COLLOQUIUM MATHEMATICUM

VOL. 111 2008 NO. 2

$C(X)$ VS. $C(X)$ MODULO ITS SOCLE

BY

F. AZARPANAH, O. A. S. KARAMZADEH and S. RAHMATI (Ahvaz)

Abstract. Let $C_F(X)$ be the socle of $C(X)$. It is shown that each prime ideal in $C(X)/C_F(X)$ is essential. For each $h \in C(X)$, we prove that every prime ideal (resp. z-ideal) of $C(X)/(h)$ is essential if and only if the set $Z(h)$ of zeros of h contains no isolated points (resp. $\operatorname{int} Z(h) = \emptyset$). It is proved that $\dim(C(X)/C_F(X)) \geq \dim C(X)$, where $\dim C(X)$ denotes the Goldie dimension of $C(X)$, and the inequality may be strict. We also give an algebraic characterization of compact spaces with at most a countable number of nonisolated points. For each essential ideal E in $C(X)$, we observe that $E/C_F(X)$ is essential in $C(X)/C_F(X)$ if and only if the set of isolated points of X is finite. Finally, we characterize topological spaces X for which the Jacobson radical of $C(X)/C_F(X)$ is zero, and as a consequence we observe that the cardinality of a discrete space X is nonmeasurable if and only if υX, the realcompactification of X, is first countable.

Introduction. Let $C(X)$ be the ring of real-valued continuous functions on an infinite completely regular Hausdorff space X, and $C^\star(X)$ be its subring of bounded functions. The socle of $C(X)$, denoted by $C_F(X)$, is the sum of all minimal ideals of $C(X)$, which is the intersection of all essential ideals in $C(X)$ (recall that, an ideal is essential if it intersects every nonzero ideal nontrivially); see [15]. It can be easily seen that $C_F(X) \neq 0$ if and only if X has isolated points, and when the set of isolated points in X is finite, then $C(X)/C_F(X) \cong C(Y)$, where Y is the set of nonisolated points of X. This implies that in this case $C(X)/C_F(X) \cong eC(X)$, where $e^2 = e \in C(X)$, as a ring will enjoy all the general algebraic properties of $C(X)$. Although in general $C(X)/C_F(X)$ may not be isomorphic to $C(Y)$ for any topological space Y, in any case, one encounters a curious similarity between the two rings $C(X)$ and $C(X)/C_F(X)$, and their common properties usually give rise to useful information about X. For example, X is a P-space (resp. an extremally disconnected P-space with only a finite number of isolated points) if and only if $C(X)$ or equivalently $C(X)/C_F(X)$ is an $\aleph_0$-self-injective (resp. self-injective) ring (see [11, Theorems 1, 2 and Lemma 3.1]). Neither of the two partially ordered rings $C(X)$ and $C(X)/C_F(X)$ (note that $C_F(X)$ is

2000 *Mathematics Subject Classification*: Primary 54C40; Secondary 13A30, 16P20.

Key words and phrases: Goldie dimension, Suslin number, socle, essential ideal, Ue-ring, z-ideal, isolated point.

convex; see [14, Theorem 5.2]) can be totally ordered, by [14, 5.4(c)] (note that $C_F(X)$ is never a prime ideal in $C(X)$; see [4], [11, Proposition 1.2]).

Our main aim in this article is to reveal more important links between these two rings and apply these links to get information about X. An outline of this article is as follows: After recalling some preliminary results in Section 1, which are frequently used in the subsequent sections, the next section deals with the behaviour of essential ideals in the factor ring $C(X)/C_F(X)$. In particular, it is shown that each prime ideal of $C(X)/C_F(X)$ is essential and the topological spaces X such that for each essential ideal E, $E/C_F(X)$ is essential in $C(X)/C_F(X)$, are characterized. The final part of Section 2 shows that the Goldie dimension of the latter ring is not smaller than that of $C(X)$. Motivated by this result and the fact that only recently the concept of infinite Goldie dimensions has received some attention (see [9]), we construct various examples of rings with arbitrary Goldie dimensions. We also observe that given any inaccessible cardinal number λ, there exists a zero-dimensional compact space X such that the Goldie dimension of $C(X)$ is λ, but unattained. The comparability of the dimensions of the two rings shows that $C(X)/C_F(X)$ as well as $C(X)$ avoids any natural finiteness conditions.

In Section 3, we study z-ideals in $C(X)/C_F(X)$ and show that, similar to $C(X)$, every ideal and its radical in $C(X)/C_F(X)$ have the same largest z-ideal. It is also shown that each countably generated ideal in $C(X)$ is essential in a principal ideal of $C(X)$, which generalizes Corollary 2.2 in [21] and immediately implies that each countably generated fixed maximal ideal in $C(X)$ is generated by an idempotent, which is well-known (see [12]). When the set of isolated points in X is finite, then $C(X)/C_F(X)$ is not different from $C(X)$, i.e., it is isomorphic to some $C(Y)$. We note that in this case $C_F(X)$ is a principal ideal. Motivated by this, we digress for a moment, and in Section 4 consider the ring $C(X)/(h)$, where (h) is an arbitrary principal ideal in $C(X)$. We prove that every prime ideal (resp. z-ideal) in $C(X)/(h)$ is essential if and only if $Z(h)$ contains no isolated point (resp. $\operatorname{int} Z(h) = \emptyset$). These facts immediately give new information about almost P-spaces.

