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SOME PROPERTIES OF α-HARMONIC MEASURE

BY

DIMITRIOS BETSAKOS (Thessaloniki)

Abstract. The α-harmonic measure is the hitting distribution of symmetric α-stable processes upon exiting an open set in $\mathbb{R}^n$ ($0 < \alpha < 2$, $n \geq 2$). It can also be defined in the context of Riesz potential theory and the fractional Laplacian. We prove some geometric estimates for α-harmonic measure.

1. Introduction. In the 1930's, O. Frostman and M. Riesz developed a potential theory on $\mathbb{R}^n$, $n \geq 2$, based on the *Riesz kernel*

$$k_\alpha(x) = \frac{\mathcal{A}(n,\alpha)}{|x|^{n-\alpha}}, \quad x \in \mathbb{R}^n \setminus \{0\}, \tag{1.1}$$

where $0 < \alpha < 2$ and $\mathcal{A}(n,\alpha)$ is a constant. When $\alpha = 2$, the Riesz kernel coincides with the kernel of the classical potential theory, the Newtonian kernel ($n \geq 3$). The α-harmonic functions are defined by a mean value property (involving the parameter α), analogous to the classical one. Equivalently, they are the solutions of the equation $\Delta^{\alpha/2}u = 0$, where $\Delta^{\alpha/2}$ is the *fractional Laplacian*, a non-local integro-differential operator.

A function $u : \mathbb{R}^n \to \mathbb{R}$ which is α-harmonic in an open set D is determined by its exterior values (its values in $D^{\mathrm{c}} := \mathbb{R}^n \setminus D$). If B is a Borel set in D^{c}, the *α-harmonic measure* of B with respect to D is the α-harmonic function u in D with exterior values $u = \chi_B$ on D^{c}. The α-harmonic measure of B with respect to D, evaluated at the point $x \in \mathbb{R}^n$, will be denoted by $\omega_\alpha^D(x,B)$. For fixed $x \in D$, $\omega_\alpha^D(x,\cdot)$ is a Borel probability measure on D^{c}.

Both classical and α-harmonic measures have symmetry properties and satisfy the Carleman principle (domain monotonicity) and the Harnack principle. The latter implies that if $\omega_\alpha^D(x,B) = 0$ for some $x \in D$, then $\omega_\alpha^D(y,B) = 0$ for all $y \in D$; we then say that B is a *D-null set*. There are, however, essential differences. The classical harmonic measure is defined (as a function) in a *domain* D and is supported (as a measure) on the boundary of D. The α-harmonic measure is defined (as a function) in the whole $\mathbb{R}^n$ and is supported (as measure) in the exterior of D. These properties be-

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come transparent when considered from the probabilistic point of view. The classical harmonic measure is the hitting distribution of a Brownian motion upon exiting D, while the α-harmonic measure is the hitting distribution of a symmetric α-stable process. This is a Hunt process with discontinuous paths. Thus its paths may jump from one component of D to another and may hit D^{c} (upon exiting D) at points of $(\overline{D})^{\mathrm{c}}$ and not necessarily at points of ∂D.

The basic facts of Riesz potential theory are presented in the book of N. S. Landkof [12]. Recently there has been a renewed interest in Riesz potential theory, mainly from the probabilistic point of view. K. Bogdan [4] proved the boundary Harnack principle for α-harmonic functions on Lipschitz open sets. R. Song and J.-M. Wu [14] proved extensions of Bogdan's results. Bogdan [5] and Z.-Q. Chen and Song [11] gave a Martin representation for non-negative α-harmonic functions. Bogdan and T. Byczkowski [6], [7] developed the theory of the Schrödinger operator based on the fractional Laplacian. Wu [15] found necessary and sufficient conditions for a boundary set to have zero α-harmonic measure. R. Bañuelos, R. Latała and P. J. Méndez-Hernández [1] proved isoperimetric type inequalities for transition probabilities, Green functions and eigenvalues associated with symmetric stable processes. Various other properties and applications of α-harmonic functions and the fractional Laplacian are presented in [10], [2], [9], [8] and the references therein. A review of the basic facts about Riesz potential theory and symmetric stable processes appears in Section 2.

In Section 3, we prove some geometric estimates for α-harmonic measure involving symmetric or polarized open sets D. Although the corresponding inequalities for the classical harmonic measure are almost trivial, we will see that the proofs for the α-harmonic measure are not simple. Theorems 1 and 2 were proved in [2] under more restrictive conditions (in [2, Theorem 3], the open set D is assumed to be bounded with boundary satisfying an exterior cone condition).

2. Background

2.1. *α-harmonic functions.* The M. Riesz kernels in $\mathbb{R}^n$, $n \geq 2$, are the functions

$$k_\alpha(x) = \frac{\mathcal{A}(n,\alpha)}{|x|^{n-\alpha}}, \qquad x \in \mathbb{R}^n \setminus \{0\}, \tag{2.1}$$

where $0 < \alpha < n$ and

$$\mathcal{A}(n,\gamma) = \frac{\Gamma((n-\gamma)/2)}{|\Gamma(\gamma/2)|\, 2^\gamma \pi^{n/2}}, \qquad -n < \gamma < n,\ \gamma \neq 0, -2, -4, \ldots. \tag{2.2}$$

These kernels include as special and limiting cases the kernels of the classical potential theory: the Newtonian kernel ($n \geq 3$, $\alpha = 2$) and the logarithmic kernel ($n = 2$, $\alpha \to 2$); see [12, Ch. I]. From now on, we assume that $0 < \alpha < 2$. We denote the n-dimensional Lebesgue measure by m_n.

DEFINITION 1. Let D be an open set in $\mathbb{R}^n$, $n \geq 2$. A function $u : \mathbb{R}^n \to \mathbb{R}$ is called *α-harmonic* in D if

(a) u is continuous in D;

(b) u is in $\mathcal{L}^1$; that is, u is locally integrable on $\mathbb{R}^n$ and

$$\int_{|x|>1} \frac{|u(x)|}{|x|^{n+\alpha}}\, m_n(dx) < \infty; \tag{2.3}$$

(c) for every ball $B(x_0, r)$ with closure in D,

$$u(x_0) = \int_{\mathbb{R}^n} u(x)\varepsilon_\alpha^{(r)}(x - x_0)\, m_n(dx), \tag{2.4}$$

where

$$\varepsilon_\alpha^{(r)}(x) = \begin{cases} \dfrac{\Gamma(n/2)\sin(\pi\alpha/2)}{\pi^{n/2+1}} \dfrac{r^\alpha}{(|x|^2 - r^2)^{\alpha/2}|x|^n}, & |x| > r, \\ 0, & |x| < r. \end{cases} \tag{2.5}$$

DEFINITION 2. Let $f \in \mathcal{L}^1$. For $\varepsilon > 0$ and $x \in \mathbb{R}^n$, we define

$$\Delta_\varepsilon^{\alpha/2} f(x) = \mathcal{A}(n, -\alpha) \int_{|y-x|>\varepsilon} \frac{f(y) - f(x)}{|y - x|^{n+\alpha}}\, m_n(dy) \tag{2.6}$$

and

$$\Delta^{\alpha/2} f(x) = \lim_{\varepsilon \downarrow 0} \Delta_\varepsilon^{\alpha/2} f(x), \tag{2.7}$$

whenever the limit exists.

