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ON A SEPARATION OF ORBITS IN THE MODULE VARIETY FOR DOMESTIC CANONICAL ALGEBRAS

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Abstract. Given a pair M, M' of finite-dimensional modules over a domestic canonical algebra Λ, we give a fully verifiable criterion, in terms of a finite set of simple linear algebra invariants, deciding if M and M' lie in the same orbit in the module variety, or equivalently, if M and M' are isomorphic.

Introduction. The problem of deciding whether or not two points in an algebraic variety X equipped with a regular action of an algebraic group G lie in the same G-orbit was intensively studied as a basic elementary question of geometric invariant theory. In case X is the affine variety

$$X = \operatorname{mod}_\Lambda(\underline{n}) \subseteq \prod_{\delta \in Q_1} \mathbb{M}_{t(\delta)\times s(\delta)}(k)$$

of Λ-modules with a fixed dimension vector $\underline{n} = (n_v)_{v\in Q_0}$ and G is the group

$$G = G(\underline{n}) := \prod_{v \in Q_0} \mathrm{Gl}_{n_v}(k)$$

associated with a finite-dimensional k-algebra $\Lambda = kQ/I$ defined by a finite quiver $Q = (Q_0, Q_1)$ and an admissible ideal I in the path algebra kQ (see [3]), this leads to the following question of purely algebraic form:

“When given Λ-modules $M, M' \in \operatorname{mod}_\Lambda(\underline{n})$ are isomorphic?”

We call it the *isomorphism question for the pair* (M, M').

There exists a rather theoretical criterion, due to Auslander [1, 2], that answers this question for modules over any finite-dimensional algebra Λ. It says that given objects M, M' in the category $\operatorname{mod}\Lambda$ of finite-dimensional Λ-modules, $M \cong M'$ if and only if $\dim_k \operatorname{Hom}_\Lambda(M, X) = \dim_k \operatorname{Hom}_\Lambda(M', X)$ for all indecomposable modules X in $\operatorname{mod}\Lambda$. If $\Lambda = kQ/I$ is a representation-finite algebra then this result can lead to an algorithmic procedure, provided

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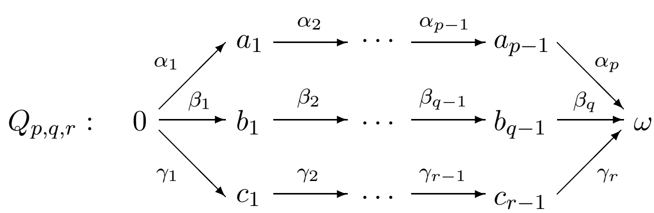
a complete classification of indecomposable Λ-modules together with a precise description of their matrix forms is known. In case Λ is representation-infinite, the result seems to be useless, in the sense that it fails to provide an effective method.

Observe that Λ-modules M and M' are isomorphic if and only if the multiplicity vectors $m(M)$ and $m(M')$ with respect to a fixed classifying set $\boldsymbol{X} = (\boldsymbol{X}, \varepsilon)$ (of invariants for the indecomposable Λ-modules) are equal (see [5]). Therefore answering the isomorphism question for M, M' can be replaced by determining $m(M)$ and $m(M')$, which is however much more difficult than the original task. Observe that just as for the polynomial algebra $k[t]$ (problem of determining eigenvalues), one cannot expect the existence of a fully verifiable procedure computing $m(M)$ in the case of representation-infinite algebras Λ. Nevertheless, a general method of attacking this problem was presented in [4] and then applied to construct algorithms determining $m(M)$ (up to finding roots of polynomials in $k[t]$) for all domestic canonical algebras (see [4, 5]).

This paper should be treated as an addendum to [5]. Its main aim is to present (applying the results of [5]) a complete, finite and fully verifiable criterion that allows us to answer the isomorphism question for any fixed pair (M, M') of modules over a domestic canonical algebra Λ, given as points in the variety $\mathrm{mod}_\Lambda(\underline{n})$; equivalently, to decide if M, M' lie in the same $G(\underline{n})$-orbit in $\mathrm{mod}_\Lambda(\underline{n})$ (see Theorem 1.5).

1. Preliminaries and the main theorem. We use well known and commonly used definitions and notation, as in [5]. We only recall some of them.

1.1. We consider finite-dimensional domestic canonical k-algebras $\Lambda_{p,q,r} = kQ_{p,q,r}/I_{p,q,r}$, $p, q, r \geq 1$, with



and $I_{p,q,r} = \langle \alpha + \beta - \gamma \rangle$, $\alpha = \alpha_1 \cdots \alpha_p$, $\beta = \beta_1 \cdots \beta_q$, $\gamma = \gamma_1 \cdots \gamma_r$, for any triple

$$(p, q, r) \in \mathcal{D} := \{(p, q, 1), p, q \geq 1;\ (p, 2, 2), p \geq 2;\ (3, 3, 2);\ (4, 3, 2);\ (5, 3, 2)\}.$$

Let $\Lambda = \Lambda_{p,q,r}$ for some $(p,q,r) \in \mathcal{D}$. Then by finite-dimensional Λ-modules we always mean matrix representations M of Λ, which can be identified with points

$$(M_\delta)_{\delta \in (Q_{p,q,r})_1} \in \operatorname{mod}_\Lambda(\underline{n})$$

of the module variety, where $\underline{n} = \underline{\dim}(M)$ (see [5, 1.4]). For simplicity, we use the notation $M = (A,B,C)$, where $A = (A_i)_{i\in[p]}$, $B = (B_j)_{i\in[q]}$, $C = (C_l)_{i\in[r]}$ and $A_i = M_{\alpha_i}$, $B_j = M_{\beta_j}$, $C_l = M_{\gamma_l}$ for $i \in [p]$, $j \in [q]$, $l \in [r]$, respectively, and $[s] := \{1,\dots,s\}$ for $s \in \mathbb{N}$. Clearly, we have $\overline{A} + \overline{B} = \overline{C}$, where $\overline{A} = A_p \dots A_1$, and similarly for $\overline{B}$ and $\overline{C}$.