Finally, in Section 5, we revisit Ue-rings and Uem-rings which have been introduced and systematically studied in [16]. A ring R is an Ue-ring (resp. Uem-ring) if R has a unique proper essential (resp. essential maximal) ideal. It is observed that $C(X)/C_F(X)$ and $C(X)$ are never Ue-rings. More generally, neither $C(X)$ nor $C(X)/C_F(X)$ has only a finite number of essential ideals. It is shown that either $C(X)$ or $C(X)/C_F(X)$ contains only a finite (resp. countable) number of essential maximal ideals if and only if X is a compact space with only a finite (resp. countable) number of nonisolated points. As a consequence, we show that being compact with at most a countable number of nonisolated points is an algebraic property. To conclude the final section, we note that if X is the one-point compactification of a discrete space, then $C(X)/C_F(X)$ is a local ring which is not a field (see [16]), i.e., its Jacobson radical, $J(C(X)/C_F(X))$, is not zero. Motivated by this fact, we characterize the topological spaces X such that $J(C(X)/C_F(X)) = (0)$.

1. Background and preliminary results. Let R be a commutative ring with unity and $A \subseteq B$ be two ideals in R. Then A is said to be *essential* in B if $A \cap (b) \neq (0)$ for all $0 \neq b \in B$. When we just say A is an essential ideal, we mean it is essential in R. We define the *annihilator* of B to be $\mathrm{Ann}(B) = \{r \in R : Br = 0\}$. It is trivial to see that if I is an ideal in a reduced ring R, i.e., $a^n = 0$ implies that $a = 0$ for all $a \in R$, then $I \oplus \mathrm{Ann}(I)$ is an essential ideal in R and therefore I is essential if and only if $\mathrm{Ann}(I) = (0)$. A set $\{I_\alpha\}_{\alpha \in S}$ of nonzero ideals in a ring R is said to be *independent* if $I_\beta \cap \sum_{\beta \neq \alpha \in S} I_\alpha = (0)$, i.e., $\sum_{\alpha \in S} I_\alpha = \bigoplus_{\alpha \in S} I_\alpha$.

The *Goldie dimension* of an ideal I of R, denoted by $\dim I$, is the smallest cardinal c such that every independent set of nonzero ideals in I has cardinality less than or equal to c. The smallest cardinal α such that every family of pairwise disjoint nonempty open subsets of X has cardinality less than or equal to α is called the *Suslin number* or the *cellularity* of X and is denoted by $\mathcal{S}(X)$ (see [10] and [23]). Clearly, if I is essential in R, then $\dim I = \dim R$. An ideal U in a ring R is *uniform* if every nonzero ideal in U is essential in U.

For any a in a ring R, the intersection of all maximal ideals containing a is denoted by M_a, and an ideal I is called a *z-ideal* if $M_a \subseteq I$ for every $a \in I$ (see [14, 4A]). Whenever $f \in C(X)$, it is easy to see that $M_f = \{g \in C(X) : Z(f) \subseteq Z(g)\}$, where $Z(h)$ denotes the set of zeros of h. Hence an ideal I in $C(X)$ is a z-ideal if $f \in I$, $g \in C(X)$ and $Z(f) \subseteq Z(g)$ imply that $g \in I$. Clearly M_f itself is a z-ideal. If $S \subseteq X$, then $M_S = \{f \in C(X) : S \subseteq Z(f)\}$ is also a z-ideal. In particular, whenever $S = Z(f)$ for some $f \in C(X)$, then $M_S = M_{Z(f)} = M_f$, and if $S = \{x\}$, then $M_S = M_x$ is a maximal ideal.

A point $x \in X$ is said to be an *almost P-point* if $\mathrm{int}\, Z(f) \neq \emptyset$ for every $f \in M_x$, and X is called an *almost P-space* if every point of X is an almost P-point. We refer the reader to [1], [20] and [22] for more details and properties of almost P-spaces and to [14] and [19] for undefined terms, notations and general information about the algebraic properties of $C(X)$.

We cite the following two results which will be frequently referred to below. The socle $C_F(X)$ of $C(X)$ was first characterized via the following proposition in [17], where it was shown that $C_F(X)$ is essential in $C(X)$ if and only if the set of isolated points of X is dense in X (see also [2] and [3]). The second result, which is in [2], gives a useful connection between $\mathcal{S}(X)$ and $\dim C(X)$.

PROPOSITION 1.1. *The socle $C_F(X)$ of $C(X)$ is a z-ideal consisting of all functions that vanish everywhere except on a finite subset of X.*

PROPOSITION 1.2.

(1) *An ideal U is a uniform ideal in $C(X)$ if and only if U is a minimal ideal.*
(2) $\dim C(X) = \mathcal{S}(X)$.

It is well-known that $x \in X$ is nonisolated if and only if M_x is an essential maximal ideal in $C(X)$ (see [17, Proposition 1.1]. The following is a generalization of this fact.

PROPOSITION 1.3. *Let A be a closed set in X. Then for each $x \in A$, the maximal ideal M_x/M_A is an essential ideal in $C(X)/M_A$ if and only if x is a limit point of A.*

Proof. Suppose that M_x/M_A is essential for some $x \in A$. Pick $f \in C(X)$ such that $x \in \mathrm{Coz}(f) = X \setminus Z(f)$. Then $f \notin M_x$ and hence $(f) \cap (M_x \setminus M_A) \neq \emptyset$. This implies that there exists $g \in C(X)$ such that $fg \in M_x$ and $A \nsubseteq Z(fg)$, i.e., $x \in Z(g)$ and $\emptyset \neq A \cap \mathrm{Coz}(fg) \neq \{x\}$; a fortiori $\emptyset \neq A \cap \mathrm{Coz}(f) \neq \{x\}$. Therefore x is a limit point of A, for the collection of cozero sets is a base for open sets.

