_{p,q,2}$).

Input: A pair of integers (p,q) such that the algebra $\Lambda = \Lambda_{p,q,2}$ is domestic canonical.

Output: The following collection of parameters for Λ described in Theorem 2.4:

(i) r; ν_j, κ_j, $[s]_j$, for $1 \leq j \leq r$; $j(i)$, $l(i)$, for $1 \leq i \leq s$, where $s = p + q + 1$; $y_{l,j}$, for $(l,j) \in \mathbb{Z}_{\nu_j} \times [r]$; $u_{l,j}$, for $(l,j) \in [\nu_j] \times [r]$;
(ii) $\varrho_j(l,i)$, for $j \in [r]$, $(l,i) \in \mathbb{Z}_{\nu_j} \times [s]_j$; $x(n,i)$, for $i \in [s]$, $n \in \mathbb{Z}_{\nu_{j(i)}}$.

(1) *Determining the set of distinct orbits* $\{\mathcal{O}(\overline{x(i_j)})\}_{j \in [r]}$ *and the parameters* (i) *connected to them, as in Theorem 2.4*(b):

```
read φ^{-1}_{Λ_{p,q,2}} from 7.2;
set r := 0;
for i := 1 to s do {
  read x(i) from 7.1.I(P)_{Λ_{p,q,2}};
  set x := overline{x(i)}; found := false; j := 1;
  while not found and j ≤ r do {
    set l := 1;
    while not found and l ≤ ν_j − 1 do {
      if x = y_{l,j} then {
        set l(i) := l; j(i) := j;
        set [s]_j := [s]_j ∪ {i};
        set found := true;
      }
      set l := l + 1;
    }
    set j := j + 1;
  }
  if not found then {
    set r := r + 1; j(i) := r; [s]_r := {i}; l(i) := 0;
    set ν_r := 0; y := x; κ_r := 0;
    do {
      set y_{ν_r,r} := overline{y};
      set y := φ^{-1}_{Λ_{p,q,2}}(overline{y}); ν_r := ν_r + 1;
      set u_{ν_r,r} := y_0; κ_r := κ_r + u_{ν_r,r};
    } while overline{y} ≠ x;
  }
}
```

(2) *Computing the parameters* $\varrho_j(l,i)$ *for* $j \in [r]$, $(l,i) \in \mathbb{Z}_{\nu_j} \times [s]_j$, *using formulas from Theorem 2.4(b):*

```
for j := 1 to r do
  for each i ∈ [s]_j {
    set ϱ_j(l(i), i) := x(i)_0;
    for l := l(i) + 1 to ν_j − 1 do
      set ϱ_j(l, i) := ϱ_j(l − 1, i) + u_{l,j};
    if l(i) > 0 then {
      set ϱ_j(0, i) := ϱ_j(ν_j − 1, i) + u_{ν_j,j};
      for l := 1 to l(i) − 1 do
        set ϱ_j(l, i) := ϱ_j(l − 1, i) + u_{l,j};
    }
  }
```

(3) *Computing the "initial" dimension vectors* $x(n,i)$ *for* $i \in [s]$, $n \in \mathbb{Z}_{\nu_{j(i)}}$, *using the first formula from Theorem 2.4(c):*

```
for i := 1 to s do
  for n := 0 to ν_{j(i)} − 1 do
    set x(n, i) := y_{n ⊕_{ν_{j(i)}} l(i)} + ϱ_{j(i)}(n ⊕_{ν_{j(i)}} l(i), i)𝟙; ■
```

REMARK.

(a) The algorithm prepares only the initial parameters for a fixed domestic canonical algebra Λ. In contrast to the next algorithms, which are invoked separately for each module M, it is executed exactly once for each algebra and clearly does not depend on M at all.

(b) After full execution of the algorithm for an algebra $\Lambda = \Lambda_{p,q,2}$, we can observe that $\nu = \max\{\nu_j : j \in [r]\}$ (cf. assertion (c)), and that $\kappa_j \leq 2$, so $\big|[s]_j\big| \leq 2$ for all $j \in [r]$. Moreover, $\nu = \nu_{j(i)}$ for $i \in [s]$ such that $x(i) = \underline{\dim}\, P(0)$.

6.2. Let M be a finite-dimensional Λ-module. We give an algorithm starting from $x = z(1)$ and computing successively, with respect to the linear order $\prec \subset \boldsymbol{P}_0 \times \boldsymbol{P}_0$ defined in 5.7, the multiplicities $m(M)_x$, $x \in \boldsymbol{P}_0$. The fact that we proceed according to the order $\prec$ has some nice consequences for managing the memory in a possible implementation (see also Remark (b)).

In the description of the algorithm we use the following conventions and constructions:

- We assume that the function ${}^{-}(\ ) : ((-\mathbb{N}\setminus\{0\})\Sigma)_0 \to 2^{(-\mathbb{N}\Sigma)_0}$, which assigns to the vertex $(-n,i) \in (-\mathbb{N}\setminus\{0\})\Sigma$ the set ${}^{-}(-n,i)$ of its direct predecessors, is already available (it can be easily implemented applying the definition of the translation quiver $-\mathbb{N}\Sigma$, see [2]).
- The string "$h(M)_x$", for $x \in \boldsymbol{P}_0$, appearing in the code can have one of the following two meanings: either

  (i) "`return` the value $h(M)_x$" if it is already determined by the algorithm (it should have been stored; it depends on a possible implementation), or

  (ii) "`form` the matrix $\mathcal{M} = \mathcal{M}(M,x)$, `compute` the value of $h(M)_x$ ($= \operatorname{cor}\mathcal{M}$, see Theorem 2.3) and then `return` it" if the integer $h(M)_x$ has not been determined yet (it also should then be stored for later use).