By [6, Theorem 3.9], a function u defined on $\mathbb{R}^n$ is α-harmonic in an open set D if and only if it is continuous in D and $\Delta^{\alpha/2} u = 0$ in D.

2.2. *The Dirichlet problem for α-harmonic functions* (see [12, Ch. IV], [3, Ch. VII], [15]). The Perron–Wiener–Brelot method can be applied for the solution of the Dirichlet problem for α-harmonic functions. Let D be an open set in $\mathbb{R}^n$. An *α-subharmonic function* in D is an $\mathcal{L}^1$ function which is upper semicontinuous in D and satisfies the inequality

$$u(x_0) \leq \int_{\mathbb{R}^n} u(x)\varepsilon_\alpha^{(r)}(x - x_0)\, m_n(dx), \tag{2.8}$$

for every ball $B(x_0, r)$ with closure in D.

Let $C(D^{\mathrm{c}})$ be the class of functions f continuous in D^{c} satisfying

$$\int_{D^{\mathrm{c}} \cap \{|x|>1\}} \frac{|f(x)|}{|x|^{n+\alpha}}\, m_n(dx) < \infty,$$

and $H(D)$ be the class of functions on $\mathbb{R}^n$, α-harmonic in D. The *lower Perron family* of a function $f \in C(D^{\mathrm{c}})$ is the family $\mathcal{P}_f$ of all functions u which are α-subharmonic in D and satisfy the inequalities $u \le f$ in $(\overline{D})^{\mathrm{c}}$ and

$$\limsup_{D \ni x \to \zeta} u(x) \le f(\zeta), \qquad \forall \zeta \in \partial D.$$

Define

$$H_f(x) := \sup\{u(x) : u \in \mathcal{P}_f\}, \qquad x \in \mathbb{R}^n.$$

Then H_f is α-harmonic in D. The definition of regular and irregular boundary points and their characterization by Wiener's criterion are similar to their classical analogs. The function H_f has limit $f(\zeta)$ at each regular boundary point ζ. We say that H_f is the *Perron solution* of the Dirichlet problem in D with exterior values f.

The operator $f \mapsto H_f$ is a positive linear operator from $C(D^{\mathrm{c}})$ into $H(D)$. Hence for each $x \in \mathbb{R}^n$, there is a measure $\omega_\alpha^D(x, \cdot)$ on D^{c} such that

$$H_f(x) = \int_{D^{\mathrm{c}}} f(y)\, \omega_\alpha^D(x, dy), \qquad x \in \mathbb{R}^n.$$

This measure is the α-harmonic measure for D evaluated at x.

In a similar manner, one can define the upper and the lower Perron family for any Borel function on D^{c} and consider the corresponding generalized solution for the Dirichlet problem; see [3] for more details.

2.3. *Symmetric stable processes* (see [4], [5], [6], [10], [11], [14], [3], [8]). The fractional Laplacian $\Delta^{\alpha/2}$ is the characteristic operator of the symmetric α-stable process $\{\mathtt{X}_t,\ t \in [0, \infty)\}$ in $\mathbb{R}^n$. This is a Lévy process (homogeneous and with independent increments) with transition density $p_t(x, y) = p_t(y, x) = p_t(x - y)$ (relative to the Lebesgue measure) uniquely determined by its Fourier transform

$$\int_{\mathbb{R}^n} e^{ix\cdot\xi} p_t(x)\, m_n(dx) = e^{-t|\xi|^\alpha}. \tag{2.9}$$

When $\alpha = 2$, we get a Brownian motion running at twice the speed. The probability measures and the corresponding expectations of the process $\{\mathtt{X}_t\}$ starting at $x \in \mathbb{R}^n$ will be denoted by $\mathbf{P}^x$ and $\mathbf{E}^x$.

The symmetric α-stable process $\{\mathtt{X}_t\}$ is a strong Feller and a Hunt process. For $A \subset \mathbb{R}^n$, we put

$$T^A = \inf\{t > 0 : \mathtt{X}_t \notin A\}, \tag{2.10}$$

the first exit time from A. A Borel function u defined on $\mathbb{R}^n$ is α-harmonic in an open set $D \subset \mathbb{R}^n$ if and only if

$$u(x) = \mathbf{E}^x u(\mathtt{X}_{T^U}), \qquad x \in U, \tag{2.11}$$

for every bounded open set U with closure $\overline{U}$ contained in D. If $D \subset \mathbb{R}^n$ is open and B is a Borel subset of D^{c}, then

$$\omega_\alpha^D(x, B) = \mathbf{P}^x(\mathtt{X}_{T^D} \in B), \qquad x \in \mathbb{R}^n. \tag{2.12}$$

2.4. *Riesz capacity* (see [12, Ch. II]). If K is a compact set in $\mathbb{R}^n$ and μ is a probability Borel measure on K, then the *α-energy* of μ is

$$I_\alpha(\mu) = \int_K \int_K k_\alpha(x - y)\, \mu(dx)\, \mu(dy). \tag{2.13}$$

The *α-capacity* of K is defined by

$$C_\alpha(K) = (\inf_\mu I_\alpha(\mu))^{-1}, \tag{2.14}$$

where the infimum is taken over all probability Borel measures on K.

For a Borel set $E \subset \mathbb{R}^n$, we define

$$C_\alpha(E) = \sup\{C_\alpha(K) : K \subset E \text{ compact}\}. \tag{2.15}$$

By the Choquet capacitability theorem [12, Theorem 2.8, p. 156],

$$C_\alpha(E) = \inf\{C_\alpha(G) : E \subset G \text{ open}\}. \tag{2.16}$$

The α-capacity is a geometric quantity because of its expression as transfinite diameter; see [12, Ch. II, §3]. It can also be characterized in terms of symmetric stable processes; see references in [2].

2.5. *Null sets.* There is no known geometric characterization of null sets for α-harmonic measure. If a boundary set has zero α-capacity, then it also has zero α-harmonic measure; see [12]. The following lemmas provide more refined necessary or sufficient conditions.

LEMMA 1 ([15, Theorem 1′]). *Let D be an open set in $\mathbb{R}^n$ and F be a subset of ∂D with $m_n(F) = 0$. Suppose that there exists $c > 0$ such that for all $x \in D$,*

$$m_n(D^{\mathrm{c}} \cap B(x, 2d(x, F))) > cd(x, F)^n.$$

Then F is D-null.

LEMMA 2 ([15, Theorem 3]). *Let D be an open set in $\mathbb{R}^n$ and F be a subset of ∂D with $C_\alpha(F) > 0$. If*

$$\lim_{r\to 0} C_\alpha(\{x \in D^{\mathrm{c}} : 0 < d(x, F) \le r\}) = 0,$$

then F is not D-null.

LEMMA 3. *Suppose that D and Ω are open sets in $\mathbb{R}^n$ with $D \subset \Omega$. Let $A = \Omega \setminus D$ and assume that A is D-null. Then $C_\alpha(A) = 0$.*

Proof. By the Choquet capacitability theorem [12, Theorem 2.8, p. 156], A is capacitable. Assume first that A is compact. Then $d(A, \partial\Omega) > 0$. For

$0 < r < d(A, \partial\Omega)$, the set

$$\{x \in D^{\mathrm{c}} : 0 < d(x, A) \leq r\}$$

is empty. By Lemma 2, $C_\alpha(A) = 0$.