Recall from [5] that if $r = 1$ then Λ is canonically isomorphic to a hereditary algebra $\Lambda_{p,q}$ and each Λ-module $M = (A,B,C)$ is uniquely determined by the pair (A,B). From now on, we identify the algebras $\Lambda_{p,q,1}$ and $\Lambda_{p,q}$, as well their module categories, via the mapping $(A,B,C) \mapsto (A,B)$.

Note that each algebra $\Lambda = \Lambda_{p,q,r}$ is canonically isomorphic to its opposite Λ^{op}; the isomorphism is given by the mapping $0 \mapsto \omega$, $a_1 \mapsto a_{p-1}$, $b_1 \mapsto b_{q-1}$, and so on. Hence, we get an equivalence $\operatorname{mod}\Lambda^{\mathrm{op}} \simeq \operatorname{mod}\Lambda$, and the standard duality $D = \operatorname{Hom}(-,k) : \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda^{\mathrm{op}}$ yields a selfduality

$$D' : \operatorname{mod}\Lambda \xrightarrow{D} \operatorname{mod}\Lambda^{\mathrm{op}} \simeq \operatorname{mod}\Lambda.$$

For any M in $\operatorname{mod}\Lambda$, given by $(M_\delta)_{\delta\in(Q_{p,q,r})_1} \in \operatorname{mod}_\Lambda(\underline{n})$, we denote by M^* the Λ-module in $\operatorname{mod}\Lambda$ given by $(M_\delta^{\mathrm{tr}})_{\delta\in(Q^{\mathrm{op}}_{p,q,r})_1} \in \operatorname{mod}_{\Lambda^{\mathrm{op}}}(\underline{n})$. Clearly, M^* is naturally isomorphic to $D'(M)$.

1.2. Let k be an algebraically closed field and $\mathbb{P}^1(k)$ the projective line over k. We identify points of $\mathbb{P}^1(k)$ with elements of $k \cup \{\infty\}$ via the standard mapping $(\lambda : 1) \mapsto \lambda$ for $\lambda \in k$, and $(1:0) \mapsto \infty$. For any homogeneous polynomial $f = f(t,u) \in k[t,u]$, the zero set of f is understood to be $V(f) = \{(x:y) \in \mathbb{P}^1(k) : f(x,y) = 0\}$. As usual, $V(f) = \{x \in k : f(x) = 0\}$ for $f = f(t) \in k[t]$.

Let $\underline{w} = (w_1,\dots,w_l) \in (\mathbb{N}\setminus\{0\})^l$, $\underline{\lambda} = (\lambda_1,\dots,\lambda_l) \in (\mathbb{P}^1(k))^l$ be a pair of sequences. Then we denote by $\mathbb{X} = \mathbb{X}(\underline{w},\underline{\lambda})$ a weighted projective line of type $(\underline{w},\underline{\lambda})$ (see [7] for a precise definition), and we view $\mathbb{X}$ as the classical projective line $\mathbb{P}^1(k)$ equipped with a function $w : \mathbb{P}^1(k) \to \mathbb{N}$, defined as follows:

$$w(\lambda) = \begin{cases} w_i & \text{if } \lambda = \lambda_i \text{ for } 1 \le i \le l, \\ 1 & \text{if } \lambda \in \mathbb{P}^1(k) \setminus \{\lambda_1,\dots,\lambda_l\}. \end{cases}$$

We set

$$\operatorname{exc}(\mathbb{X}) = \{\lambda \in \mathbb{P}^1(k) : w(\lambda) > 1\}, \quad \operatorname{ord}(\mathbb{X}) = \{\lambda \in \mathbb{P}^1(k) : w(\lambda) = 1\}.$$

The elements of these sets are called respectively the *exceptional* and *ordinary* points of $\mathbb{X}$.

It is well known that with any domestic canonical algebra $\Lambda = \Lambda_{p,q,r}$ we can associate the weighted projective line $\mathbb{X}(\Lambda) = \mathbb{X}(\underline{w}, \underline{\lambda})$, where $(\underline{w}, \underline{\lambda}) = ((p,q,r),(0,\infty,1))$, in such way that the collection $(n_\lambda)_{\lambda\in\mathbb{P}^1(k)}$ of ranks for the 1-parameter family $\mathcal{T}^\Lambda = (\mathcal{T}^\Lambda_\lambda)_{\lambda\in\mathbb{P}^1(k)}$ of stable tubes describing the structure of the category $\mathcal{R}$ of all regular Λ-modules, satisfies

$$(n_\lambda)_{\lambda\in\mathbb{P}^1(k)} = (w(\lambda))_{\lambda\in\mathbb{P}^1(k)}.$$

Moreover, for each tube $\mathcal{T}^\Lambda_\lambda$, $\lambda \in \mathbb{P}^1(k)$, we have fixed in [5, 2.1 and 3.3] a system of tubular coordinates given by a precise selection of one quasi-simple Λ-module in $\mathcal{T}^\Lambda_\lambda$. This leads to specification of the classifying set

$$\boldsymbol{T} = \bigsqcup_{\lambda\in\mathbb{P}^1(k)} \boldsymbol{T}_\lambda$$

for regular indecomposable Λ-modules, where

$$\boldsymbol{T}_\lambda = \{[\lambda, s, l] : s \in \mathbb{Z}_{w(\lambda)},\ l \geq 1\}$$

for $\lambda \in \mathbb{P}^1(k)$. (Note that if $w(\lambda) = 1$ then $\mathbb{Z}_{w(\lambda)} = \{0\}$, the tube $\mathcal{T}^\Lambda_\lambda$ is homogeneous and each triple $[\lambda, s, l] \in \boldsymbol{T}_\lambda$ is in fact a pair $[\lambda, l]$.) Since postprojective and preinjective indecomposable Λ-modules are fully described by their dimension vector sets $\boldsymbol{P}$ and $\boldsymbol{Q}$, respectively, the set

$$\boldsymbol{X}(\Lambda) := \boldsymbol{P} \sqcup \boldsymbol{T} \sqcup \boldsymbol{Q}$$

is a classifying set of invariants for indecomposable Λ-modules (see [5, 1.6, 2.1]).

Recall that given a Λ-module M, we set $h_x = \dim_k \operatorname{Hom}_\Lambda(M, X_x)$, where X_x is any module from the isomorphism class $\varepsilon(x)$, for $x \in \boldsymbol{X}(\Lambda)$.