- The function `compute` is realized by applying the standard Gaussian-row elimination. The function `form`, given a Λ-module $M = (A,B,C)$ and $x \in \boldsymbol{P}$, constructs the matrix $\mathcal{M}(M,x)$ using the matrix $U(x)$, chosen from a finite list (see Tables 1 and 2 in 2.3) and next "combined" with C_2, and some matrices from the finite list $A_{p,1},\dots,A_{p,p}$; $B_{q,1},\dots,B_{q,q}$. Selection of the matrices depends on x and is done according to the rules from Theorem 2.3. The list above consists of consecutive "partial products" and can be computed only once for the module M.
- We set $h(M)_0 = 0$; also, $\tau x = 0$ if it is not defined for $x \in \boldsymbol{P}_0^0 \cup \Sigma_0$.

ALGORITHM (computing the restricted multiplicity vector $m(M)_{|\boldsymbol{P}}$ for a module M over a fixed domestic canonical algebra Λ). Fix a pair (p,q) of integers such that the canonical algebra $\Lambda=\Lambda_{p,q,2}$ is domestic.

Input: A finite-dimensional Λ-module M given by the triple (A,B,C).

Output: The restricted multiplicity vector $m(M)_{|\boldsymbol{P}}$.

(0) *Preparation*: loading the following collection of parameters for $\Lambda=\Lambda_{p,q,2}$ described in Theorem 2.4 (it can be computed by applying Algorithm 6.1 for the pair (p,q)):

(i) ν_j, κ_j, for $1\le j\le r$; $j(i)$, for $1\le i\le s$, where $s=p+q+1$.
(ii) $x(n,i)$ for $i\in[s]$, $n\in\mathbb{Z}_{\nu_{j(i)}}$.

(1) *Computing the vector $m(M)_{|\boldsymbol{P}^0\cup\Sigma}$*:

```
read |P_0^0| from 7.1.I(P)_{Λ_{p,q,2}};
for i := 1 to |P_0^0| do {
  read z(i), τz(i), ⁻z(i) from 7.1.I(P)_{Λ_{p,q,2}};
  set m(M)_{z(i)} := h(M)_{z(i)} + h(M)_{τz(i)} − Σ_{y∈⁻z(i)} h(M)_y;
}
for i := 1 to s do {
  read x(i), τx(i), ⁻x(i) from 7.1.I(P)_{Λ_{p,q,2}};
  set m(M)_{x(i)} := h(M)_{x(i)} + h(M)_{τx(i)} − Σ_{y∈⁻x(i)} h(M)_y;
}
```

(2) *Computing the vector $m(M)_{|\boldsymbol{P}''}$*:

```
read ν_{Λ_{p,q,2}} from Proposition 5.2;
set m := dim M;
set η := min{κ_j/ν_j : j ∈ [r]};
set ξ := min{m_i/η + ν_{Λ_{p,q,2}} : i ∈ (Q_{p,q,2})_0};
set n := 1;
while n < ξ do {
  for i := 1 to s do {
    if n < ν_{j(i)} then set x := x(n,i);
      else set x := x(rem_{ν_{j(i)}}(n), i) + quo_{ν_{j(i)}}(n) κ_{j(i)} 𝟙;
    set ⁻x := ∅;
    for each (−n′, i′) ∈ ⁻(−n, i) do
      if n′ < ν_{j(i′)} then set ⁻x := ⁻x ∪ {x(n′, i′)};
        else set ⁻x := ⁻x ∪ {x(rem_{ν_{j(i′)}}(n′), i′) + quo_{ν_{j(i′)}}(n′) κ_{j(i′)} 𝟙};
    if n − 1 < ν_{j(i)} then set τx := x(n − 1, i);
      else set τx := x(rem_{ν_{j(i)}}(n − 1), i) + quo_{ν_{j(i)}}(n − 1) κ_{j(i)} 𝟙;
    set m(M)_x := h(M)_x + h(M)_{τx} − Σ_{y∈⁻x} h(M)_y;
    set m := m − m(M)_{x(n,i)} · x(n, i);
    set ξ := min{m_i/η + ν_{Λ_{p,q,2}} : i ∈ (Q_{p,q,2})_0};
  }
  set n := n + 1;
} ∎
```

Remark.

(a) The correctness of the algorithm follows from Theorem 2.4 and formulas $(*)$ from the Introduction. Note that after the stop of loops in step (2) we obtain the multiplicities for all postprojective direct summands of M. Since the stop condition in (2) is based on Theorem 2.4(f), the possible next run of these loops would test an indecomposable postprojective module X whose dimension vector x does not satisfy the inequality $x \leq \underline{\dim}\, M - \underline{\dim}\, P$, so clearly X could not be a direct summand of M (P is a postprojective summand of M detected up to that stage).