Next assume that A is bounded. Let

$$A_k = \{x \in A : d(x, \partial\Omega) \geq 1/k\}, \quad k \in \mathbb{N}.$$

Then A_k is compact. Hence $C_\alpha(A_k) = 0$ for all k. By the subadditivity of α-capacity, $C_\alpha(A) = 0$. Finally, for unbounded A we consider the sequence of bounded sets

$$A_m = \{x \in A : |x| \leq m\}, \quad m \in \mathbb{N},$$

and conclude as above that $C_\alpha(A) = 0$. ∎

2.6. *The minimum principle in Riesz potential theory.* We will need some extensions of the minimum principle for α-superharmonic functions; see [12, pp. 115, 183].

LEMMA 4. *Let D be an open set in $\mathbb{R}^n$ and $u : \mathbb{R}^n \to (-\infty, \infty]$ be a function which is α-superharmonic in D and lower semicontinuous on $\overline{D}$. Suppose that there exists a constant $M \in \mathbb{R}$ such that $u \geq M$ in D^{c}. Then $u \geq M$ in $\mathbb{R}^n$. If $u(x_0) = M$ for some $x_0 \in D$, then $u = M$ in $\mathbb{R}^n$.*

Proof. Define $v(x) = u(x) - M$, $x \in \mathbb{R}^n$. Then v is lower semicontinuous on $\overline{D}$. Also, for $\zeta \in \partial D$,

$$\liminf_{D \ni x \to \zeta} v(x) = \liminf_{D \ni x \to \zeta} u(x) - M \geq u(\zeta) - M \geq 0. \tag{2.17}$$

Suppose that there exists a point $x_0 \in D$ such that

$$\min_{\overline{D}} v = v(x_0) < 0. \tag{2.18}$$

Take $r > 0$ sufficiently small so that the ball of radius r, centered at x_0, lies in D. Then $v(x_0) < \varepsilon^{(r)}_{\alpha,x_0} v$; indeed, if $v(x_0) = \varepsilon^{(r)}_{\alpha,x_0} v$, then $v = v(x_0) < 0$ a.e. in $\{|x - x_0| > r\}$, and therefore

$$\liminf_{x \to \zeta \in D} v(x) \leq v(x_0) < 0,$$

contradicting (2.17). Hence

$$v(x_0) < \varepsilon^{(r)}_{\alpha,x_0} v = \varepsilon^{(r)}_{\alpha,x_0} u - M \leq u(x_0) - M = v(x_0), \tag{2.19}$$

which is absurd. We conclude that the minimum of v on $\overline{D}$ is non-negative and therefore $u(x) \geq M$ for all $x \in \mathbb{R}^n$.

If $u(x_0) = M$ for some $x_0 \in D$, then for all sufficiently small $r > 0$,

$$0 = v(x_0) \geq \varepsilon^{(r)}_{\alpha,x_0} v. \tag{2.20}$$

This implies $v = 0$ a.e. in $\mathbb{R}^n$; that is, $u = M$ a.e. in $\mathbb{R}^n$. If $x \in D$, then [12, p. 114]

$$u(x) = \lim_{r\to 0} \varepsilon_{\alpha,x}^{(r)} u = M.$$

Hence $u = M$ in D. ∎

LEMMA 5. *Let D be an open set in $\mathbb{R}^n$ and $u : \mathbb{R}^n \to (-\infty, \infty]$ be a function α-superharmonic in D. Assume that*

(i) *u is bounded below in D;*
(ii) *u is lower semicontinuous in $\overline{D} \setminus E$, where E is a subset of ∂D with $\infty \notin E$ and $C_\alpha(E) = 0$ (of course, if $E \subset \mathbb{R}^n$ then $\infty \notin E$);*
(iii) $\liminf_{D\ni x\to\zeta} u(x) \geq M$ *for some $M \in \mathbb{R}$ and all $\zeta \in \partial D \setminus E$;*
(iv) *$u(x) \geq M$ for all $x \in (\overline{D})^{\mathrm{c}}$.*

Then $u(x) \geq M$ for all $x \in D$. Moreover, if $u(x_0) = M$ for some $x_0 \in D$, then $u = M$ in D.

Proof. For $n \in \mathbb{N}$, let A_n be an open set such that $E \subset A_n$ and $C_\alpha(A_n) \leq 1/n$. Then $E_1 := \bigcap_{n=1}^{\infty} A_n$ is a G_δ-set such that $E \subset E_1$ and $C_\alpha(E_1) = 0$.

There exists a measure λ on $\mathbb{R}^n$ such that the Riesz potential U_α^λ of λ has the following properties (see [12, p. 179]):

$$U_\alpha^\lambda(x) = \infty, \quad \forall x \in E_1 \cap \partial D, \quad \text{and} \quad U_\alpha^\lambda(x) < \infty, \quad \forall x \notin E_1 \cap \partial D.$$

For $\varepsilon > 0$, define

$$u_1(x) = u(x) + \varepsilon U_\alpha^\lambda(x), \qquad x \in \mathbb{R}^n.$$

The function u_1 is α-superharmonic in D. Moreover,

$$\liminf_{D\ni x\to\zeta} u_1(x) \geq M, \qquad \forall \zeta \in \partial D, \tag{2.21}$$

because $U_\alpha^\lambda(x) \geq 0$, $\forall x \in \mathbb{R}^n$, and $U_\alpha^\lambda(x) = \infty$, $\forall x \in E_1 \cap \partial D$. Also, since U_α^λ is lower semicontinuous in $\mathbb{R}^n$ and

$$\liminf_{\overline{D}\ni x\to\zeta\in E} [u(x) + \varepsilon U_\alpha^\lambda(x)] = \infty = u(\zeta) + \varepsilon U_\alpha^\lambda(\zeta),$$

we see that u_1 is lower semicontinuous in $\overline{D}$.

We apply Lemma 4 to the function u_1 and conclude

$$u_1(x) = u(x) + \varepsilon U_\alpha^\lambda(x) \geq M, \qquad \forall x \in D.$$

Since $\varepsilon > 0$ is arbitrary and $U_\alpha^\lambda < \infty$ in D, it follows that $u \geq M$ in D.

Suppose next that $u(x_0) = M$ for some $x_0 \in D$. By the α-mean value inequality, $M = u(x_0) \geq \varepsilon_{\alpha,x_0}^{(r)} u$ for all sufficiently small $r > 0$. It follows that $u = M$ a.e. in $\mathbb{R}^n$. If $x \in D$, then [12, p. 114]

$$u(x) = \lim_{r\to 0} \varepsilon_{\alpha,x}^{(r)} u = M.$$

Hence $u = M$ in D. ∎

3. Some geometric properties of α-harmonic measure Let $\Pi = \{(x_1,\dots,x_n) \in \mathbb{R}^n : x_n = 0\}$. For $E \subset \mathbb{R}^n$, we denote by $\widehat{E}$ the reflection of E in the $(n-1)$-dimensional plane Π. Thus we have

$$\widehat{E} = \{(x_1,\dots,x_{n-1},x_n) : (x_1,\dots,x_{n-1},-x_n) \in E\}.$$

We will also use the following notation: if $x = (x_1,\dots,x_{n-1},x_n)$ then $\widehat{x} := (x_1,\dots,x_{n-1},-x_n)$; $E_+ := \{(x_1,\dots,x_{n-1},x_n) \in E : x_n > 0\}$; $E_0 := E \cap \Pi$; $E_- = \{(x_1,\dots,x_{n-1},x_n) \in E : x_n < 0\}$.