We assume that the set $\boldsymbol{X} = \boldsymbol{X}(\Lambda)$ carries the structure of a translation quiver transported from the Auslander–Reiten quiver Γ_Λ of Λ. Moreover, each section Σ in the connected component $\boldsymbol{P} = \boldsymbol{P}(\Lambda)$ induces a splitting $\boldsymbol{P} = \boldsymbol{P}^0 \cup \boldsymbol{P}'$, where $\boldsymbol{P}^0 = \boldsymbol{P}^0(\Sigma)$ is finite and $\boldsymbol{P}' = \boldsymbol{P}'(\Sigma) = -\mathbb{N}\Sigma$ (see [5, 2.4] for $r = 2$). Note that if $r = 1$, we can take for Σ the full subquiver formed by the dimension vectors of all indecomposable projective Λ-modules and in the splitting above the part $\boldsymbol{P}^0$ is empty. The "consecutive" vertices of $\boldsymbol{P}'$ are denoted by $x(n,i)$ (see [5, 5.1] for details). Following [5, 7.1], for each Λ there exists a section Σ such that the vertex set $\boldsymbol{P}_0$ admits some ordering $\prec$ with nice properties with respect to the splitting above.

From now on we assume that $\Sigma = \Sigma(\Lambda)$ is as in [5, 7.1] if $r = 2$, and is the section mentioned above if $r = 1$.

1.3. Following [6], given a matrix $\mathcal{A} \in \mathbb{M}_{x\times y}(k[t_1,\dots,t_l])$ and an integer $j \le r = \mathrm{r}(\mathcal{A})$, we denote by $D_j = D_j(\mathcal{A})$ the polynomial in $k[t_1,\dots,t_l]$ which is the greatest common divisor of all $j\times j$ minors of $\mathcal{A}$, where $\mathrm{r}(\mathcal{A})$ denotes the rank of $\mathcal{A}$ over the quotient field $k(t_1,\dots,t_l)$. Note that the polynomials D_j are determined uniquely up to scalars from $k\setminus\{0\}$, and that $D_{j-1}\mid D_j$ for all $j=1,\dots,r$ (we set $D_0(\mathcal{A})=1$). In case $l=1$, the elements $\mathcal{A}\in\mathbb{M}_{x\times y}(k[t])$ are called simply *t-matrices*. We say that t-matrices $\mathcal{A},\mathcal{A}'\in\mathbb{M}_{x\times y}(k[t])$ are *equivalent* (and write $\mathcal{A}\sim\mathcal{A}'$) if

$$\mathcal{A}' = \mathcal{B}\mathcal{A}\mathcal{C}$$

for some invertible $\mathcal{B}\in\mathbb{M}_{x\times x}(k[t])$ and $\mathcal{C}\in\mathbb{M}_{y\times y}(k[t])$; equivalently, if $\mathcal{A}$ can be transformed to $\mathcal{A}'$ by applying a finite sequence of elementary row and column transformations "over $k[t]$". It is well known that each equivalence class $[\mathcal{A}]_\sim$, $\mathcal{A}\in\mathbb{M}_{x\times y}(k[t])$, contains precisely one t-matrix $\varDelta(\mathcal{A})$ in the so-called canonical diagonal form

$$\begin{bmatrix} E_1 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & & \vdots \\ 0 & \dots & E_r & \dots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{bmatrix}$$

where $E_j\in k[t]$, $j=1,\dots,r$, are nonzero monic polynomials satisfying $E_1\mid E_2,\dots,E_{r-1}\mid E_r$, and all other entries are zero. Moreover, there exists a precise algorithm determining $\varDelta(\mathcal{A})$. On the other hand, one can compute $\varDelta(\mathcal{A})$ directly, by applying the formulas

$$E_j(\mathcal{A}) = \frac{D_j(\mathcal{A})}{D_{j-1}(\mathcal{A})}, \qquad j=1,\dots,r,$$

provided we assume that all $D_j(\mathcal{A})$ are monic polynomials.

1.4. Let $\varLambda=\varLambda_{p,q,r}$ be an arbitrary domestic canonical algebra. Given a finite-dimensional $\varLambda$-module $M=(A,B,C)$, we set

$$\overline{M} = \begin{cases} \mathrm{res}(M) = (\bar{A},\bar{B}) & \text{if } r=1,\\ \varPsi(M) = (\bar{A},-\bar{B}) & \text{if } r=2.\end{cases}$$

Clearly, $\overline{M}$ is a $\varLambda_{1,1}$-module and $\overline{M}=M$ if $p=q=r=1$. Recall that if $r=1$ then we identify $\operatorname{mod}\varLambda$ with the module category for the hereditary algebra $\varLambda_{p,q}$ (see 1.1).

Following [5], for a $\varLambda$-module M, we denote by $\mathrm{rk}_{\mathcal{P}}(M)$ the rank of a maximal postprojective direct summand of M.

Lemma. *Let M be a module over a domestic canonical algebra Λ.*

(a) *If $\Lambda = \Lambda_{1,1}$ and $M = (A, B)$ with $A, B \in \mathbb{M}_{n_\omega \times n_0}(k)$, $n_\omega, n_0 \geq 0$, then $\mathrm{rk}_{\mathcal{P}}(M) = n_\omega - \mathrm{r}(M_P^{i_0}) + \mathrm{r}(M_P^{i_0-1})$ for any $i_0 \geq n_0$, where*

$$M_P^i = \begin{bmatrix} -A & B & 0 & 0 & \cdots & 0 \\ 0 & -A & B & 0 & \cdots & 0 \\ 0 & 0 & -A & B & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -A & B \end{bmatrix} \in \mathbb{M}_{in_\omega \times (i+1)n_0}(k)$$

for $i \geq 0$.