(b) The loops in steps (1) and (2) are constructed in such a way that the multiplicities $m(M)_x$, $x \in \boldsymbol{P}$, are computed in the algorithm successively according to the order $\prec$. Consequently, by Proposition 5.7, when the instruction "`set` $m(M)_x := h(M)_x + h(M)_{\tau x} - \sum_{y\in{}^-x} h(M)_y$" is being executed, the integers $h(M)_{\tau x}$, $h(M)_y$, for $y \in {}^-x$, are already determined. So, determining $h(M)_x$ is the only computation that is executed in this step (see also the comments before Algorithm 6.2). Thus, in a possible computer implementation, some data structure for storing the integers $h(M)_x$ already computed should be used. Note that if $x = x(n,i)$, $n \geq 1$, then $\tau x = x(n',i')$, $y = x(n'',i'')$ and $n',n'' \in \{n-1,n\}$, for $y \in {}^-x$. Consequently, to compute $m(M)_x$, only the integers $h(M)_{x(n''',i''')}$ for $n''' \in \{n-1,n\}$, $i''' \in [s]$ should be stored.

(c) Algorithm 6.2 can also be applied to compute the integer $\mathrm{rk}_{\mathcal{P}}(M)$.

Now we give a first estimate of the complexity of Algorithm 6.2.

Lemma. *Let M be a finite-dimensional module with $\dim_k M = n$ over a fixed domestic canonical algebra $\Lambda = \Lambda_{p,q,2}$. Then the pessimistic complexity of Algorithm* 6.2 *is $\mathcal{O}(n^7)$.*

Proof. Let $\underline{n} = \underline{\dim}\, M$. We start by estimating the complexity of determining $m(M)_x$, $x \in \boldsymbol{P}(d)$, for a fixed vector $d \in \boldsymbol{L} = \overline{\boldsymbol{P}}$, where

$$\boldsymbol{P}(d) = \{y \in \boldsymbol{P} : \overline{y} = d\}.$$

By the stop condition in part (2), the integers $m(M)_x$, $x \in \boldsymbol{P}(d)$, are computed only at most for those vectors x that belong to the finite set

$$\boldsymbol{P}(d)_n = \{y \in \boldsymbol{P}(d) : y = x(i,l),\ l < n/s\eta + \nu_{\Lambda_{p,q,2}}\} \cup \{d\}.$$

Let us arrange all elements of $\boldsymbol{P}(d)_n$ in a chain $x^{(0)} \prec x^{(1)} \prec \cdots \prec x^{(t)}$, $t \geq 0$. From the definition of $\prec$ and Theorem 2.4(f), we have $x^{(i)}_0 = i$, so $x^{(i)} = d[i]$, and $t < n/s\eta + \nu_{\Lambda_{p,q,2}}$. Note that determining the multiplicity $m(M)_{x^{(i)}}$, for a fixed i, relies in fact only on computing the integer $h(M)_{x^{(i)}}$ (see Remark (b)). By Theorem 2.3, $h(M)_{x^{(i)}} = \operatorname{cor} \mathcal{M}$, where $\mathcal{M} = \mathcal{M}(M, x^{(i)})$ is a matrix of size $(i+r)n_\omega \times ((r_{\beta_q} n_{b_{q-1}} + \cdots + r_{\beta_1} n_0) + in_0 + (r_{\alpha_1} n_0 + \cdots + r_{\alpha_p} n_{a_{p-1}}) + r_{\gamma_2} n_{c_1})$, $r = \operatorname{rk}(d)$, $1 \le r \le 6$. On the other hand, obviously

$$(i+r)n_\omega \le (2r + r_{\gamma_2} + i)n$$

and

$$(r_{\beta_q} n_{b_{q-1}} + \cdots + r_{\beta_1} n_0) + in_0 + (r_{\alpha_1} n_0 + \cdots + r_{\alpha_p} n_{a_{p-1}}) + r_{\gamma_2} n_{c_1} \le (2r + r_{\gamma_2} + i)n.$$

Recall also that the complexity of standard Gaussian row elimination for an $m \times m$-matrix (equivalently, determining the corank) is just $\mathcal{O}(m^3)$. Consequently, once we know the matrix $\mathcal{M}$ (in the sense of concrete values for all entries), the complexity of computing the integer $h(M)_{x^{(i)}}$ by applying the function `compute`, and hence of computing $m(M)_{x^{(i)}}$, in the step corresponding to $x^{(i)}$, is $\mathcal{O}((in)^3)$, since the integer $2r + r_{\gamma_2}$ depends neither on n nor on i.

The integers $\eta, s, \nu_{\Lambda_{p,q,2}}$ are constant, so to estimate the pessimistic complexity we can assume that $t = \theta n$ for some constant integer $\theta \ge 0$. Therefore, the complexity of determining all $m(M)_x$, $x \in \boldsymbol{P}(d)_n = \{x^i\}_{i=1}^t$, by Algorithm 6.2 is

$$\mathcal{O}(1^3 n^3 + 2^3 n^3 + \cdots + (\theta n)^3 n^3) = \mathcal{O}(n^3(1^3 + 2^3 + \cdots + (\theta n)^3)) = \mathcal{O}(n^7).$$

Note that the process of forming the matrices $\mathcal{M}(M, x)$, $x \in \boldsymbol{P}(d)_n = \{x^i\}_{i=1}^t$, does not affect this estimation, since the complexity of total computations executed by `form` is $\mathcal{O}(n^3)$ (cf. 6.2, the introductory comment).

Since $\boldsymbol{P}_0 = \bigcup_{d \in \boldsymbol{L}} \boldsymbol{P}(d)$ and $\boldsymbol{L}$ is finite, the pessimistic complexity of Algorithm 6.2 is also $\mathcal{O}(n^7)$. ∎

We show in 6.5 that this complexity can be reduced to $\mathcal{O}(n^4)$ and in this way we complete the proof of Theorem 2.2(c).