Conversely, let $x \in A$ be a limit point of A and suppose that M_x/M_A is not essential. Then there exists $f \notin M_x$ such that $(f) \cap (M_x \setminus M_A) = \emptyset$. Since $x \notin Z(f)$, we have $A \cap \mathrm{Coz}(f) \neq \emptyset$. Let $x \neq y \in \mathrm{Coz}(f) \cap A$ and choose $g \in C(X)$ such that $g(x) = 0$ and $g(y) = 1$. Then $fg \in M_x$ and $A \nsubseteq Z(fg)$, for $f(y)g(y) \neq 0$. This implies that $fg \in M_x \setminus M_A$, which is absurd. ∎

EXAMPLE 1.4. Let $X = \mathbb{R}$ and $A = [0,1] \cup \{2\}$. Since $x = 2$ is not the limit point of A, M_2/M_A is not essential in $C(\mathbb{R})/M_A$, but clearly M_2 is essential in $C(\mathbb{R})$.

The next characterization of essential ideals in $C(X)$ shows that all free ideals and certain fixed ideals are always essential (see [2, Theorem 3.1]).

PROPOSITION 1.5. *Let E be a nonzero ideal in $C(X)$. Then the following are equivalent*:

(1) *E is essential in $C(X)$.*
(2) $\mathrm{Ann}(E) = (0)$.
(3) *E intersects every nonzero z-ideal in $C(X)$ nontrivially.*
(4) *$Z(E) = \bigcap_{f \in E} Z(f)$ has empty interior.*

It is well-known and very easy to prove that in any ring R, every prime ideal is either essential or a minimal prime ideal which is also an annihilator ideal. The following result (see the proof of Corollary 3.3 in [2]) shows that in $C(X)$, we have a much stronger result.

PROPOSITION 1.6. *Let P be a prime ideal in $C(X)$. Then either P is essential or $P = eC(X)$, where e is an idempotent in $C(X)$ (i.e., P is a maximal, minimal prime ideal and at the same time an annihilator ideal).*

Finally, we cite the following lemmas from [5]. The proof of the first one is evident and the second one is Sublemma 3.2 in [5].

LEMMA 1.7. *If R is a reduced ring and $A \subseteq B$ are ideals in R, then A is essential in B if and only if* $\mathrm{Ann}(A) = \mathrm{Ann}(B)$.

LEMMA 1.8. *If $S \subseteq C(X)$ and $\bigcap_{f\in S} Z(f) = Z(g)$, where $g \in C(X)$, then* $\mathrm{Ann}(S) = \mathrm{Ann}(g)$.

2. Essential ideals in $C(X)$ modulo its socle. Whenever E/I is essential in R/I, it is trivial to see that E is essential in R, but the converse is rarely true (see Example 1.4). Since every essential ideal of a ring R contains the socle of R, it is natural to ask: is $E/\mathrm{soc}(R)$ an essential ideal in $R/\mathrm{soc}(R)$ for every essential ideal E of R? There is also the same natural question about prime ideals of $R/\mathrm{soc}(R)$: is every prime ideal of $R/\mathrm{soc}(R)$ an essential ideal? In this section, we answer these questions for $C(X)$ modulo its socle $C_F(X)$.

Essential ideals and the Goldie dimension of a ring R are two related concepts and we may have either $\dim R \le \dim(R/I)$ or $\dim(R/I) \le \dim R$, where I is an ideal of R. In this section we prove that $\dim(C(X)/C_F(X)) \ge \dim C(X)$ and present some useful examples.

Pegging a particular topological property to an algebraic property has always been of main interest to the workers in the area of $C(X)$. Perhaps the starting point here is the simple fact that X is connected if and only if $C(X)$ has no nontrivial idempotent, and so X is connected if and only if $\beta X(\upsilon X)$ is, where βX is the Stone–Čech compactification of X.

To start with, let us generalize this starting point, which determines the connectedness of $\beta X \setminus X$, where X is a discrete space.

PROPOSITION 2.1. *Let D be the set of isolated points of X. Then $\beta X \setminus D = Y$ is connected if and only if for each idempotent $\overline{f} = f + C_F(X)$ in $C^\star(X)/C_F(X)$ we have either $Y \subseteq Z(f^\beta)$ or $Y \subseteq Z(1 - f^\beta)$, where $f \mapsto f^\beta$ is the isomorphism of $C^\star(X)$ onto $C(\beta X)$.*

Proof. Clearly under the isomorphism $f \mapsto f^\beta$, the socle $C_F(X)$ is sent to the socle $C_F(\beta X)$ of $C(\beta X)$. Hence $C^\star(X)/C_F(X) \cong C(\beta X)/C_F(\beta X)$. Now assume that Y is connected and let $\overline{f} = f + C_F(X)$ be idempotent in $C^\star(X)/C_F(X)$, i.e., $f - f^2 \in C_F(X)$. This implies that $f^\beta - {f^\beta}^2 \in C_F(\beta X)$, which means that $Z(f^\beta - {f^\beta}^2) = \beta X \setminus A$, where $A \subseteq D$ is a finite set of isolated points of X. Now $\beta X = Z(f^\beta) \cup Z(1 - f^\beta) \cup A$, Y is connected and $Y \cap A = \emptyset$, i.e., $Y \subseteq Z(f^\beta)$ or $Y \subseteq Z(1 - f^\beta)$.