(b) *If $\Lambda = \Lambda_{p,q,r}$ then $\mathrm{rk}_{\mathcal{P}}(M) = \mathrm{rk}_{\mathcal{P}}(\overline{M})$.*

Proof. (a) Note that $\mathrm{rk}(P) = 1$ for any indecomposable postprojective Λ-module P. Therefore $\mathrm{rk}_{\mathcal{P}}(M) = \sum_{i=1}^{i_0} m(M)_{P_i}$, where $i_0 \geq n_0$ is a fixed integer and P_i denotes an indecomposable postprojective Λ-module with $\underline{\dim}\, P_i = [i, i+1]$, for $i \geq 1$. Consequently, by [4, Lemmata 4.2(i), 4.6(i)], we obtain the following equalities:

$$\begin{aligned} \mathrm{rk}_{\mathcal{P}}(M) &= [M, P_1] + ([M, P_2] - 2[M, P_1]) \\ &\quad + \sum_{i=3}^{i_0}([M, P_i] - 2[M, P_{i-1}] + [M, P_{i-2}]) \\ &= [M, P_{i_0}] - [M, P_{i_0-1}] \\ &= i_0 n_\omega - \mathrm{r}(M_P^{i_0}) - ((i_0 - 1)n_\omega - \mathrm{r}(M_P^{i_0-1})) \\ &= n_\omega - \mathrm{r}(M_P^{i_0}) + \mathrm{r}(M_P^{i_0-1}) \end{aligned}$$

and the proof of (a) is complete.

(b) follows immediately from [5, Theorem 2.2]. ■

Given a $\Lambda_{1,1}$-module $M = (A, B)$ with $A, B \in \mathbb{M}_{n_\omega \times n_0}(k)$, $n_\omega, n_0 \geq 0$, we set

$$M(t) = A - tB \in \mathbb{M}_{n_\omega \times n_0}(k[t]), \qquad M(t, u) = uA - tB \in \mathbb{M}_{n_\omega \times n_0}(k[t, u]).$$

Note that $D_j(M(t, u))$ is a homogeneous polynomial in $k[t, u]$ for any $j \leq \mathrm{r}(M(t, u))$.

Definition. Let M be a module over a domestic canonical algebra $\Lambda = \Lambda_{p,q,r}$. Then the polynomial

$$\chi_M = D_j(\overline{M}(t, u)) \in k[t, u]$$

where $j = j(M) := n_\omega - \mathrm{rk}_{\mathcal{P}}(M)$ $(= \mathrm{r}(\overline{M}_P^{n_0}) - \mathrm{r}(\overline{M}_P^{n_0-1}))$, $\underline{\dim}\, \overline{M} = [n_0, n_\omega]$, is called the *characteristic polynomial* of the M. The set

$$\operatorname{spec}(M) = V(\chi_M) \cap \operatorname{ord}(\mathbb{X}(\Lambda))$$

is called the *ordinary point spectrum* (or simply the *spectrum*) of M.

REMARK. (a) If $(p,q,r) \neq (1,1,1)$ then $\operatorname{spec}(M)$ is an affine variety. In case $q \neq 1$, $\operatorname{spec}(M) = V(\chi_M(t,1)) \cap \operatorname{ord}(\mathbb{X}(\Lambda)) \subseteq k$; in case $q = 1$, we have $p \neq 1$ and $\operatorname{spec}(M) = V(\chi_M(1,u)) \cap \operatorname{ord}(\mathbb{X}(\Lambda))$.

(b) $\operatorname{spec}(M) = \operatorname{spec}(\overline{M}) \setminus \operatorname{exc}(\mathbb{X}(\Lambda))$ for any Λ-module M.

(c) If one uses alternatively the rank of maximal preinjective direct summand of a Λ-module M in the definition above, the result does not change since $(\underline{\dim}\, M)_\omega - \operatorname{rk}_{\mathcal{P}}(M) = (\underline{\dim}\, M)_0 - \operatorname{rk}_{\mathcal{P}}(M^*)$.

1.5. Now we formulate the main result of this paper. To do this, with any domestic canonical algebra Λ we associate a pair $\theta_0 = \theta_0(\Lambda), \theta_1 = \theta_1(\Lambda)$ of integers, as in the following table:

Λ	$\theta_1(\Lambda)$	$\theta_0(\Lambda)$
$\Lambda_{p,q}$, $p \geq q \geq 1$	$\lceil \frac{pq}{p+q} \rceil$	$\operatorname{lcm}(p,q)$
$\Lambda_{p,2,2}$, p even	p	p
$\Lambda_{p,2,2}$, p odd	p	$2p$
$\Lambda_{3,3,2}$	6	6
$\Lambda_{4,3,2}$	12	12
$\Lambda_{5,3,2}$	30	30

THEOREM. *Let $\Lambda = \Lambda_{p,q,r}$ be an arbitrary domestic canonical algebra and $\theta_0 = \theta_0(\Lambda)$, $\theta_1 = \theta_1(\Lambda)$ be as above. Then for any pair $M = (A,B,C)$, $M' = (A',B',C')$ of finite-dimensional Λ-modules with $\underline{\dim}\, M = \underline{\dim}\, M' = \underline{n} = (n_v)_{v \in (Q_{p,q,r})_0}$, conditions* (a), (b) *and* (c) *below are equivalent*:

(a) *$M \cong M'$.*

(b) *The following equalities hold*:

- *$h(M)_x = h(M')_x$ and $h(M^*)_x = h(M'^*)_x$ for any $x \in \boldsymbol{P}^0 \cup \boldsymbol{P}'_{n_*}$, where $\boldsymbol{P}'_{n_*} = \{x(n,i) \in \boldsymbol{P}' : i \in \Sigma_0, n < \theta_1 n_* + \theta_0\}$, $n_* = \min\{n_v : v \in (Q_{p,q,r})_0\}$;*
- $\operatorname{spec}_\Lambda(M) = \operatorname{spec}_\Lambda(M')$;
- *$h(M)_{[\lambda,s,l]} = h(M')_{[\lambda,s,l]}$ for any $\lambda \in \operatorname{spec}_\Lambda(M) \cup \operatorname{exc}(\mathbb{X}(\Lambda))$, $s \in \mathbb{Z}_{w(\lambda)}$ and $1 \leq l \leq (n_* + 1)w(\lambda)$.*

(c) *The following equalities hold*:

- *$\operatorname{r}(\mathcal{M}(M,x)) = \operatorname{r}(\mathcal{M}(M',x))$ and $\operatorname{r}(\mathcal{M}(M^*,x)) = \operatorname{r}(\mathcal{M}(M'^*,x))$ for any $x \in \boldsymbol{P}^0 \cup \boldsymbol{P}'_{n_*}$* (see [5, 2.3] for definition of $\mathcal{M}(N,y)$);
- *$\Delta(\overline{M}(t)) = \Delta(\overline{M'}(t))$, or equivalently, $\operatorname{r}(\overline{M}(t)) = \operatorname{r}(\overline{M'}(t))$ $(=: r)$ and $D_j(\overline{M}(t)) = D_j(\overline{M'}(t))$ for all $j = 1, \ldots, r$;*