6.3. Now we describe the "local version" of the algorithm above which determines the multiplicity for a fixed, single postprojective root. In the algorithm we use the conventions established for Algorithm 6.2.

ALGORITHM (for a given $x \in \boldsymbol{P}$, computing the multiplicity $m(M)_x$ for a module M over a fixed domestic canonical algebra Λ). Fix a pair (p, q) of integers such that the canonical algebra $\Lambda = \Lambda_{p,q,2}$ is domestic.

Input: A finite-dimensional Λ-module M given by the triple (A, B, C) and a vector $x \in \boldsymbol{P}$.

Output: The multiplicity $m(M)_x$.

(0) *Preparation*: loading the following collection of parameters for Λ described in Theorem 2.4 (it can be computed using Algorithm 6.1 for the pair (p,q)):

(i) r; $\nu_j, \kappa_j, [s]_j$ for $1 \le j \le r$; $l(i), j(i)$, for $1 \le i \le s$, where $s = p+q+1$; $y_{l,j}$ for $(l,j) \in \mathbb{Z}_{\nu_j} \times [r]$;

(ii) $\varrho_j(l,i)$, for $j \in [r]$, $(l,i) \in \mathbb{Z}_{\nu_j} \times [s]_j$; $x(n,i)$, for $i \in [s]$, $n \in \mathbb{Z}_{\nu_{j(i)}}$.

(1) *Determining the triple* $(j,l,i) \in [r] \times \mathbb{Z}_{\nu_j} \times [s]_j$ *as in Theorem 2.4(d)*:

```
set j := 0; found := false;
while not found do {
  set j := j + 1; l := 0;
  while l < ν_j and not found do
    if x̄ = y_{l,j} then set found := true;
    else set l := l + 1;
}
set i := first in [s]_j; found := false;
while not found do
  if rem_{κ_j}(ϱ_j(l,i)) = rem_{κ_j}(x_0) then set found := true;
  else set i := next in [s]_j;
```

(2) *Finding out if x lies in $\boldsymbol{P}^0$ or in $\boldsymbol{P}'$ and determining the "appropriate coordinates" of the vectors from the set $^{-}x \cup \{\tau x\}$ by applying parts* (c), (d) *of Theorem 2.4*:

```
if x_0 < ϱ_j(l,i) then
  read τx and ⁻x from 7.1.I(P)_{Λ_{p,q,2}};
else {
  if l ≥ l(i) then set n := l − l(i) + (x_0 − ϱ_j(l,i))ν_j/κ_j;
  else set n := l − l(i) + ν_j + (x_0 − ϱ_j(l,i))ν_j/κ_j;
  if n = 0 then read τx and ⁻x from 7.1.I(P)_{Λ_{p,q,2}};
  else {
    if n − 1 < ν_{j(i)} then set τx := x(n − 1, i);
      else set τx := x(rem_{ν_{j(i)}}(n − 1), i) + quo_{ν_{j(i)}}(n − 1)κ_{j(i)} 𝟙;
    set ⁻x := ∅
    for each (−n', i') ∈ ⁻(−n, i) do
      if n' < ν_{j(i')} then set ⁻x := ⁻x ∪ {x(n', i')};
        else set ⁻x := ⁻x ∪ {x(rem_{ν_{j(i')}}(n'), i') + quo_{ν_{j(i')}}(n')κ_{j(i')} 𝟙};
  }
}
```

(3) *Determining the multiplicity* $m(M)_x$:

```
set m(M)_x := h(M)_x + h(M)_{τx} − Σ_{y∈⁻x} h(M)_y; ■
```

6.4. Now we show how to improve the efficiency of Algorithm 6.2 and to decrease the complexity of computations from $\mathcal{O}(n^7)$ to $\mathcal{O}(n^4)$. We start by a general construction.

Let $r, \theta \in \mathbb{N}$ be a fixed pair of positive integers. For any positive $n \in \mathbb{N}$ and the triple

$$S = (D_1, \dots, D_t; E_1, \dots, E_t; F_1, \dots, F_t)$$

consisting of sequences of matrices $D_i, E_i \in \mathbb{M}_{n\times n}(k)$ and $F_i \in \mathbb{M}_{n\times rn}(k)$, respectively, $1 \leq t \leq \theta n + r$, we define a family $K(S) = \{K_i\}_{i\in[t]}$ of matrices $K_i = K_i(S) \in \mathbb{M}_{in\times(\theta n+3r)n}(k)$ by setting

$$K_i = \begin{bmatrix} D_1 & & & E_1 & & & F_1 \\ & \ddots & & & \ddots & & \vdots \\ & & D_i & & & E_i & F_i \end{bmatrix},$$

where the jth block row of K_i has the shape

$$[0_{(j-1)n} \,|\, D_j \,|\, 0_{(r-1)n} \,|\, E_j \,|\, 0 \,|\, F_j]$$

for $j \in [i]$. (For any $s \in \mathbb{N}$, 0_s denotes the zero matrix in $\mathbb{M}_{n\times s}(k)$.)