Conversely, suppose that Y is disconnected, i.e. $Y = X_1 \cup X_2$, where X_1 and X_2 are two disjoint nonempty closed subsets of Y. Then X_1, X_2 are compact subsets of βX, i.e., there are two disjoint open sets G_1 and G_2 containing X_1 and X_2, respectively. Now $\beta X \setminus (G_1 \cup G_2) \subseteq D$ is compact, i.e., $\beta X = G_1 \cup G_2 \cup H$, where $H \subseteq D$ is a finite set. We can now pick $f_1^\beta \in C(\beta X)$ such that $f_1^\beta(G_1) = \{0\}$ and $f_1^\beta(G_2 \bigcup H) = \{1\}$. We may assume that G_2 is infinite, i.e., $f_1^\beta \notin C_F(\beta X)$. Clearly f_1^β is an idempotent, i.e., $\bar{f}_1 = f_1 + C_F(X)$ is an idempotent in $C^\star(X)/C_F(X)$. Then $f_1^\beta(X_1) = \{0\}$ and $f_1^\beta(X_2) = \{1\}$ imply that $Y \nsubseteq Z(f_1^\beta)$ and $Y \nsubseteq Z(1 - f_1^\beta)$. ∎

In [11, Proposition 1.2], it is shown that every maximal ideal of $C(X)$ modulo its socle is essential. In the following theorem we generalize this result.

THEOREM 2.2. *Every prime ideal in $C(X)/C_F(X)$ is an essential ideal.*

Proof. Let P be a prime ideal in $C(X)$ containing $C_F(X)$, and $f \notin P$. First, suppose that there exist at least two nonisolated points in $\mathrm{Coz}(f)$. Let $x \in \mathrm{Coz}(f) \setminus Z(P)$ be a nonisolated point (note that $Z(P)$ is at most a singleton) and therefore there exists $g \in P$ such that $g(x) \neq 0$. Now $\mathrm{Coz}(f) \cap \mathrm{Coz}(g)$ is an open set containing the nonisolated point x, which is an infinite set, i.e., $fg \notin C_F(X)$. So $fg \in (P \setminus C_F(X)) \cap (f)$, i.e., $P/C_F(X)$ is essential in $C(X)/C_F(X)$. Next, as $f \notin C_F(X)$, $\mathrm{Coz}(f)$ is infinite, i.e., we may assume that all points of $\mathrm{Coz}(f)$ but at most one are isolated. Thus, we may choose two infinite disjoint sets $A = \{x_n\}_{n\in\mathbb{N}}$ and $B = \{y_n\}_{n\in\mathbb{N}}$ of isolated points in $\mathrm{Coz}(f)$. Now, we define

$$h(x) = \begin{cases} 0, & x \neq x_n, \\ 1/n, & x = x_n, \end{cases} \qquad k(x) = \begin{cases} 0, & x \neq y_n, \\ 1/n, & x = y_n. \end{cases}$$

Clearly $h, k \in C(X)$ (note that for all $\varepsilon > 0$, the set $\{x \in X : |h(x)| < \varepsilon\}$ is cofinite, i.e., it is open (even clopen); this means that h is continuous at each point of $X \setminus A$). But $hk = 0 \in P$ implies that either $h \in P$ or $k \in P$; say $h \in P$. Then $0 \neq hf \in (P \setminus C_F(X)) \cap (f)$, i.e., $P/C_F(X)$ is essential in $C(X)/C_F(X)$. ∎

THEOREM 2.3. *For every essential ideal E in $C(X)$, $E/C_F(X)$ is an essential ideal in $C(X)/C_F(X)$ if and only if the set of isolated points of X is finite.*

Proof. Let $A = \{x_1, \ldots, x_k\}$ be the set of isolated points of X and E be an essential ideal in $C(X)$. If $E/C_F(X)$ is not an essential ideal in $C(X)/C_F(X)$, then there exists $f \notin E$ such that $(f) \cap E \subseteq C_F(X)$. Now for each $g \in E$, $X \setminus (Z(f) \cup Z(g)) \subseteq A$ and hence $\emptyset \neq (X \setminus Z(f)) \setminus A \subseteq Z(g)$. So $\emptyset \neq (X \setminus Z(f)) \setminus A \subseteq Z(E)$, and by Proposition 1.5, E is not essential, a contradiction.

Conversely, suppose that for every essential ideal E in $C(X)$, $E/C_F(X)$ is an essential ideal in $C(X)/C_F(X)$ and the set of isolated points of X is infinite, say contains $\{x_1, x_2, \ldots\}$. Let H be the set of nonisolated points of X and consider two cases:

CASE 1: $\operatorname{int} H \neq \emptyset$. Set $A = X \setminus \operatorname{int} H$ and $F = \{f \in C(X) : A \setminus Z(f)$ is finite$\}$. Clearly $Z(F) \subseteq H \setminus \operatorname{int} H$ and hence $\operatorname{int} Z(F) = \emptyset$, i.e., F is an essential ideal in $C(X)$, by Proposition 1.5. Now choose $f \in C(X)$ with $f(A) = \{0\}$ and $f(x) \neq 0$ for some $x \in \operatorname{int} H$. Then $f \in F$, but $f \notin C_F(X)$, for f is different from zero at every point of a neighbourhood of x which is infinite. This shows that $F/C_F(X)$ is a non-trivial ideal in $C(X)/C_F(X)$. Now we show that the ideal $F/C_F(X)$ is not essential. Pick $g_n \in C(X)$ such that $g_n(x_n) > 0$ and $g_n(X \setminus \{x_n\}) = \{0\}$. Set

$$g = \sum_{n=1}^{\infty} \frac{1}{2^n}\,(g_n^2 \wedge 1).$$

Then $g \in C(X)$, $H \subseteq Z(g)$ and $g(x_n) \neq 0$ for every $n = 1, 2, \ldots,$ thus $g \notin F$. Now whenever there exists $h \in C(X)$ such that $gh \in F$, then $A \setminus (Z(h) \cup Z(g)) = X \setminus (Z(h) \cup Z(g))$ should be finite, i.e., $gh \in C_F(X)$. This means that

$$\frac{F}{C_F(X)} \cap \frac{C_F(X) + (g)}{C_F(X)} = (0),$$

i.e., $F/C_F(X)$ is not essential in $C(X)/C_F(X)$, a contradiction.