- $\mathrm{r}(\mathcal{M}(\lambda, M, s, l)) = \mathrm{r}(\mathcal{M}(\lambda, M', s, l))$ *for any* $\lambda \in \mathrm{exc}(\mathbb{X}(\Lambda)) \cup \{\infty\}$, $s \in \mathbb{Z}_{w(\lambda)}$ *and* $1 \le l \le (n_* + 1)w(\lambda)$, *where*
$$\mathcal{M}(\lambda, M, s, l) = \begin{cases} \mathcal{M}^{\mu_p(s,l)}(\bar{B}, A) & \text{if } \lambda = 0, \\ \mathcal{M}^{\mu_2(s,l)}(-\bar{B}, C) & \text{if } \lambda = 1, \\ \mathcal{M}^{\mu_q(s,l)}(\bar{A}, B) & \text{if } \lambda = \infty, \end{cases}$$
and similarly for $\mathcal{M}(\lambda, M', s, l)$ (see [5, 2.2] for definition of the indexing function $\mu_{(-)}$ and the matrices in the formulas above).

Remark. (a) Condition (b) is rather theoretical. In comparison with the Auslander theorem, it restricts the class of indecomposable modules in $\mathrm{mod}\,\Lambda$ for which one has to test the equality of dimensions to members of a finite, precisely described set of connected components in the Auslander–Reiten quiver Γ_Λ, in fact, to a finite set of isoclasses. Nevertheless, because of the necessity of solving polynomial equations, we should not expect that one can determine this set effectively.

(b) Condition (c) says, in particular, that the multiplicity vectors, for M and M', restricted to all components which are not homogeneous tubes, are equal. In contrast to (b), all ingredients of (c) have algorithmic and "fully computable" character (see [5] for details). Therefore, (c) can be effectively used in practice. Moreover, it can be converted into a computer program.

2. Proof of the main result. In this section we give the full proof of Theorem 1.5, which we precede by some preparatory facts.

2.1. We start with a lemma concerning the main property of the spectra of modules over domestic canonical algebras.

Lemma. *Let M be a module over a domestic canonical algebra Λ. Then, for any $\lambda \in \mathrm{ord}(\mathbb{X}(\Lambda))$, λ belongs to $\mathrm{spec}_\Lambda(M)$ if and only if M contains a direct summand from the tube $\mathcal{T}_\lambda^\Lambda$.*

Proof. Assume first that $\Lambda = \Lambda_{1,1}$ and $M = (A, B)$. Then clearly $\mathrm{ord}(\mathbb{X}(\Lambda)) = \mathbb{P}^1(k)$. We set $\chi_M^t = \chi_M(t, 1)$ and $\chi_M^u = \chi_M(1, u)$. Observe that $\chi_M^t = D_{j(M)}(M(t))$, since the mapping $k[t, u] \ni f \mapsto f(t, 1) \in k[t]$ is an algebra homomorphism which preserves irreducibility for homogeneous polynomials $f \ne u$, and sends u to 1. Analogously, $\chi_M^u = -D_{j(M)}(M'(u))$, where $M' = (B, A)$. Then $k \cap \mathrm{spec}_\Lambda(M) = V(\chi_M^t) = V(D_{j(M)}(M(t)))$, where the embedding $k \subseteq \mathbb{P}^1(k)$ is as in 1.2. Consequently, by [4, Proposition 4.4], $\lambda \in \mathrm{spec}_\Lambda(M)$ if and only if M contains a direct summand from the tube $\mathcal{T}_\lambda^\Lambda$, for $\lambda \in k$. In case $\lambda = \infty \in \mathbb{P}^1(k)$, we have $\infty \in \mathrm{spec}_\Lambda(M)$ if and only if $0 \in V(\chi_M^u) = V(D_{j(M)}(M'(u)))$, M' contains a $\mathcal{T}_0^\Lambda$ if and only if M contains a direct summand from $\mathcal{T}_\infty^\Lambda$, and we again apply [4, Proposition 4.4]. (Note that $j(M') = j(M)$ since the autoequivalence of $\mathrm{mod}\,\Lambda$ given by $(A, B) \mapsto (B, A)$ preserves the dimension vectors so it acts invariantly on the isoclasses of postprojectives.) Consequently, the proof of the assertion for a Kronecker algebra is complete.

Let $\Lambda = \Lambda_{p,q,2}$. Then by [5, 3.4(∗)], for $\lambda \in \mathrm{ord}(\mathbb{X}(\Lambda)) = k \setminus \{0, 1\}$, M contains a direct summand from $\mathcal{T}_\lambda^\Lambda$ if and only if the $\Lambda_{1,1}$-module $\Psi(M) = \overline{M}$ contains a direct summand from $\mathcal{T}_\lambda^{\Lambda_{1,1}}$. Moreover, by the Kronecker algebra case, the second equivalent condition holds exactly when $\lambda \in \mathrm{spec}_\Lambda(M) = \mathrm{spec}_{\Lambda_{1,1}}(\overline{M}) \setminus \{0, 1, \infty\}$.

Finally, assume that $\Lambda = \Lambda_{p,q}$. Then similarly $\mathrm{spec}_\Lambda(M) = \mathrm{spec}_{\Lambda_{1,1}}(\overline{M}) \setminus \mathrm{exc}(\mathbb{X}(\Lambda))$. Moreover, one can easily show that the functor $\mathrm{res} : \mathrm{mod}\Lambda \to \Lambda_{1,1}$ has analogous properties to those of the functor Ψ.

In this way the assertion is proven for any domestic canonical algebra Λ. ■

2.2. To formulate our next result precisely we need some extra notation. We consider the pairs $\mathcal{E} = (\mathcal{E}, \varepsilon)$ consisting of subsets $\mathcal{E} \subseteq k[t]$ and functions $\varepsilon : \mathcal{E} \to \mathbb{N}$. Note that each such pair (sometimes called a "multiset") can be treated as a sequence $(\varepsilon(f) \times f)_{f\in\mathcal{E}}$ of tuples $\varepsilon(f) \times f = (f, \ldots, f) \in k[t]^{\varepsilon(f)}$.