Let E be any set in $\mathbb{R}^n$. We divide E into three subsets S, U, V:

$$\begin{aligned} S &= S_E = \{x \in E : \widehat{x} \in E\} = E \cap \widehat{E}, \\ U &= U_E = \{x \in E : x \in E_+,\ \widehat{x} \notin E\} = E_+ \setminus S_E, \\ V &= V_E = \{x \in E : x \in E_-,\ \widehat{x} \notin E\} = E_- \setminus S_E. \end{aligned}$$

S is the *symmetric part* of E, U is the *upper non-symmetric part* of E, and V is the *lower non-symmetric part* of E. The sets S, U, V are disjoint and $E = S \cup U \cup V$. Note that if E is open, then its symmetric part S is always open, while the sets U, V are not necessarily open. We say that E is *symmetric with respect to Π* if $U = V = \emptyset$ and hence $E = S$. We say that E is *polarized with respect to Π* if $V = \emptyset$ and hence $E = S \cup U$.

THEOREM 1. *Let S be an open set in $\mathbb{R}^n$. Suppose that S is symmetric with respect to Π. Let $B \subset \mathbb{R}^n_+ \cap S^{\mathrm{c}}$ be a Borel set. Then*:

(i) $\omega_\alpha^S(x,B) \ge \omega_\alpha^S(\widehat{x},B),\ x \in \mathbb{R}^n_+$;
(ii) $\omega_\alpha^S(x,B) \ge \omega_\alpha^S(x,\widehat{B}),\ x \in \mathbb{R}^n_+$.

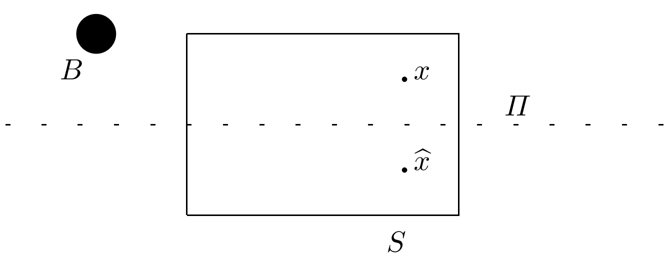

Fig. 1. An illustration for Theorem 1

Proof. For $x \in \mathbb{R}^n_+ \setminus S_+$, the inequalities (i) and (ii) are trivial. So we prove them for $x = s \in S_+$. Because of symmetry, (i) and (ii) are equivalent. So we only prove (i). By the inner regularity of α-harmonic measure, we may and do assume that B is a compact set in $\mathbb{R}^n_+ \cap S^{\mathrm{c}}$. Take a decreasing sequence of compactly supported continuous functions $f_k : S^{\mathrm{c}} \to [0,1]$ with

$\operatorname{supp} f_k \downarrow B$, $f_k \downarrow \chi_B$ and $f_k = 0$ in $(S^{\mathrm{c}})_-$. Then for the sequence of functions

$$H_{f_k}(x) := \int_{S^{\mathrm{c}}} f_k(y)\, \omega_\alpha^S(x, dy), \qquad x \in \mathbb{R}^n,$$

we have $H_{f_k}(s) \downarrow \omega_\alpha^S(s, B),\ s \in S$. It therefore suffices to prove that

$$H_{f_k}(s) \geq H_{f_k}(\widehat{s}), \qquad s \in S_+,\, k \in \mathbb{N}. \tag{3.1}$$

Let E be the set of irregular points of ∂S. By a classical result (see e.g. [12, p. 296]), $C_\alpha(E) = 0$. There exists a G_δ-set $E_1 \supset E$ with $C_\alpha(E_1) = 0$ and a measure λ on $\mathbb{R}^n$ such that (see [12, p. 179])

$$U_\alpha^\lambda(x) = \infty, \quad \forall x \in E_1 \cap \partial D, \quad \text{and} \quad U_\alpha^\lambda(x) < \infty, \quad \forall x \in \mathbb{R}^n \setminus (E_1 \cup \partial D).$$

Because of symmetry, we may also assume that $U_\alpha^\lambda(\widehat{x}) = U_\alpha^\lambda(x)$.

Fix $k \in \mathbb{N}$ and $\varepsilon > 0$ and define

$$v(x) = H_{f_k}(x) - H_{f_k}(\widehat{x}) + \varepsilon U_\alpha^\lambda(x), \qquad x \in \mathbb{R}^n. \tag{3.2}$$

We look at the boundary values of v in S_+. Let $\zeta \in \partial(S_+)$.

Case 1: $\zeta \in S_0$. Then

$$\liminf_{S_+ \ni s \to \zeta} v(s) = \liminf_{S_+ \ni s \to \zeta} \varepsilon U_\alpha^\lambda(s) \geq 0.$$

Case 2: $\zeta \in \partial(S_+) \setminus (S_0 \cup E_1)$. Then

$$\liminf_{S_+ \ni s \to \zeta} v(s) = f_k(\zeta) - 0 + \liminf_{S_+ \ni s \to \zeta} \varepsilon U_\alpha^\lambda(s) \geq 0.$$

Case 3: $\zeta \in E_1$. Then by the lower semicontinuity of U_α^λ,

$$\liminf_{S_+ \ni s \to \zeta} v(s) = \varepsilon U_\alpha^\lambda(\zeta) = \infty.$$

Case 4: S is unbounded and $\zeta = \infty$. Let B_1 be the support of f_k. For $s \in S$, we have

$$\begin{aligned} H_{f_k}(s) = \int_{S^{\mathrm{c}}} f_k(y)\, \omega_\alpha^S(s, dy) &\leq \int_{B_1} \omega_\alpha^S(s, dy) = \omega_\alpha^S(s, B_1) \\ &\leq \omega_\alpha^{B_1^{\mathrm{c}}}(s, B_1) = \mathbf{P}^s(T^{B_1^{\mathrm{c}}} < \infty). \end{aligned}$$

By a formula of S. Port [13],

$$C_\alpha(B_1) = \lim_{s \to \infty} \mathcal{A}(n, \alpha)^{-1}\, |s|^{n-\alpha}\, \mathbf{P}^s(T^{B_1^{\mathrm{c}}} < \infty).$$

Hence $\lim_{s \to \infty} H_{f_k}(s) = 0$. This implies that

$$\liminf_{S_+ \ni s \to \infty} v(s) = \liminf_{S_+ \ni s \to \infty} \varepsilon U_\alpha^\lambda(s) \geq 0. \tag{3.3}$$