CASE 2: $\operatorname{int} H = \emptyset$. Put $B = \{x_1, x_3, \ldots, x_{2n-1}, \ldots\}$ and consider the ideal F consisting of all functions f such that $B \setminus Z(f)$ is finite. Since $Z(F) \subseteq H$, we have $\operatorname{int} Z(F) = \emptyset$ and hence F is an essential ideal in $C(X)$, by Proposition 1.5. Now we choose $f_n \in C(X)$ such that $f_n(X \setminus \{x_{2n}\}) = \{0\}$, $f_n(x_{2n}) = 1$ for every $n = 1, 2, \ldots$ and set $f = \sum_{n=1}^{\infty} 2^{-n}(f_n^2 \wedge 1)$. Then $f(B) = \{0\}$ and $f(x_{2n}) \neq 0$ for every $n = 1, 2, \ldots,$ and consequently $f \in F \setminus C_F(X)$. This means that $F/C_F(X)$ is not trivial. Now according to our hypothesis, $F/C_F(X)$ is an essential ideal in $C(X)/C_F(X)$. As in Case 1, if we consider $g \in C(X)$ such that $X \setminus B \subseteq Z(g)$ and $g(x_{2n-1}) \neq 0$ for every $n = 1, 2, \ldots,$ then

$$\frac{F}{C_F(X)} \cap \frac{C_F(X) + (g)}{C_F(X)} = (0),$$

for if there is $h \in C(X)$ such that $gh \in F \setminus C_F(X)$, then $B \setminus Z(gh)$ is finite, which implies that $X \setminus Z(gh) = B \setminus Z(gh)$ is finite, i.e., $gh \in C_F(X)$, a contradiction. ∎

REMARK 2.4. Since X is infinite, we have $C_F(X) \neq C(X)$. Suppose that A is the set of isolated points of X. If A is finite, then $M_{X\setminus A}$ is not essential in $C(X)$ (as $M_{X\setminus A} \cap M_A = (0)$) and contains $C_F(X)$. Hence

$M_{X\setminus A}/C_F(X)$ is not essential in $C(X)/C_F(X)$. If A is infinite, then by Theorem 2.3, $C(X)/C_F(X)$ has at least one nonzero nonessential ideal. In any case, $C(X)/C_F(X)$ has a nonzero nonessential ideal. Now, this immediately yields another proof of the fact that $C_F(X)$ is never a prime ideal. For otherwise, $C(X)/C_F(X)$ is a domain (i.e., each nonzero ideal is essential), which contradicts our previous argument; see also [4] and [11].

Next we investigate the comparability of the dimensions of the rings in the title of this article.

Theorem 2.5.

(1) *$C(X)/C_F(X)$ has no uniform ideal.*
(2) *The socle of $C(X)/C_F(X)$ is the zero ideal of $C(X)/C_F(X)$.*
(3) $\dim(C(X)/C_F(X)) \geq \dim C(X)$.

Proof. (1) Let $U/C_F(X)$ be a uniform ideal in $C(X)/C_F(X)$ and $f \in U \setminus C_F(X)$. Then $X \setminus Z(f)$ is an infinite open set. Suppose that G and H are two disjoint infinite open subsets of $X \setminus Z(f)$ and let $\{x_n\}$ and $\{y_n\}$ be two infinite sequences in G and H, respectively. Choose $g_n, h_n \in C(X)$ such that $g_n(x_n) > 0$, $g_n(X \setminus G) = \{0\}$ and $h_n(y_n) > 0$, $h_n(X \setminus H) = \{0\}$ for every $n \in \mathbb{N}$. Let

$$g = \sum_{n=1}^{\infty} \frac{1}{2^n}(g_n^2 \wedge 1) \quad \text{and} \quad h = \sum_{n=1}^{\infty} \frac{1}{2^n}(h_n^2 \wedge 1).$$

Then $g(X \setminus G) = \{0\}$, $h(X \setminus H) = \{0\}$ and $g(x_n) \neq 0 \neq h(y_n)$ for every $n \in \mathbb{N}$. Now $fg, fh \in U \setminus C_F(X)$ and $(fg) \cap (fh) = (0)$. This means that the subideals $(C_F(X) + (fg))/C_F(X)$ and $(C_F(X) + (fh))/C_F(X)$ of $U/C_F(X)$ do not intersect nontrivially, a contradiction (note that if $k_1, k_2 \in C(X) \setminus C_F(X)$ and $k_1 - k_2 \in C_F(X)$, then $k_1 - k_2 \in eC(X) \subseteq C_F(X)$ for some idempotent $e \in C(X)$, i.e., $0 \neq k_1(1-e) = k_2(1-e) \in (k_1) \cap (k_2)$).

(2) This is evident by Proposition 1.2 in [11].