Given $\mathcal{E}_1 = (\mathcal{E}_1, \varepsilon_1)$, $\mathcal{E}_2 = (\mathcal{E}_2, \varepsilon_2)$ as above we define the union $\mathcal{E}_1 \uplus \mathcal{E}_2$ by setting

$$(\mathcal{E}_1, \varepsilon_1) \uplus (\mathcal{E}_2, \varepsilon_2) = (\mathcal{E}, \varepsilon)$$

where $\mathcal{E} = \mathcal{E}_1 \cup \mathcal{E}_2$ and $\varepsilon : \mathcal{E} \to \mathbb{N}$ is defined as follows:

$$\varepsilon(f) = \begin{cases} \varepsilon_1(f) & \text{if } f \in \mathcal{E}_1 \setminus \mathcal{E}_2, \\ \varepsilon_2(f) & \text{if } f \in \mathcal{E}_2 \setminus \mathcal{E}_1, \\ \varepsilon_1(f) + \varepsilon_2(f) & \text{if } f \in \mathcal{E}_1 \cap \mathcal{E}_2. \end{cases}$$

Following [6], with any t-matrix $\mathcal{A} \in \mathbb{M}_{x,y}(k[t])$ we associate the system $\mathcal{E}(\mathcal{A})$ of *elementary divisors* of $\mathcal{A}$. Recall that $\mathcal{E}(\mathcal{A})$ is the collection of all polynomials $f_i^{u_{j,i}} \neq 1$ from the decompositions $E_j(\mathcal{A}) = f_1^{u_{j,1}} \cdot \ldots \cdot f_v^{u_{j,v}}$ of the monic polynomials $E_j = E_j(\mathcal{A})$, $j = 1, \ldots, r = \mathrm{r}(\mathcal{A})$, into products of powers of pairwise different irreducible monic polynomials $f_1, \ldots, f_v \in k[t]$. Note that $D_r(\mathcal{A}) = E_r \cdot \ldots \cdot E_1$ and $E_1 \,|\, E_2, \ldots, E_{r-1} \,|\, E_r$, in particular, $u_{r,i} \geq \cdots \geq u_{1,i} \geq 0$ for every $i = 1, \ldots, v$. Clearly, $\mathcal{E}(\mathcal{A})$ carries a canonical structure of a pair $(\mathcal{E}, \varepsilon)$ as above: $\mathcal{E}$ is the set of all elements in $\mathcal{E}(\mathcal{A})$ and ε is $\mathcal{E} \ni f \mapsto |\{(j, i) : f_i^{u_{j,i}} = f\}| \in \mathbb{N}$.

Now we formulate all properties of t-matrices assigned to $\Lambda_{1,1}$-modules, which are necessary in the proof of our main result.

PROPOSITION. *Given a pair $M = (A, B)$, $M' = (A', B')$ of $\Lambda_{1,1}$-modules with $A, B \in \mathbb{M}_{n_\omega \times n_0}(k)$, $A', B' \in \mathbb{M}_{n'_\omega \times n'_0}(k)$, the following assertions hold*:

(a) *If $M \cong M'$ then $M(t) \sim M'(t)$.*

(b) *If $M(t) \sim M'(t)$ and the matrices B, B' are invertible then $M \cong M'$.*

(c) *If* $\mathcal{E}(M(t)) = \mathcal{E}(M'(t))$, $n_\omega = n_0 = n'_\omega = n'_0$ *and* $D_{n_0}(M(t)) \neq 0$, $D_{n_0}(M'(t)) \neq 0$ *then* $M(t) \sim M'(t)$.

(d) $\mathcal{E}(M(t) \oplus M'(t)) = \mathcal{E}(M(t)) \uplus \mathcal{E}(M'(t))$, *where*

$$M(t) \oplus M'(t) = \begin{bmatrix} M(t) & 0 \\ 0 & M'(t) \end{bmatrix}.$$

Proof. Assertion (a) is clear. For the proof of (b) and a more general version of (d), we refer to [6]. It remains to prove (c).

Let M and M' be as in (c). Then $M(t)$ and $M'(t)$ are square matrices of maximal rank, i.e. they belong to $\mathbb{M}_{r\times r}(k[t])$, where $r = \mathrm{r}(M(t)) = \mathrm{r}(M'(t))$. We can assume that $D_r(M(t))$ and $D_r(M'(t))$ are polynomials of positive degree, or equivalently, that $\mathcal{E}(M(t)) \neq \emptyset$ and $\mathcal{E}(M'(t)) \neq \emptyset$. Otherwise, $\mathcal{E}(M(t)) = \mathcal{E}(M'(t)) = \emptyset$, so $D_r(M(t)) = D_r(M'(t)) = 1$, and hence $\Delta(M(t)) = I_r = \Delta(M'(t))$, since $E_j((M(t)) = E_j((M'(t)) = 1$ for every $j = 1, \ldots, r$.

Now, the collection $\mathcal{E}(M(t)) = \mathcal{E}(M'(t))$ has the form $(f_i^{w_{j,i}})_{i=1,\ldots,v;\, j=1,\ldots,r_i}$ for some monic irreducible polynomials $f_1, \ldots, f_v \in k[t]$, where $r_i \leq r$ and $w_{1,i} \geq \cdots \geq w_{r_i,i} \geq 1$ for every $i = 1, \ldots, v$. We set $w_{j,i} = 0$ for $j > r_i$, $i = 1, \ldots, v$. Then

$$\begin{aligned} E_r(M(t)) &= f_1^{w_{1,1}} \cdot \ldots \cdot f_v^{w_{1,v}} = E_r(M'(t)), \\ E_{r-1}(M(t)) &= f_1^{w_{2,1}} \cdot \ldots \cdot f_v^{w_{2,v}} = E_{r-1}(M'(t)), \\ &\vdots \\ E_1(M(t)) &= f_1^{w_{r,1}} \cdot \ldots \cdot f_v^{w_{r,v}} = E_1(M'(t)). \end{aligned}$$

Consequently, $\Delta(M(t)) = \Delta(M'(t))$ and $M(t) \sim M'(t)$. ∎

2.3. *Proof of Theorem 1.5.* First we prove jointly the implications (a)⇒(b) and (a)⇒(c), next the implication (b)⇒(a), and finally (c)⇒(a).