Note here that we cannot apply the minimum principle of Subsection 2.6 because the condition $v \geq 0$ in $(S_+)^{\mathrm{c}}$ is not satisfied. Nevertheless, we will prove that $v \geq 0$ in S_+. Suppose that v takes on strictly negative values in S_+. Let

$$\beta := \inf\{v(s) : s \in S_+\}.$$

Take a sequence $\{s_k\}$ in S_+ such that $v(s_k) \to \beta$. By passing to a subsequence if necessary, we may assume that $\{s_k\}$ converges in $\overline{S}_+$. By Cases 1–4 examined above, we may assume that $\lim_k s_k = s_0 \in S_+$. The measure λ is not necessarily concentrated on E (see [12, p. 181]). However, λ may be taken so that its support is as close to E as we wish (see the proof of Theorem 3.1 in [12]). It is also known [12, Ch. I, §6] that the potential U_α^λ is an α-harmonic function in the complement of the support of λ. Hence v is α-harmonic in a neighborhood of s_0. Hence

$$\begin{aligned}(3.4)\qquad 0 = \Delta^{\alpha/2} v(s_0) &= \int_{\mathbb{R}^n} \frac{v(x)-v(s_0)}{|x-s_0|^{n+\alpha}}\, m_n(dx) \\ &= \int_{\mathbb{R}^n_+} \frac{v(x)-v(s_0)}{|x-s_0|^{n+\alpha}}\, m_n(dx) + \int_{\mathbb{R}^n_+} \frac{v(\widehat{x})-v(s_0)}{|\widehat{x}-s_0|^{n+\alpha}}\, m_n(dx) \\ &\geq \int_{\mathbb{R}^n_+} \left[\frac{v(x)-v(s_0)}{|x-s_0|^{n+\alpha}} - \frac{v(x)+v(s_0)}{|x-\widehat{s}_0|^{n+\alpha}}\right] m_n(dx) =: I_1.\end{aligned}$$

We used above the equalities $v(\widehat{x}) = -v(x) + 2\varepsilon U_\alpha^\lambda(x)$, $U_\alpha^\lambda(\widehat{x}) = U_\alpha^\lambda(x)$, and $|x-\widehat{s}_0| = |\widehat{x}-s_0|$, which come from symmetry. Now we set $A_1 = \{x \in \mathbb{R}^n_+ : v(x)+v(s_0) \geq 0\}$ and $A_2 = \{x \in \mathbb{R}^n_+ : v(x)+v(s_0) < 0\}$. Using also the obvious inequality $|x-\widehat{s}_0| > |x-s_0|$, we get

$$\begin{aligned} I_1 &= \int_{A_1} \left[\frac{v(x)-v(s_0)}{|x-s_0|^{n+\alpha}} - \frac{v(x)+v(s_0)}{|x-\widehat{s}_0|^{n+\alpha}}\right] m_n(dx) \\ &\quad + \int_{A_2} \left[\frac{v(x)-v(s_0)}{|x-s_0|^{n+\alpha}} - \frac{v(x)+v(s_0)}{|x-\widehat{s}_0|^{n+\alpha}}\right] m_n(dx) \\ &\geq \int_{A_1} \left[\frac{v(x)-v(s_0)}{|x-s_0|^{n+\alpha}} - \frac{v(x)+v(s_0)}{|x-s_0|^{n+\alpha}}\right] m_n(dx) + \int_{A_2} \frac{v(x)-v(s_0)}{|x-s_0|^{n+\alpha}}\, m_n(dx) \\ &= \int_{A_1} \frac{-2v(s_0)}{|x-s_0|^{n+\alpha}}\, m_n(dx) + \int_{A_2} \frac{v(x)-v(s_0)}{|x-s_0|^{n+\alpha}}\, m_n(dx).\end{aligned}$$

Since $v(s_0) < 0$, the first integrand is positive. The second integrand is non-negative; indeed, if $x \in \mathbb{R}^n{}_+ \setminus S_+$, then $v(x) - v(s_0) = f_k(x) + \varepsilon U_\alpha^\lambda(x) - v(s_0) \geq 0$, and if $x \in S_+$, then $v(x) - v(s_0) \geq 0$, by the definition of s_0. Because of (3.4), we conclude that $m_n(A_1) = 0$ and $v = v(s_0)$ a.e. in A_2. Hence $v = v(s_0) < 0$ a.e. in $\mathbb{R}^n_+$.

We proved above that the function v is equal to a negative constant a.e. in $\mathbb{R}^n_+$. This is absurd; indeed: (a) if $m_n(\mathbb{R}^n_+ \setminus S_+) > 0$ and $x \in \mathbb{R}^n_+ \setminus S_+$, then

$v(x) = f_k(x) + \varepsilon U_\alpha^\lambda(x) \ge 0$, (b) if $m_n(\mathbb{R}^n_+ \setminus S_+) = 0$, then S is unbounded and, by (3.3), $\liminf_{S_+ \ni s \to \infty} v(s) \ge 0$.

The contradiction shows that $v(s) \ge 0$ for all $s \in S_+$. Since $\varepsilon > 0$ is arbitrary, (3.1) is proved. ∎

THEOREM 2. *Let D be an open set in $\mathbb{R}^n$. Suppose that D is polarized with respect to the plane Π. Let $B \subset \mathbb{R}^n_+ \cap D^c$ be a Borel set. Then*:

(i) $\omega_\alpha^D(x,B) \ge \omega_\alpha^D(\widehat{x},B)$, $x \in \mathbb{R}^n_+ \cup \Pi$;
(ii) $\omega_\alpha^D(x,B) \ge \omega_\alpha^D(x,\widehat{B})$, $x \in \mathbb{R}^n_+ \cup \Pi$;
(iii) $\omega_\alpha^D(x,B) + \omega_\alpha^D(\widehat{x},B) \ge \omega_\alpha^D(x,\widehat{B}) + \omega_\alpha^D(\widehat{x},\widehat{B})$, $x \in \mathbb{R}^n$;
(iv) $\omega_\alpha^D(x,B) + \omega_\alpha^D(x,\widehat{B}) \ge \omega_\alpha^D(\widehat{x},B) + \omega_\alpha^D(\widehat{x},\widehat{B})$, $x \in \mathbb{R}^n$.

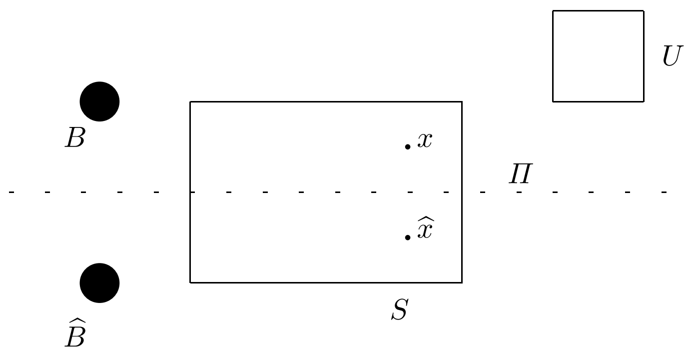

Fig. 2. An illustration for Theorem 2

Proof. Since D is polarized, the lower non-symmetric part of D is empty. Hence $D = S \cup U$, where S is the symmetric part of D, and U is the upper non-symmetric part of D.

(i) If $x \in (\mathbb{R}^n_+ \cup \Pi) \setminus S_+$, the inequality (i) is trivial. So we assume that $x = s \in S_+$. By the strong Markov property,

$$\omega_\alpha^D(s,B) = \omega_\alpha^S(s,B) + \int_U \omega_\alpha^S(s,du)\,\omega_\alpha^D(u,B),$$
$$\omega_\alpha^D(\widehat{s},B) = \omega_\alpha^S(\widehat{s},B) + \int_U \omega_\alpha^S(\widehat{s},du)\,\omega_\alpha^D(u,B).$$

By Theorem 1, $\omega_\alpha^S(s,B) \ge \omega_\alpha^S(\widehat{s},B)$ and $\omega_\alpha^S(s,du) \ge \omega_\alpha^S(\widehat{s},du)$. So (i) is proved.