(3) By Proposition 1.2, it is enough to show that $\dim(C(X)/C_F(X)) \geq \mathcal{S}(X)$. Let $F = \{G_\alpha : \alpha \in \mathcal{K}\}$ be any infinite collection of disjoint open sets in X. We may assume that every G_α is infinite, for otherwise we can construct a new collection with this property and the same cardinality. To see this, clearly if G_α contains any nonisolated point, then G_α is infinite. Thus we may assume that each G_α consists entirely of isolated points. By Zorn's lemma, we may write $F = \bigcup_{j\in J} F_j$ such that $F_i \cap F_j = \emptyset$, for every $i \neq j$, and each F_j is an infinite countable set. Clearly, $|J| = |F|$. For each $j \in J$ put $H_j = \bigcup_{G_\alpha \in F_j} G_\alpha$, which is clearly an infinite open set and $H_i \cap H_j = \emptyset$ for all $i \neq j$. Setting $F' = \{H_j : j \in J\}$, we have $|F'| = |F|$ and so F' is the collection that we were to construct.

Next, using the complete regularity of X we choose $f_\alpha \in C(X)$ such that $f_\alpha(X \setminus G_\alpha) = \{0\}$ and f_α is different from zero at infinitely many points of G_α. Now we claim that the collection $\{(C_F(X) + (f_\alpha))/C_F(X) : \alpha \in \mathcal{K}\}$ is an independent set of nonzero ideals in $C(X)/C_F(X)$. If we prove our claim, then we are done, for in that case $\dim(C(X)/C_F(X)) \geq \mathcal{S}(X)$. Since $f_\alpha \notin C_F(X)$ for every $\alpha \in \mathcal{K}$, every member of this collection is a nonzero ideal in $C(X)/C_F(X)$. Therefore we must only show that

$$I = \frac{C_F(X) + (f_\alpha)}{C_F(X)} \cap \sum_{\alpha \neq \beta \in \mathcal{K}} \frac{C_F(X) + (f_\beta)}{C_F(X)} = (0).$$

Let $f + C_F(X) \in I$. Then $f - f_\alpha g = f - f_{\alpha_1} g_1 - \cdots - f_{\alpha_n} g_n \in C_F(X)$, where $g, g_i \in C(X)$ and $\alpha \neq \alpha_i$ for every $i = 1, \ldots, n$. But clearly $f_\alpha f_{\alpha_i} = 0$ for every $i = 1, \ldots, n$ implies that $f_\alpha^2 g \in C_F(X)$, and hence $f_\alpha g \in C_F(X)$, i.e., $f \in C_F(X)$. ∎

REMARK 2.6. Part (2) of the previous result shows that $C(X)$ is never a Loewy ring (recall that a ring R is Loewy if it is a Loewy R-module, and an R-module M is Loewy if each of its nonzero homomorphic image has essential socle). One can show that a $C(X)$-module is Loewy if and only if it is semisimple.

The next remark shows that $\dim(C(X)/C_F(X))$ might be strictly greater than $\dim C(X)$.

REMARK 2.7. Let X be an infinite countable discrete space. Then $C(X) \cong \prod_{i\in\mathbb{N}} \mathbb{R}_i$, $\mathbb{R}_i = \mathbb{R}$ for every $i = 1, 2, \ldots$, where $\mathbb{R}$ is the field of real numbers. Clearly,

$$\sum_{i\in\mathbb{N}} \oplus \mathbb{R}_i = \operatorname{soc}\Big(\prod_{i\in\mathbb{N}} \mathbb{R}_i\Big) \quad \text{and} \quad C(X)/C_F(X) \cong \prod_{i\in\mathbb{N}} \mathbb{R}_i / \sum_{i\in\mathbb{N}} \oplus \mathbb{R}_i.$$

But $\sum_{i\in\mathbb{N}} \oplus \mathbb{R}_i$ is essential in $\prod_{i\in\mathbb{N}} \mathbb{R}_i$, i.e., $\dim \prod_{i\in\mathbb{N}} \mathbb{R}_i = \dim \sum_{i\in\mathbb{N}} \oplus \mathbb{R}_i$. Now we claim that $\dim S = \aleph_0$ and $\dim \overline{R} = 2^{\aleph_0}$, where $S = \sum_{i\in\mathbb{N}} \oplus \mathbb{R}_i$, $R = \prod_{i\in\mathbb{N}} \mathbb{R}_i$ and $\overline{R} = R/S$. To see this, let $F = \{s_k R : s_k \in S\}_{k\in K}$ be a maximal independent collection of principal ideals in S. Then for each s_k, we have $s_k R = \mathbb{R}_{s_1} \oplus \cdots \oplus \mathbb{R}_{s_n}$. But the set of all finite subsets of $\mathbb{N}$ is countable, i.e., $|K| = \aleph_0$. Finally, we are to show that $\dim(R/S) = 2^{\aleph_0}$. It is well known [18, Theorem 1.2] that there exists $F \subseteq \mathcal{P}(\mathbb{N})$ such that $|F| = 2^{\aleph_0}$ and whenever $A, B \in F$, then $A \cap B$ is a finite set and $|A| = \aleph_0$ for each $A \in F$; see also [14, 5I]. Now for each $A \in F$ we put $x_A = (x_i) \in R$ such that $x_i = 1$ for all $i \in A$ and $x_i = 0$ otherwise.