(a)⇒(b),(c): Recall first that $h(M)_x = h(M')_x$ if and only if $\mathrm{r}(\mathcal{M}(M, x)) = \mathrm{r}(\mathcal{M}(M', x))$ for $x \in \boldsymbol{P}$ (see [5, Theorem 2.3] for $\Lambda = \Lambda_{p,q,2}$, and [4, Lemma 5.6(i)] for $\Lambda = \Lambda_{p,q}$). Similarly, $h(M)_{[\lambda,s,l]} = h(M')_{[\lambda,s,l]}$ if and only if $\mathcal{M}(\lambda, M, s, l) = \mathcal{M}(\lambda, M', s, l)$, for $[\lambda, s, l] \in \boldsymbol{T}$ (see [4, Lemma 5.6] and [5, 3.4] for $\Lambda_{p,q}$ and $\Lambda_{p,q,2}$, respectively). Next note that clearly the integers $h(N)_x$, $x \in \mathbb{X}$, are invariants of isomorphism classes of $\Lambda_{p,q,r}$-modules N. Moreover, by Lemma 2.1 and Proposition 2.2(a), so also are the sets $\mathrm{spec}_\Lambda(N)$ and the matrices $\Delta(\overline{N}(t))$, respectively. Now, the implications (a)⇒(b) and (a)⇒(c) follow immediately.

(b)⇒(a): Assume that (b) holds. To prove $M \cong M'$, we show that $m(M)_{|\boldsymbol{Y}} = m(M')_{|\boldsymbol{Y}}$ for $\boldsymbol{Y} = \boldsymbol{P}, \boldsymbol{Q}, \boldsymbol{T}$, respectively.

We start by showing $m(M)_{|\boldsymbol{P}} = m(M')_{|\boldsymbol{P}}$. Fix $v_0 \in (Q_{p,q,2})_0$ such that $n_{v_0} = n_*$. Consider first the case $\Lambda = \Lambda_{p,q,2}$. Let $\eta = \eta(\Lambda)$ and $\nu = \nu(\Lambda)$ be the constants defined in [5, Theorem 2.4]. Then by [5, Theorem 2.4(e)] we have $x(n,i)_{v_0} > n_* = n_{v_0}$, and hence $m(M)_{x(n,i)} = 0$ for all $n \geq n_*/\eta + \nu$ and $i \in \Sigma_0$. From [5, Proposition 5.2] it follows that $\nu = \theta_0$. To determine η we use the formula $\eta = \min\{\kappa_j/\nu_j : j \in [r]\}$ (see [5, Theorem 2.4(e)]). First, applying [5, Algorithm 6.1], we compute the constants κ_j and ν_j, $j \in [r]$, and then we find directly that $1/\eta = \theta_1$. As a consequence, $m(M)_x = 0 = m(M')_x$ for all $x \in \boldsymbol{P}' \setminus (\boldsymbol{P}^0 \cup \boldsymbol{P}'_{n_*})$.

Next consider the case $\Lambda = \Lambda_{p,q}$. We define η and ν as in [5, Theorem 2.4]. The arguments from the proof of [5, Theorem 2.4(e)] show again that $x(n,i)_{v_0} > n_* = n_{v_0}$, so $m(M)_{x(n,i)} = 0$ for all $n \geq n_*/\eta + \nu$ and $i \in \Sigma_0$. From [8, Section XIII.1] we know that $\nu = \operatorname{lcm}(p,q)$ and $\partial(x) = -(p+q)/\gcd(p,q)$ for $x \in \boldsymbol{P}$, where $\partial = \partial_\Lambda$ denotes the defect function for the hereditary algebra Λ. On the other hand, by the proof of [5, Theorem 2.4(e)], we have $\min\{-\partial(x) : x \in \Sigma_0\} = \eta\nu$. Consequently,

$$\frac{1}{\eta} = \frac{\operatorname{lcm}(p,q)\gcd(p,q)}{p+q} = \frac{pq}{p+q},$$

so $\lceil 1/\eta \rceil = \theta_1$, and again $m(M)_x = 0 = m(M')_x$ for all $x \in \boldsymbol{P}' \setminus (\boldsymbol{P}^0 \cup \boldsymbol{P}'_{n_*})$.

It remains to show that $m(M)_{\boldsymbol{P}^0 \cup \boldsymbol{P}'_{n_*}} = m(M')_{\boldsymbol{P}^0 \cup \boldsymbol{P}'_{n_*}}$ for any domestic canonical algebra Λ.

Fix any $x \in \boldsymbol{P}^0 \cup \boldsymbol{P}'_{n_*}$. Recall the formulas

$$(*)_M \qquad m(M)_x = \begin{cases} h(M)_x + h(M)_{\tau x} - \sum_{y \in {}^-x} d_{y,x}\, h(M)_y & \text{if } X_x \text{ is non-projective,} \\ h(M)_x - \sum_{y \in {}^-x} d_{y,x}\, h(M)_y & \text{if } X_x \text{ is projective,} \end{cases}$$

and an analogous one $(*)_{M'}$ for $m(M')_x$ (see [5, Introduction]). Then by [5, Proposition 5.7], we have $\tau x \prec x$ and $y \prec x$ for any $y \in {}^-x$. Hence, by definition of the order $\prec$, the vectors τx and $y \in {}^-x$ belong to $\boldsymbol{P}^0 \cup \boldsymbol{P}'_{n_*}$ (see [5, 5.7]). Consequently, $h(M)_{\tau x} = h(M')_{\tau x}$ and $h(M)_y = h(M')_y$ for $y \in {}^-x$, so $m(M)_x = m(M')_x$.

Concluding, we have $m(M)_{|\boldsymbol{P}} = m(M')_{|\boldsymbol{P}}$.

Next consider the case $\boldsymbol{Y} = \boldsymbol{Q}$. Applying the equalities $h(M^*)_x = h(M'^*)_x$ for $x \in \boldsymbol{P}^0 \cup \boldsymbol{P}'_{n_*}$, (b) and dual arguments, we obtain $m(M)_{|\boldsymbol{Q}} = m(M')_{|\boldsymbol{Q}}$.