LEMMA. *Let S be an arbitrary triple as above, for fixed r, θ, and $L(S) = \{L_i\}_{i\in[t]}$ be the family of matrices $L_i = L_i(S)$, with $(\theta n + 3r)n$ columns, defined inductively as follows:*

$$L_1 = \widehat{J}_1,$$

where $J_1 = [D_1 \,|\, 0_{(r-1)n} \,|\, E_1 \,|\, 0 \,|\, F_1]$, and

$$L_{i+1} = \begin{bmatrix} L_{11}^{(i)} & L_{12}^{(i)} \\ 0 & \widehat{J}_{i+1} \end{bmatrix}$$

for $i < t$, where

$$L_i = \begin{bmatrix} L_{11}^{(i)} & L_{12}^{(i)} \\ L_{21}^{(i)} & L_{22}^{(i)} \end{bmatrix},$$

with maximal zero block $L_{21}^{(i)}$ containing $i \cdot n$ columns, and

$$J_{i+1} = \begin{bmatrix} L_{22}^{(i)} \\ D_{i+1} \,|\, 0_{(r-1)n} \,|\, E_{i+1} \,|\, 0 \,|\, F_{i+1} \end{bmatrix}.$$

Then

$$\mathrm{r}(K_j) = \mathrm{r}(L_j) = \mathrm{r}(L_{1,1}^{(j-1)}) + \mathrm{r}(\widehat{J}_j)$$

and all matrices J_j, $j \in [t]$, have at most $(2r + 1)n$ rows and $(2r + 1)n$ nonzero columns, where $K(S) = \{K_j\}_{j\in[t]}$.

Proof. The first assertion follows immediately by the construction of the families $K(S)$ and $L(S)$. To prove the second, note that the number of rows in $L_{22}^{(j-1)}$ coincides with the number of leading coefficients in $L_{22}^{(j-1)}$, so it is bounded by the number of columns of $L_{22}^{(j-1)}$. The last integer is not greater than $2rn$, since the columns with numbers $(j-1)n+1,\ldots,jn$ in $L_{22}^{(j-1)}$ and clearly those with numbers $jn+1,\ldots,(2r+n)n$ in J_j are zero columns. Now the claim follows immediately from the definition of the matrix J_j. ∎

Remark.

(a) Letting $\mathrm{K}_0 = L_0$ be equal to the trivial matrix in $\mathbb{M}_{0\times(\theta n+3r)n}(k)$, we can extend the inductive definition of $L(S)$ to $\{L_i\}_{i\in\{0,\ldots,t\}}$, starting from L_0. Clearly, the assertion of the lemma remains valid.

(b) The lemma remains valid if in the sequences defining the triple S we allow also rectangle matrices of sizes suitable for the construction of the family $K(S)$ and bounded by $n\times n$ (resp. $n\times rn$).

(c) Let S be a triple as above. Then, for any $1\le s\le t$, we have
$$L(S_{|s}) = \{L_1(S),\ldots,L_s(S)\},$$
where
$$S_{|s} = (D_1,\ldots,D_s;E_1,\ldots,E_s;F_1,\ldots,F_s).$$

6.5. *Proof of Theorem 2.2(c).* To show assertion (c) we apply the general idea of the proof of Lemma 6.2 and keep the notation established there. Clearly, it suffices to modify the algorithm computing, for a Λ-module $M = (A,B,C)$, the integers $h(M)_x = \operatorname{cor}\mathcal{M}(M,x)$, $x\in \boldsymbol{P}(d)$ for a fixed $d\in \boldsymbol{L} = \overline{\boldsymbol{P}}$, in such a way that the new one already has complexity $\mathcal{O}(n^4)$, where $n=\dim_k M$. In fact, we have to compute the ranks $\mathrm{r}_l = \mathrm{r}(\mathcal{M}(M,x))$ for $l=0,\ldots,\theta n$, where $x=d[l]$, i.e. $x_0=l$, and θ is as in the proof of Lemma 6.2.

We assume first that either $\operatorname{char} k\neq 2$, or $\operatorname{char} k = 2$ and $\operatorname{rk}(d)\le 5$. Then, for any $l=0,\ldots,\theta n$, we set
$$S_l = (r_{\beta_q}\times B_{q,q},\ldots,r_{\beta_1}\times B_{q,1},\, l\times\bar{B};\, l\times\bar{A},\, r_{\alpha_1}\times A_{p,1},\ldots,r_{\alpha_p}\times A_{p,p};\, F_1,\ldots,F_{l+r}),$$
where $r=\operatorname{rk}(d)=d_\omega$, $m\times N$ denotes the sequence consisting of m copies of N for any $m\in\mathbb{N}$ and matrix N, and the matrices $F_1,\ldots,F_{l+r}\in\mathbb{M}_{n\times r_{\gamma_2}}(k)$ are determined by the formula
$$\begin{bmatrix} F_1\\ \vdots\\ F_{\theta n+r}\end{bmatrix} = \begin{bmatrix} -I_{r_{\gamma_2}}\\ U(d)\end{bmatrix}_{(\infty_{|\theta n+r})}\otimes C_2.$$
By Lemma 6.4, we have
$$\mathrm{r}_l = \mathrm{r}(L_{l+r}(S_l))$$

for $l = 0, \ldots, \theta n$, since $\mathcal{M}(M, x) = K_{l+r}(S_l)$. Observe that $(S_l)_{|l} = S_{|l}$ for all $l = 0, \ldots, \theta n$, where

$$S = (r_{\beta_q} \times B_{q,q}, \ldots, r_{\beta_1} \times B_{q,1},\ \theta n \times \overline{B};\ (\theta n + r) \times \overline{A};\ F_1, \ldots, F_{\theta n+r})$$