We claim that $\{(x_A + S)\overline{R}\}_{A\in F}$ is an independent family of ideals in R/S, which finishes the proof. To this end, we must show that $(x_{A_1} + S)\overline{R} \cap \sum_{i=2}^{n} (x_{A_i} + S)\overline{R} = (0)$, where $A_1, \ldots, A_n \in F$. First, note that x_A is an idempotent in R for all $A \in F$, and $x_A x_B \in S$ for all $A, B \in F$ such that

$A \neq B$. Thus if $x_{A_1}\bar{r}_1 - \sum_{i=2}^n x_{A_i}\bar{r}_i \in S$, then multiplying $x_{A_1}\bar{r}_1 - \sum_{i=2}^n x_{A_i}\bar{r}_i$ by each x_{A_i} we immediately have $x_{A_i}\bar{r}_i \in S$, i.e., $x_{A_i}\bar{r}_i = S = \overline{0}$ for each i, and this completes the proof.

More generally, if X is a discrete space with cardinality $|X| = a$ say, where a is a regular cardinal, and $S = \{f \in C(X) : |X \setminus Z(f)| < a\}$, then $\dim(C(X)/S) > a = \dim C(X)$ (note S is an ideal containing $C_F(X)$). To see this, we may use the same set theory result in the book by K. Kunen to get a subset F of $\mathcal{P}(X)$ such that $|F| \geq a^+$ and whenever $A, B \in F$, then $|A \cap B| < a$ and $|A| = a$ for each $A \in F$ (note that we work in ZFC, i.e., Zermelo–Fraenkel set theory with the Axiom of Choice, and by a^+ we mean the least cardinal greater than a). Now, everything is ready to just imitate the previous proof to show that $\dim(C(X)/S) \geq a^+$.

Finally, the observations made in the previous two parts of this remark can be extended with the same proofs to more general rings: if $\{R_i\}_{i\in I}$ is an infinite collection of rings with $\dim R_i \leq \aleph_0$ for all $i \in I$ and $R = \prod_{i\in I} R_i$ and $S = \sum_{i\in I} \oplus R_i$, then $\dim(R/S) \geq \dim R$. Moreover, if I is an infinite countable set, then $\dim(R/S) > \dim R$. We also note that if $|I| = a$ is a regular cardinal and $S' = \{\langle x_i \rangle \in R : |T| < a\}$, where $T = \{i \in I : x_i \neq 0\}$, then $\dim(R/S') \geq a^+ > \dim R$.

Remark 2.8. If X is a discrete space with $|X| = a$, where a is a regular cardinal, and $S^\star = \{f \in C^\star(X) : |X \setminus Z(f)| < a\} \subseteq S = \{f \in C(X) : |X \setminus Z(f)| < a\}$, then

$$\dim(C(\beta X)/S^\beta) > a = \dim C(\beta X),$$

where S^β is the ideal corresponding to $S^\star$ under the isomorphism $f \mapsto f^\beta$ of $C^\star(X)$ onto $C(\beta X)$ (note that if $a = \aleph_0$, then $S = S^\star = C_F(X)$).

Motivated by the proof of Remark 2.7, we present a case that yields the equality $\dim(C(X)/C_F(X)) = \dim C(X)$.

Theorem 2.9. *If $\mathcal{S}(X) = \aleph_0$, then the following are equivalent.*

(1) $C(X)/C_F(X) \cong C(Y)$, *where Y is a space with $\mathcal{S}(Y) = \aleph_0$.*
(2) $\dim(C(X)/C_F(X)) = \dim C(X)$.
(3) *X has at most a finite number of isolated points.*

Proof. We only show that (2)⇒(3), for other implications are clear. First, we note that $\dim(C(X)/C_F(X)) = \dim C(X) = \mathcal{S}(X) = \aleph_0$. Next, suppose X contains an infinite countable set $D = \{x_1, x_2, \ldots\}$ of isolated points. Now, as in Remark 2.7, let $F = \{A_k\}_{k\in K}$ be an uncountable collection of infinite subsets of D such that $A_k \cap A_{k'}$ is finite for $k \neq k'$. As each A_k is countable, we may use the complete regularity of X and for each k pick $f_k \in C(X)$ such that $f_k(x) \neq 0$ for all $x \in A_k$ and $f_k(X \setminus A_k) = \{0\}$. Clearly, $f_k \notin C_F(X)$ for all $k \in K$.

We claim that $\{\bar{f}_k C(X)\}_{k\in K}$, where

$$\bar{f}_k C(X) = (f_k C(X) + C_F(X))/C_F(X),$$

is an independent collection of ideals in $C(X)/C_F(X)$, and this is the required contradiction. To this end, suppose $\bar{f}_{k_1} C(X) \cap (\sum_{k\neq k_1} \bar{f}_k C(X)) \neq (0)$. Clearly, there must exist $g_1, \ldots, g_n \in C(X)$ such that $h = f_{k_1} g_1 - \sum_{i=2}^{n} f_{k_i} g_i \in C_F(X)$ and $f_{k_1} g_1 \notin C_F(X)$. Now, since $X \setminus Z(h)$ is finite and f_{k_i} is zero at all points of A_{k_1} except possibly a finite number for all $i \geq 2$, for all $i \geq 2$, we immediately infer that $f_{k_1} g_1$ must be zero at all points of A_{k_1} except possibly a finite number. This immediately shows that $f_{k_1} g_1 \in C_F(X)$, which is absurd. ∎

REMARK 2.10. In [9, Theorem 6], it is shown that unless the Goldie dimension of a ring R is an inaccessible cardinal, R contains a direct sum of $\dim R$ nonzero ideals (i.e., $\dim R$ is attained). By using an example of Erdős and Tarski one can easily show that there is a Boolean ring whose unattained Goldie dimension is any given inaccessible cardinal μ (see the last remark in [9]). In Examples 8 and 9 of [9], it is also shown that rings with arbitrary Goldie dimensions exist. We observe that the cardinality of each of the latter rings is the same as its Goldie dimension. Motivated by these facts, we are going to present more general examples in the context of $C(X)$.