Finally, we show $m(M)_{|\boldsymbol{T}_\lambda} = m(M')_{|\boldsymbol{T}_\lambda}$. Fix $\lambda \in \operatorname{spec}_\Lambda(M) \cup \operatorname{exc}(\mathbb{X}(\Lambda))$. It is well known [8] that $(\underline{\dim}\, X_{[\lambda,s,l]})_i \geq n_* + 1$ for all $i \in (Q_{p,q,r})_0$, $s \in \mathbb{Z}_{w(\lambda)}$ and $l \geq (n_*+1)w(\lambda)$. Hence, $m(M)_{[\lambda,s,l]} = 0 = m(M')_{[\lambda,s,l]}$ for all $s \in \mathbb{Z}_{w(\lambda)}$ and $l \geq (n_*+1)w(\lambda)$.

Consider $x = [\lambda, s, l]$ given by the pair $(s,l) \in \mathbb{Z}_{w(\lambda)} \times \mathbb{N}$ such that $l < (n_*+1)w(\lambda)$. Clearly, $\tau x = [\lambda, s \ominus 1, l]$ and ${}^-x = \{[\lambda, s\ominus 1, l-1], [\lambda, s, l+1]\}$ if $l \geq 2$, and ${}^-x = \{[\lambda, s, 2]\}$ otherwise, where $\ominus = \ominus_{w(\lambda)}$. Since $l, l-1, l+1 \leq (n_*+1)w(\lambda)$, we have $h(M)_{\tau x} = h(M')_{\tau x}$ and $h(M)_y = h(M')_y$ for $y \in {}^-x$, so $m(M)_x = m(M')_x$, from $(*)$. Consequently, $m(M)_{|\boldsymbol{T}_\lambda} = m(M')_{|\boldsymbol{T}_\lambda}$ for all $\lambda \in \operatorname{spec}_\Lambda(M) \cup \operatorname{exc}(\Lambda)$.

Finally, note that by Lemma 2.1, $m(M)_{|\boldsymbol{T}_\lambda} = m(M')_{|\boldsymbol{T}_\lambda} = 0$ for all $\lambda \in \operatorname{ord}(\mathbb{X}(\Lambda)) \setminus \operatorname{spec}_\Lambda(M)$. Thus, $m(M)_{|\boldsymbol{T}} = m(M)_{|\boldsymbol{T}}$.

In this way $m(M)_x = m(M')_x$ for all $x \in \boldsymbol{X}$, and the proof of (b)⇒(a) is complete.

(c)⇒(a): Assume that (c) holds. By the previous parts of the proof,

$$(**)\qquad m(M)_{|\boldsymbol{Y}} = m(M')_{|\boldsymbol{Y}}, \quad \text{where} \quad \boldsymbol{Y} = \boldsymbol{P} \sqcup \boldsymbol{Q} \sqcup \bigsqcup_{\lambda \in \operatorname{exc}(\mathbb{X}(\Lambda)) \cup \{\infty\}} \boldsymbol{T}_\lambda.$$

So it remains to show that $R \cong R'$, where R (resp. R') denotes the maximal direct summand of M (resp. M') belonging to $\operatorname{add}(\bigcup_{\lambda \in \operatorname{ord}(\mathbb{X}(\Lambda)) \setminus \{\infty\}} \mathcal{T}_\lambda^\Lambda)$. By the properties of the functors Ψ and res, the $\Lambda_{1,1}$-modules $\overline{R} = (R_1, R_2)$ and $\overline{R'} = (R'_1, R'_2)$ belong to $\operatorname{add}(\bigcup_{\lambda \in \operatorname{ord}(\mathbb{X}(\Lambda)) \setminus \{\infty\}} \mathcal{T}_\lambda^{\Lambda_{1,1}})$; moreover, $R \cong R'$ if and only if $\overline{R} \cong \overline{R'}$ (see [8, 5]). Therefore, we now show that the equality $\Delta(\overline{M}(t)) = \Delta(\overline{M'}(t))$ implies the required isomorphism $\overline{R} \cong \overline{R'}$.

By $(**)$, $\underline{\dim}\, R = \underline{\dim}\, R'$, so $\underline{\dim}\, \overline{R} = \underline{\dim}\, \overline{R'}$, since $\underline{\dim}\, M = \underline{\dim}\, M'$. We can assume that $\overline{R}, \overline{R'}$ are non-zero modules (otherwise, there is nothing to show). By the description of indecomposable modules in the category $\operatorname{mod} \Lambda_{1,1}$, Proposition 2.2(a) and an elementary calculation, we have the following:

(i) R_1, R_2, R'_1, R'_2 are square $l \times l$ matrices for some $l \geq 1$,
(ii) R_2, R'_2 are invertible,
(iii) $D_l(\overline{R}(t)), D_l(\overline{R'}(t)) \neq 0$,
(iv) for an indecomposable N in $\operatorname{mod} \Lambda_{1,1}$, the set $\mathcal{E}(N(t))$ is empty, provided N is postprojective, preinjective or belongs to $\mathcal{T}_\infty^{\Lambda_{1,1}}$.

The equality $\Delta(\overline{M}(t)) = \Delta(\overline{M'}(t))$ implies $\mathcal{E}(\overline{M}(t)) = \mathcal{E}(\overline{M'}(t))$, so $\mathcal{E}(\overline{R}(t)) = \mathcal{E}(\overline{R'}(t))$ by (iv) and Proposition 2.2(d), since $m(M)_{|\boldsymbol{Z}} = m(M')_{|\boldsymbol{Z}}$, where $\boldsymbol{Z} = \bigsqcup_{\lambda \in \operatorname{exc}(\mathbb{X}(\Lambda))} \boldsymbol{T}_\lambda$. Then, by (i), (iii) and Proposition 2.2(c), the matrices $\overline{R}(t)$ and $\overline{R'}(t)$ are equivalent. Hence, by (ii) and Proposition 2.2(b), the modules $\overline{R}$ and $\overline{R'}$ are isomorphic (and so are R and R').

Summarizing, (c) implies $(**)$ and the isomorphism $R \cong R'$, so $M \cong M'$. The proof of the theorem is complete. ■

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