(ii) As in the proof of (i), we may assume that $x = s \in S_+$. Set $S_1 := S \cup U \cup \widehat{U}$. Then S_1 is an open set which is symmetric with respect to Π and contains D. By the strong Markov property,

$$\omega_\alpha^D(s,B) = \omega_\alpha^{S_1}(s,B) - \int\limits_{\widehat{U}} \omega_\alpha^D(s,du)\,\omega_\alpha^{S_1}(u,B),$$
$$\omega_\alpha^D(s,\widehat{B}) = \omega_\alpha^{S_1}(s,\widehat{B}) - \int\limits_{\widehat{U}} \omega_\alpha^D(s,du)\,\omega_\alpha^{S_1}(u,\widehat{B}).$$

By Theorem 1, $\omega_\alpha^{S_1}(s,B) \geq \omega_\alpha^{S_1}(s,\widehat{B})$ and $\omega_\alpha^{S_1}(u,\widehat{B}) \geq \omega_\alpha^{S_1}(u,B), u \in \widehat{U}$. So (ii) is proved.

(iii) By the inner regularity of α-harmonic measure, we may and do assume that B is a compact set in $\mathbb{R}^n_+ \cap D^{\mathrm{c}}$. Take a decreasing sequence of continuous functions $f_k : D^{\mathrm{c}} \to [0,1]$ with $\mathrm{supp} f_k \downarrow B$, $f_k \downarrow \chi_B$ and $f_k = 0$ in $(D^{\mathrm{c}})_-$. Let $\widehat{f_k}(x) = f_k(\widehat{x}), x \in D^{\mathrm{c}}$ (with $\widehat{f_k} = 0$ in $\widehat{U}$). Consider the sequences of functions

$$H_{f_k}(x) := \int\limits_{D^{\mathrm{c}}} f_k(y)\,\omega_\alpha^D(x,dy), \qquad x \in \mathbb{R}^n,$$
$$H_{\widehat{f_k}}(x) := \int\limits_{D^{\mathrm{c}}} \widehat{f_k}(y)\,\omega_\alpha^D(x,dy), \qquad x \in \mathbb{R}^n.$$

We have $H_{f_k}(x) \downarrow \omega_\alpha^D(x,B)$ and $H_{\widehat{f_k}}(x) \downarrow \omega_\alpha^D(x,B)$, $x \in \mathbb{R}^n$. Therefore it suffices to prove that

$$H_{f_k}(x) + H_{f_k}(\widehat{x}) \geq H_{\widehat{f_k}}(x) + H_{\widehat{f_k}}(\widehat{x}), \qquad x \in \mathbb{R}^n,\ k \in \mathbb{N}.$$

Fix $k \in \mathbb{N}$ and define

$$v(x) = H_{f_k}(x) + H_{f_k}(\widehat{x}) - H_{\widehat{f_k}}(x) - H_{\widehat{f_k}}(\widehat{x}), \qquad x \in \mathbb{R}^n.$$

It is clear that v is α-harmonic in S. Note that for $u \in U$, $v(u) = H_{f_k}(u) - H_{\widehat{f_k}}(u)$. So v is α-harmonic in D. It is also continuous in $\overline{D} \setminus E$, where E is the set of irregular points of ∂D. We will apply the minimum principle (Lemma 5) to the function v in the domain D.

Let $\zeta \in D^{\mathrm{c}}$.

Case 1: $\zeta \in \partial D \setminus (E \cup \widehat{U})$. Then

$$\lim_{D \ni x \to \zeta} v(x) = f_k(\zeta) + f_k(\widehat{\zeta}) - \widehat{f_k}(\zeta) - \widehat{f_k}(\widehat{\zeta}) = 0.$$

Case 2: $\zeta \in (\partial D \cap \widehat{U}) \setminus E$. Then

$$\begin{aligned}\lim_{D \ni x \to \zeta} v(x) &= f_k(\zeta) + H_{f_k}(\widehat{\zeta}) - \widehat{f_k}(\zeta) - H_{\widehat{f_k}}(\widehat{\zeta}) = H_{f_k}(\widehat{\zeta}) - H_{\widehat{f}}(\widehat{\zeta}) \\ &= \int\limits_{D^{\mathrm{c}}} f_k(y)\,\omega_\alpha^D(\widehat{\zeta},dy) - \int\limits_{D^{\mathrm{c}}} \widehat{f_k}(y)\,\omega_\alpha^D(\widehat{\zeta},dy) \\ &= \int\limits_{D^{\mathrm{c}}} f_k(y)\,\omega_\alpha^D(\widehat{\zeta},dy) - \int\limits_{D^{\mathrm{c}}} f_k(y)\,\omega_\alpha^D(\widehat{\zeta},\widehat{dy}) \\ &= \int\limits_{(D^{\mathrm{c}})_+} f_k(y)\,[\omega_\alpha^D(\widehat{\zeta},dy) - \omega_\alpha^D(\widehat{\zeta},\widehat{dy})] \geq 0.\end{aligned}$$

Here $\omega_\alpha^D(\widehat{\zeta},\widehat{dy})$ is the measure μ on $(D^{\mathrm{c}})_+$ defined by $\mu(E):=\omega_\alpha^D(\widehat{\zeta},\widehat{E})$. The last equality holds because f_k is supported in $(D^{\mathrm{c}})_+$. The inequality comes from part (ii) of Theorem 2.

CASE 3: $\zeta\in(\overline{D})^{\mathrm{c}}\setminus\widehat{U}$. Then $v(\zeta)=f_k(\zeta)+f_k(\widehat{\zeta})-\widehat{f}_k(\zeta)-\widehat{f}_k(\widehat{\zeta})=0$.

CASE 4: $x=u\in\widehat{U}\setminus\partial D$. Then we work as in Case 2.

By Lemma 5, we conclude that $v\geq 0$ on D.

(iv) The proof is similar to that of (iii). ∎

THEOREM 3. *Let D be an open set in $\mathbb{R}^n$. Suppose that D is polarized with respect to the plane Π. Let $B\subset\mathbb{R}^n_+\cap D^{\mathrm{c}}$ be a Borel set. Then*:

(i) $\omega_\alpha^D(x,B)\leq 1/2$, $x\in D_-\cup D_0$;
(ii) $\omega_\alpha^D(x,\widehat{B})\leq 1/2$, $x\in D_+\cup D_0$;
(iii) $\omega_\alpha^{\widehat{D}}(x,B)\leq 1/2$, $x\in(\widehat{D})_-\cup D_0$.

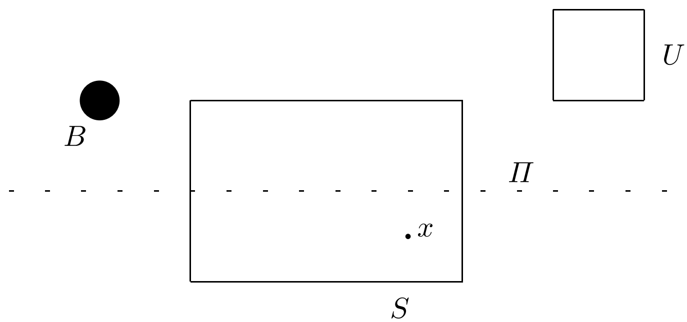

Fig. 3. An illustration for Theorem 3

Proof. We will prove only the inequality (ii). The proof of (i) is similar and (iii) is equivalent to (ii) because of symmetry.