Hence,

$$L_l(S_l) = L_l(S)$$

for every l (see Remark 6.4(b)). Therefore, by Lemma 6.4, to compute the integer r_l, we have to execute Gaussian row elimination on the matrices $J_{l+1}(S_l), \ldots, J_{l+r}(S_l)$, provided we know the matrix $L_l(S)$. By analogous arguments, to compute all matrices $L_l(S)$, $l = 0, \ldots, \theta n$, we have to execute only the Gaussian row elimination of $J_l(S_l)$ in each step. Thus, in computation of all integers r_l, $l = 0, \ldots, \theta n$, we execute $(\theta n + 1)r + \theta n$ eliminations of matrices with row and column dimensions bounded by $(2r+1)n$. Consequently, the total number of arithmetic operations is bounded by $((r + 1)\theta n + r)(2r + 1)^3 n^3$, so the pessimistic complexity is $\mathcal{O}(n^4)$. Note that just as in the proof of Lemma 6.2, the process of forming the sequences S_l, $l = 0, \ldots, \theta n$, does not affect this estimation.

In the remaining case, $\operatorname{char} k = 2$ and $\operatorname{rk}(d) = 6$, the algorithm computing r_l, $l = 0, \ldots, \theta n$, can be constructed similarly, although in a slightly more complicated way. Nevertheless, the difficulties have only a technical character, and therefore we do not give any extra details.

In this way the proof of Theorem 2.2 is complete. ∎

Remark.

(a) The problem of determining the restricted multiplicity vector $m(M)_{|\boldsymbol{Q}}$ (resp. multiplicity $m(M)_x$, for a single $x \in \boldsymbol{Q}$) for a module M over a fixed domestic canonical algebra Λ is equivalent to that of determining $m(D(M))_{|\boldsymbol{P}(\Lambda^{\mathrm{op}})}$ (resp. $m(D(M))_x$) for x regarded as an element of $\boldsymbol{P}(\Lambda^{\mathrm{op}})$ for the opposite algebra Λ^{op}, which is again a domestic canonical algebra of the same type. In fact, to compute m_x, $x \in \boldsymbol{Q}$, we have to apply the formula dual to $(*)$ in the Introduction and use the dimensions $h'_x = \dim_k \operatorname{Hom}_\Lambda(X_x, M)$, $x \in \boldsymbol{Q}$ (see [9, 2.3]).

(b) Decreasing the pessimistic complexity of Algorithm 6.2 as above, one should also take into account some "negative effects". Namely, in a possible implementation of the improved version of the algorithm, at each step of computations we have to store much more information than the in algorithm without optimizations as above (see Remark 6.2(b)).

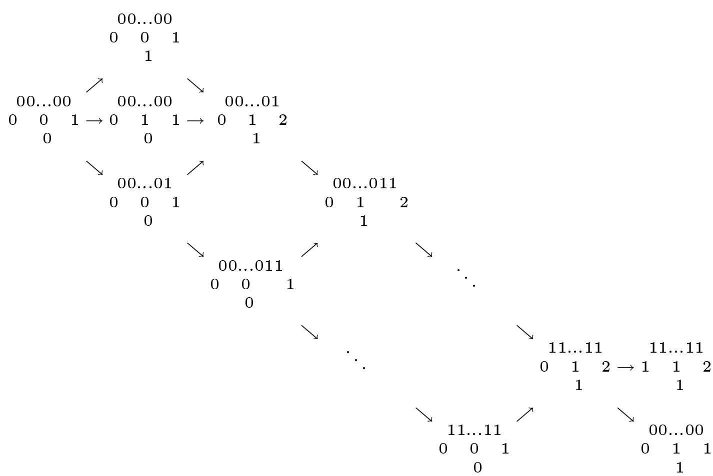
A final comment. The algorithmic method of determining multiplicity vectors for modules, proposed in [9], should be possible to adapt for a larger class of tame algebras with an appropriate shape of the Auslander–Reiten quiver; in particular, for all concealed algebras of Euclidean type. We expect to prove that in this situation there exist algorithms with pessimistic complexities similar to those considered here. We strongly believe that to achieve this we do not need a precise description of the canonical forms for all indecomposables. We have already obtained some results in this direction. They will be contained in a forthcoming publication.

7. TABLES

In this section we give the finite sets of discrete data used in the algorithms from Section 6. They can be easily computed by applying the definitions and standard techniques.

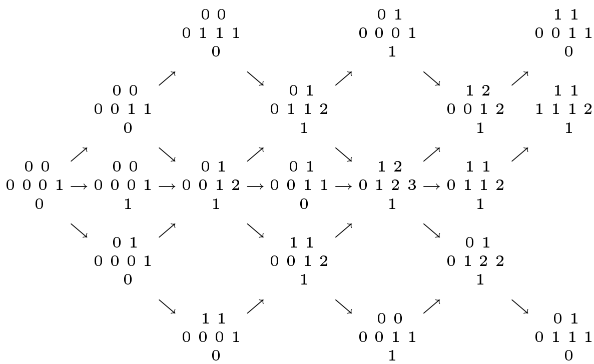
7.1. We give, using the standard graphic convention, the list of initial parts $I(\boldsymbol{P}) = I(\boldsymbol{P})_\Lambda$ of postprojective components, more precisely of the translation quivers $\boldsymbol{P} = \boldsymbol{P}(\Lambda)$, for all domestic canonical algebras Λ. Each $I(\boldsymbol{P})$ is a full subquiver of $\boldsymbol{P}$, formed by the sets $\boldsymbol{P}_0^0 \cup \Sigma_0$, where Σ is a suitable section in $\boldsymbol{P}$ and $\boldsymbol{P}^0 = \boldsymbol{P}^0(\Sigma)$. They are obtained by applying the standard "knitting" technique (cf. [23]). Below the quivers $I(\boldsymbol{P})$ we fix the notation which is used in the algorithms. We list the names for all consecutive vertices in $I(\boldsymbol{P})_0$ in the form of a "scheme" reflecting the shape of $I(\boldsymbol{P})$. The vertices $z(i)$ constitute the part $\boldsymbol{P}^0$, the vertices $x(i)$ belong to the section Σ and form there an admissible sequence of sources. The enumeration of vertices in $I(\boldsymbol{P})$ is crucial for the definition of the order relation $\prec$ in $\boldsymbol{P}$ (see 6.2).

(a) $I(\boldsymbol{P})_{\Lambda_{p,2,2}}$:



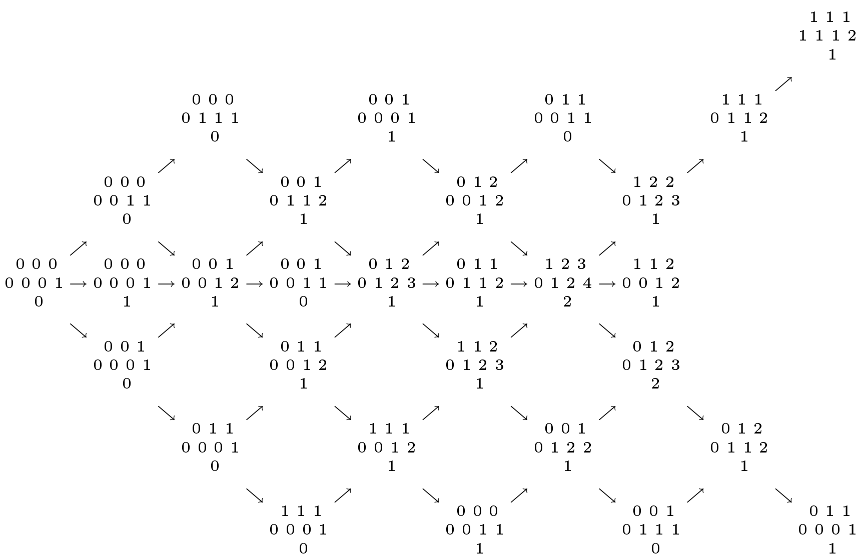
$$\begin{array}{ccccccc}
 & x(1) & & & & & \\
z(1) & x(2) & x(3) & & & & \\
 & z(2) & & x(4) & & & \\
 & & z(3) & & \ddots & & \\
 & & & \ddots & & x(p{+}1) & x(p{+}2) \\
 & & & & z(p) & & x(p{+}3)
\end{array}$$

(b) $I(\boldsymbol{P})_{\Lambda_{3,3,2}}$:



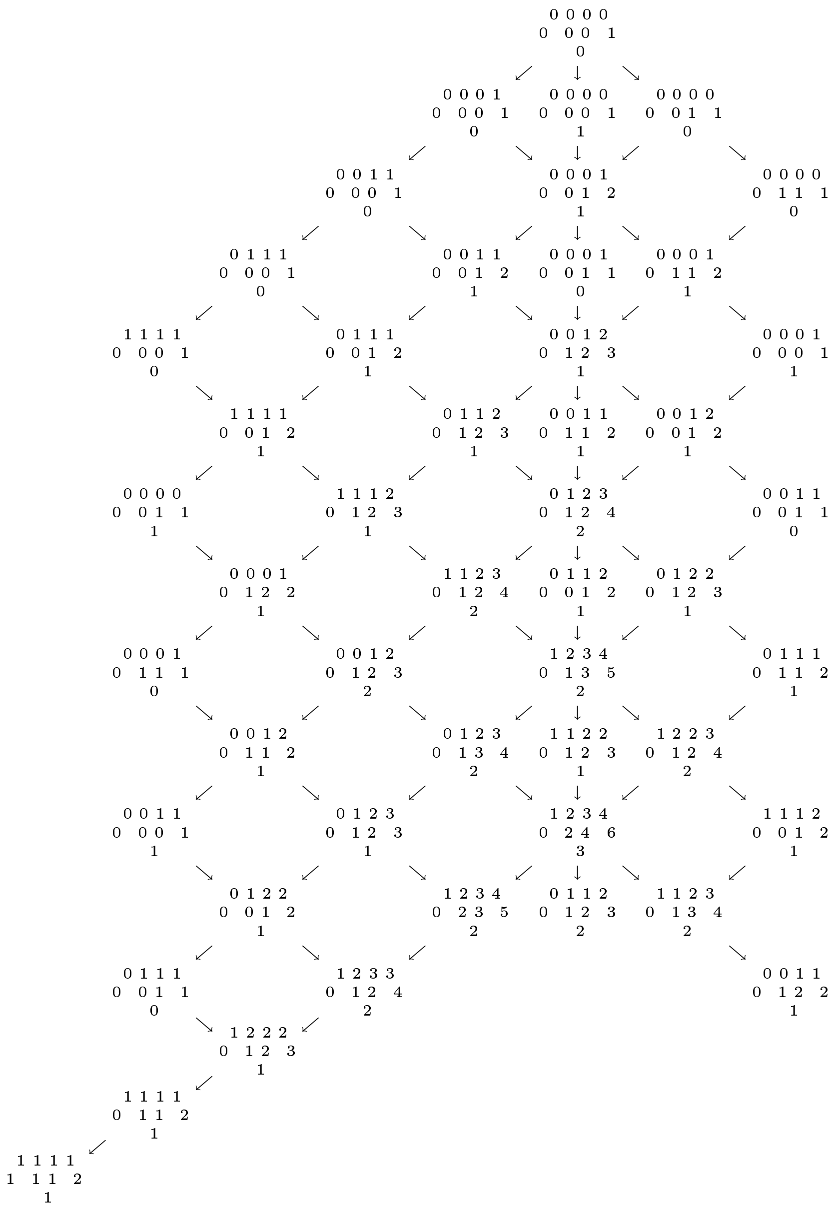
$$\begin{array}{ccccccc}
 & & z(5) & & z(11) & & x(5) \\
 & z(2) & & z(8) & & x(4) & x(3) \\
z(1) & z(3) & z(6) & z(9) & x(1) & x(2) & \\
 & z(4) & & z(10) & & x(6) & \\
 & & z(7) & & z(12) & & x(7)
\end{array}$$