We write $D=S\cup U$, where S is the symmetric part of D and U is the upper non-symmetric part of D. Set $S_1:=D\cup\widehat{U}$. Then S_1 is an open set, symmetric with respect to Π, and $D\subset S_1$. Using Theorem 1 we obtain

$$\omega_\alpha^D(x,\widehat{B})\leq\omega_\alpha^{S_1}(x,\widehat{B})\leq\omega_\alpha^{S_1}(x,B),\qquad x\in D_+\cup D_0.$$

Hence

$$\omega_\alpha^D(x,\widehat{B})\leq\frac{1}{2}\,[\omega_\alpha^{S_1}(x,\widehat{B})+\omega_\alpha^{S_1}(x,B)]=\frac{1}{2}\,\omega_\alpha^{S_1}(x,B\cup\widehat{B})\leq\frac{1}{2}.\ ∎$$

We now turn to a sharp form of Theorem 2.

THEOREM 4. *Let D be an open set in $\mathbb{R}^n$. Suppose that D is polarized with respect to the plane Π. Let $B\subset\mathbb{R}^n_+\cap D^{\mathrm{c}}$ be a Borel set which is not*

D-null. Then for $x \in D_+$, we have

$$\omega_\alpha^D(x,B) > \omega_\alpha^D(\widehat{x},B), \tag{3.5}$$

$$\omega_\alpha^D(x,B) > \omega_\alpha^D(x,\widehat{B}). \tag{3.6}$$

Proof. First we prove (3.5). We write $D = S \cup U$, where S is the symmetric part of D and U is the upper non-symmetric part of D. If $x = u \in U$, then $\omega_\alpha^D(u,B) > 0$ because B is not D-null. On the other hand, $\omega_\alpha^D(\widehat{u},B) = 0$ because $\widehat{u} \notin B$. Therefore (3.5) is proved in this case. So it remains to prove (3.5) for $x = s \in S_+$.

Consider the function

$$v(x) = \omega_\alpha^D(x,B) - \omega_\alpha^D(\widehat{x},B), \qquad x \in \mathbb{R}^n.$$

Then v is α-harmonic in D and by Theorem 2,

$$v(x) \geq 0, \qquad x \in \mathbb{R}^n_+. \tag{3.7}$$

Also, it is obvious that

$$v(x) + v(\widehat{x}) = 0, \qquad x \in \mathbb{R}^n_+. \tag{3.8}$$

We want to prove that

$$v(s) > 0, \qquad s \in S_+. \tag{3.9}$$

Suppose that $v(s_0) = 0$ for some $s_0 \in S_+$. Since v is α-harmonic in D,

$$\begin{aligned} 0 = \Delta^{\alpha/2} v(s_0) &= \int_{\mathbb{R}^n} \frac{v(x) - v(s_0)}{|x - s_0|^{n+\alpha}}\, m_n(dx) = \int_{\mathbb{R}^n} \frac{v(x)}{|x - s_0|^{n+\alpha}}\, m_n(dx) \\ &= \int_{\mathbb{R}^n_+} \left[\frac{v(x)}{|x - s_0|^{n+\alpha}} - \frac{v(x)}{|x - \widehat{s}_0|^{n+\alpha}} \right] m_n(dx) \\ &= I_1 + I_2 + I_3 + I_4, \end{aligned}$$

where

$$\begin{aligned} I_1 &:= \int_{S_+} v(s) \left[\frac{1}{|s - s_0|^{n+\alpha}} - \frac{1}{|s - \widehat{s}_0|^{n+\alpha}} \right] m_n(ds), \\ I_2 &:= \int_U v(u) \left[\frac{1}{|u - s_0|^{n+\alpha}} - \frac{1}{|u - \widehat{s}_0|^{n+\alpha}} \right] m_n(du) \\ &= \int_U \omega_\alpha^D(u,B) \left[\frac{1}{|u - s_0|^{n+\alpha}} - \frac{1}{|u - \widehat{s}_0|^{n+\alpha}} \right] m_n(du), \end{aligned}$$

$$
\begin{aligned}
I_3 &:= \int\limits_B v(x)\left[\frac{1}{|x-s_0|^{n+\alpha}} - \frac{1}{|x-\widehat{s}_0|^{n+\alpha}}\right] m_n(dx),\\
&= \int\limits_B \left[\frac{1}{|x-s_0|^{n+\alpha}} - \frac{1}{|x-\widehat{s}_0|^{n+\alpha}}\right] m_n(dx),\\
I_4 &:= \int\limits_{(D_+)^{\mathrm{c}}\setminus B} v(x)\left[\frac{1}{|x-s_0|^{n+\alpha}} - \frac{1}{|x-\widehat{s}_0|^{n+\alpha}}\right] m_n(dx).
\end{aligned}
$$

Since $v = 0$ in $(D_+)^{\mathrm{c}} \setminus B$, we have $I_4 = 0$. Because of the obvious inequality

$$|x - s_0| < |x - \widehat{s}_0|, \qquad x \in \mathbb{R}^n_+,$$

the integrands in I_1, I_2, I_3 are non-negative. Therefore $I_1 = I_2 = I_3 = 0$. We conclude that $m_n(U) = 0$, $m_n(B) = 0$ and $v = 0$ m_n-a.e. in S. Since v is continuous in D, we conclude that $v = 0$ in S, which means that

$$\omega_\alpha^D(s, B) = \omega_\alpha^D(\widehat{s}, B), \qquad s \in S. \tag{3.10}$$

The fact that $m_n(B) = 0$ implies that (see [4], [15]) the set $B \cap (\overline{D})^{\mathrm{c}}$ is D-null; hence the set $B \cap \partial D$ is not D-null. Thus, by [15, Lemma 1], we have

$$\sup_{x\in D} \omega_\alpha^D(x, B) = 1.$$

Take a sequence $\{x_k\}$ in D such that

$$\lim_{k\to\infty} \omega_\alpha^D(x_k, B) = 1. \tag{3.11}$$

By Theorem 3, we may assume that $\{x_k\} \subset D_+$. Since D_+ is an open set and $m_n(U) = 0$, every neighborhood of x_k contains a point $s_k \in S_+$, $k \in \mathbb{N}$. So, by the continuity of α-harmonic measure in D, we can choose a sequence s_k in S_+ such that

$$\lim_{k\to\infty} \omega_\alpha^D(s_k, B) = 1. \tag{3.12}$$

Then, again by Theorem 3,

$$\limsup_{k\to\infty} \omega_\alpha^D(\widehat{s}_k, B) \le \frac{1}{2}.$$

This together with (3.12) contradicts (3.10). So (3.9) is proved.

We now turn to the proof of (3.6). We consider the function

$$h(x) = \omega_\alpha^D(x, B) - \omega_\alpha^D(x, \widehat{B}), \qquad x \in \mathbb{R}^n.$$

We know from Theorem 2 that

$$h(x) \ge 0, \quad h(x) + h(\widehat{x}) \ge 0, \qquad x \in \mathbb{R}^n_+. \tag{3.13}$$

We want to prove that

$$h(x) > 0, \quad x \in D_+. \tag{3.14}$$

Suppose that $h(x_0) = 0$ for some $x_0 \in D_+$. Since h is α-harmonic in D,

$$\begin{aligned}
0 = \Delta^{\alpha/2} h(x_0) &= \int_{\mathbb{R}^n} \frac{h(x) - h(x_0)}{|x - x_0|^{n+\alpha}}\, m_n(dx) \\
&= \int_{\mathbb{R}^n_+} \frac{h(x)}{|x - x_0|^{n+\alpha}}\, m_n(dx) + \int_{\mathbb{R}^n_-} \frac{h(x)}{|x - x_0|^{n+\alpha}}\, m_n(dx) \\
&= \int_{\mathbb{R}^n_+} \frac{h(x)}{|x - x_0|^{n+\alpha}}\, m_n(dx) + \int_{\mathbb{R}^n_+} \frac{h(\widehat{x})}{|\widehat{x} - x_0|^{n+\alpha}}\, m_n(dx) \\
&= \int_{\mathbb{R}^n_+} \left\{ \frac{h(x) + h(\widehat{x})}{|x - \widehat{x}_0|^{n+\alpha}} + h(x)\left[\frac{1}{|x - x_0|^{n+\alpha}} - \frac{1}{|x - \widehat{x}_0|^{n+\alpha}} \right] \right\} m_n(dx) =: J.
\end{aligned}$$

As in the proof of (3.5), we find that $J = J_1 + J_2 + J_3$, where

$$\begin{aligned}
J_1 &:= \int_{S_+} \left\{ \frac{h(s) + h(\widehat{s})}{|x - \widehat{x}_0|^{n+\alpha}} + h(s)\left[\frac{1}{|s - x_0|^{n+\alpha}} - \frac{1}{|s - \widehat{x}_0|^{n+\alpha}} \right] \right\} m_n(ds), \\
J_2 &:= \int_U \frac{h(u)}{|u - x_0|^{n+\alpha}}\, m_n(du), \\
J_3 &:= \int_B \left[\frac{1}{|x - x_0|^{n+\alpha}} - \frac{1}{|x - \widehat{x}_0|^{n+\alpha}} \right] m_n(dx).
\end{aligned}$$

Using (3.13) we conclude that $m_n(B) = 0$ and that $v = 0$ in S, which means that

$$\omega^D_\alpha(s, B) = \omega^D_\alpha(s, \widehat{B}), \qquad s \in S. \tag{3.15}$$

By (3.15) and the fact that B is not D-null we infer that $\widehat{B}$ is not D-null. As $m_n(\widehat{B}) = m_n(B) = 0$, the set $\widehat{B} \cap \partial D$ is not D-null. By [15, Lemma 1], we thus have

$$\sup_{x \in D} \omega^D_\alpha(x, \widehat{B}) = 1.$$

Take a sequence $\{y_k\}$ in D with $\omega^D_\alpha(y_k, \widehat{B}) \to 1$. As $\widehat{B} \subset \mathbb{R}^n_-$, Theorem 3 implies that we may assume $y_k \in D_- = S_-, k \in \mathbb{N}$. Then (3.15) gives $\omega^D_\alpha(y_k, B) \to 1$. But Theorem 3 implies $\omega^D_\alpha(y_k, B) \le 1/2$. This contradiction proves (3.14). ∎

THEOREM 5. *Let D be an open set in $\mathbb{R}^n$. Suppose that D is polarized with respect to the hyperplane Π, i.e. $D = S \cup U$, where S is the symmetric part of D and U is the upper non-symmetric part of D. Let $B \subset \mathbb{R}^n_+ \cap D^c$ be a Borel set which is not D-null. Then we have*:

$$\omega^D_\alpha(s_0, B) + \omega^D_\alpha(\widehat{s}_0, B) = \omega^D_\alpha(s_0, \widehat{B}) + \omega^D_\alpha(\widehat{s}_0, \widehat{B}) \tag{3.16}$$

for some $s_0 \in S$ *if and only if* $C_\alpha(U) = 0$;

$$\omega_\alpha^D(s_0, B) + \omega_\alpha^D(s_0, \widehat{B}) = \omega_\alpha^D(\widehat{s}_0, B) + \omega_\alpha^D(\widehat{s}_0, \widehat{B}) \tag{3.17}$$

for some $s_0 \in S$ *if and only if* $C_\alpha(U) = 0$;

$$\omega_\alpha^D(s_0, B) = \omega_\alpha^D(s_0, \widehat{B}) \tag{3.18}$$

for some $s_0 \in S_0 := S \cap \Pi$ *if and only if* $C_\alpha(U) = 0$.

Proof. We only prove the first equivalence; the proofs of the remaining ones are similar.

Suppose that (3.16) holds for some $s_0 \in S$. By the strong Markov property,

$$\omega_\alpha^D(s_0, B) = \omega_\alpha^S(s_0, B) + \int_U \omega_\alpha^S(s_0, du)\, \omega_\alpha^D(u, B), \tag{3.19}$$

$$\omega_\alpha^D(\widehat{s}_0, B) = \omega_\alpha^S(\widehat{s}_0, B) + \int_U \omega_\alpha^S(\widehat{s}_0, du)\, \omega_\alpha^D(u, B), \tag{3.20}$$

$$\omega_\alpha^D(s_0, \widehat{B}) = \omega_\alpha^S(s_0, \widehat{B}) + \int_U \omega_\alpha^S(s_0, du)\, \omega_\alpha^D(u, \widehat{B}), \tag{3.21}$$

$$\omega_\alpha^D(\widehat{s}_0, \widehat{B}) = \omega_\alpha^S(\widehat{s}_0, \widehat{B}) + \int_U \omega_\alpha^S(\widehat{s}_0, du)\, \omega_\alpha^D(u, \widehat{B}). \tag{3.22}$$

Hence

$$\int_U [\omega_\alpha^S(s_0, du) + \omega_\alpha^S(\widehat{s}_0, du)]\, \omega_\alpha^D(u, B) = \int_U [\omega_\alpha^S(s_0, du) + \omega_\alpha^S(\widehat{s}_0, du)]\, \omega_\alpha^D(u, \widehat{B}).$$

By Theorem 4, $\omega_\alpha^D(u, B) > \omega_\alpha^D(u, \widehat{B})$ for all $u \in U$. Hence $\omega_\alpha^S(s_0, du) + \omega_\alpha^S(\widehat{s}_0, du)$ is the zero measure on U. This implies $\omega_\alpha^S(s_0, U) = 0$, i.e. U is S-null. By Lemma 3, $C_\alpha(U) = 0$.

Conversely, if $C_\alpha(U) = 0$, then U is S-null. Therefore (3.19)–(3.22) imply

$$\omega_\alpha^D(s, B) + \omega_\alpha^D(\widehat{s}, B) = \omega_\alpha^D(s, \widehat{B}) + \omega_\alpha^D(\widehat{s}, \widehat{B})$$

for all $s \in S$. ∎

THEOREM 6. *Let D be an open set in $\mathbb{R}^n$. Suppose that D is polarized with respect to the hyperplane Π, i.e. $D = S \cup U$, where S is the symmetric part of D and U is the upper non-symmetric part of D. Let $B \subset D^{\mathrm{c}}$ be a Borel set which is symmetric with respect to Π and is not D-null. Then*

$$\omega_\alpha^D(s, B) = \omega_\alpha^D(\widehat{s}, B)$$

for some $s \in S_+$ *if and only if* $C_\alpha(U) = 0$.

Proof. Similar to the proof of Theorem 5.

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Department of Mathematics
Aristotle University of Thessaloniki
54124 Thessaloniki, Greece
E-mail: betsakos@math.auth.gr

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