(c) $I(\boldsymbol{P})_{\Lambda_{4,3,2}}$:



$$\begin{array}{cccccccccc}
 & & & & & & & & & x(5) \\
 & & z(5) & & z(12) & & z(19) & & x(4) & \\
 & z(2) & & z(8) & & z(15) & & x(3) & & \\
z(1) & z(3) & z(6) & z(9) & z(13) & z(16) & x(1) & x(2) & & \\
 & z(4) & & z(10) & & z(17) & & x(6) & & \\
 & & z(7) & & z(14) & & z(20) & & x(7) & \\
 & & & z(11) & & z(18) & & z(21) & & x(8)
\end{array}$$

(d) $I(\boldsymbol{P})_{\Lambda_{5,3,2}}$:



$$
\begin{array}{cccccccc}
 & & & & & z(1) & & \\
 & & & & z(4) & z(3) & z(2) & \\
 & & & z(7) & & z(6) & & z(5) \\
 & & z(11) & & z(10) & z(9) & z(8) & \\
 & z(15) & & z(14) & & z(13) & & z(12) \\
 & & z(19) & & z(18) & z(17) & z(16) & \\
 & z(23) & & z(22) & & z(21) & & z(20) \\
 & & z(27) & & z(26) & z(25) & z(24) & \\
 & z(31) & & z(30) & & z(29) & & z(28) \\
 & & z(35) & & z(34) & z(33) & z(32) & \\
 & z(38) & & z(37) & & x(1) & & z(36) \\
 & & z(39) & & x(5) & x(2) & x(3) & \\
 & z(40) & & x(6) & & & & x(4) \\
 & & x(7) & & & & & \\
 & x(8) & & & & & & \\
x(9) & & & & & & &
\end{array}
$$

7.2. Finally, we list the inverses ϕ_{Λ}^{-1} of the Coxeter transformations ϕ_{Λ} : $\mathrm{K}_0(\Lambda) \to \mathrm{K}_0(\Lambda)$ for the domestic canonical algebras $\Lambda = \Lambda_{p,q,2}$. They are presented as matrices from $\mathbb{M}_{s\times s}(\mathbb{Z})$, under the identification of $\mathrm{K}_0(\Lambda) = \mathbb{Z}^{(Q_{p,q,2})_0}$ with $\mathbb{Z}^s$ via the mapping

$$n = (n_v) \mapsto [n_\omega, n_{a_{p-1}}, \ldots, n_{a_1}, n_{b_{q-1}}, \ldots, n_{b_1} n_{c_1} n_0]^t,$$

where $s = |(Q_{p,q,2})_0| = p + q + 1$. The following matrices are computed by applying the formula given in 2.4.

$$
\phi^{-1}_{\Lambda_{p,2,2}} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \\
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & -1 \\
-1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -1
\end{bmatrix},
$$

$$
\phi^{-1}_{\Lambda_{3,3,2}} = \begin{bmatrix}
0 & 0 & 1 & 0 & 1 & 1 & -2 \\
0 & 0 & 0 & 0 & 1 & 1 & -1 \\
-1 & 1 & 0 & 0 & 1 & 1 & -1 \\
0 & 0 & 1 & 0 & 0 & 1 & -1 \\
-1 & 0 & 1 & 1 & 0 & 1 & -1 \\
0 & 0 & 1 & 0 & 1 & 0 & -1 \\
-1 & 0 & 1 & 0 & 1 & 1 & -1
\end{bmatrix},
$$

$$\phi_{\Lambda_{4,3,2}}^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \\ -1 & 1 & 0 & 0 & 0 & 1 & 1 & -1 \\ -1 & 0 & 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 & 1 & 1 & -1 \end{bmatrix},$$

$$\phi_{\Lambda_{5,3,2}}^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & -1 \\ -1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & -1 \\ -1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 \\ -1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & -1 \end{bmatrix}.$$

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Received 4 January 2007;
revised 3 October 2007 (4860)