First, in view of the Stone representation theorem, let X be a zero-dimensional (i.e., having a base of clopen subsets) compact space whose Boolean algebra of clopen subsets is isomorphic to the above Boolean ring R such that $\mu = \dim R$ is not attained. Now the example of Erdős and Tarski (see [9]) easily shows that the cellularity of X is μ and is not attained, i.e., by Proposition 1.2, $\dim C(X) = \mu$ is not attained.

Let X be a discrete space with cardinality $a = |X|$. Clearly, $\mathcal{S}(X) = \mathcal{S}(\beta X)$, i.e., $\dim C(X) = \dim C(\beta X) = a$, and it is evident that $|C(\beta X)| \geq 2^a$, for $|\beta X| = 2^{2^a}$. Clearly $|C(\beta X)/C_F(\beta X)| \geq 2^a$ and in view of Theorem 2.5, we also have $\dim(C(\beta X)/C_F(\beta X)) \geq a$. Next, we show that there are rings as large as we want whose Goldie dimensions are the least infinity, i.e., $\aleph_0$. To see this, let $\lambda \geq \aleph_0$ be any cardinal and D_λ be the Cantor cube of weight λ, i.e., $D_\lambda = \prod_{i\in I} D_i$, $|I| = \lambda$ and $D_i = \{0, 1\}$ the two-point discrete space. It is well known that the cellularity of D_λ is $\aleph_0$; see [10, Corollary 2.3.18] (note that a more general result is true: if the countability of the cellularity of a space X is preserved by products with two factors, it is preserved by arbitrary products). Thus, $\dim C(D_\lambda) = \aleph_0$ and $|C(D_\lambda)| \geq \lambda$. Finally, using these facts, one can show that there is a ring R such that $\dim R = a$ and $|R| \geq b$ where $a < b$ are arbitrary infinite cardinals. To this end, let $R_1 = C(D_b)$ and $R_2 = C(X)$, where X is a discrete space with $|X| = a \geq \aleph_0$. Clearly $\dim R_1 = \aleph_0$ and $\dim R_2 = a$. Now put $R = R_1 \oplus R_2$. Clearly, $\dim R = \dim R_1 + \dim R_2 = a$ and $|R| \geq |R_1| \geq b$.

By using the sophisticated fact of infinite additivity of the Goldie dimensions (see [9, Theorem 3]), one can construct still more examples similar to R. To see this, let I be a set with $|I| = a$ and for each $i \in I$ let R_i be a domain with $|R_i| = b$ (for example, one may take $R_i = K[X]$, where K is a finite field and X is a set of indeterminates with $|X| = b$). Now put $S = \sum_{i \in I} \oplus R_i$ and $R = \prod_{i \in I} R_i$. Then it is clear that S is essential in R and $\dim R = \dim S = a$ (note that $\dim R_i = 1$ for all $i \in I$) and $|R| = b^a \geq b$.

Remark 2.11. In [11], it is shown that $C(X)$ is $\aleph_0$-injective if and only if $C(X)/C_F(X)$ is $\aleph_0$-injective. We also observe trivially that $C(X)$ is regular (i.e., X is a P-space) if and only if $C(X)/C_F(X)$ is regular. To see this, assume $C(X)/C_F(X)$ is regular; then each prime ideal in $C(X)/C_F(X)$ is maximal. Now for each prime ideal P in $C(X)$, either P is maximal and at the same time minimal prime which is not essential, or P is essential (see Proposition 1.6). But if P is essential in $C(X)$, then $P/C_F(X)$ is maximal in $C(X)/C_F(X)$, i.e., in this case P is also maximal in $C(X)$. Thus we have shown that every prime ideal in $C(X)$ is maximal, i.e., $C(X)$ is regular. The converse is trivial. The reader should be reminded that one can apply some algebraic results to show that if R is any reduced ring such that $R/\mathrm{soc}(R)$ is regular, then R is also regular.

Remark 2.12. It is well known folklore that $C(X)$ generally avoids any finiteness condition, such as being Noetherian (i.e., *acc* on finitely generated ideals), perfect (i.e., *dcc* on principal ideals), finitely embedded (i.e., having a finitely generated essential socle), and more generally having finite Goldie dimension. Perfect rings are Loewy rings, but $C(X)$ is never Loewy (see [11, Proposition 1.2]). Clearly, any of the above conditions in any ring forces the ring to have finite Goldie dimension. Now, part (3) of Theorem 2.5 shows that $C(X)/C_F(X)$, similarly to $C(X)$, avoids all these finiteness conditions.

3. z-ideals in $C(X)$ modulo its socle. In [21, Lemma 2.1], it is shown that for any $f_1, \dots, f_n$ in $C(X)$, there exists $g \in C(X)$ such that g^n divides every f_i for any natural number n and $Z(g) = Z(f_1) \cap \cdots \cap Z(f_n)$. Consequently, each finitely generated ideal in $C(X)$ is contained in a principal ideal. In [5, Theorem 3.3], it is observed that X is basically disconnected if and only if every nonzero countably generated ideal in $C(X)$ is essential in a principal ideal generated by an idempotent. The following is a more general result.

Proposition 3.1. *Let $I = (f_1, f_2, \dots,)$ be a countably generated ideal in $C(X)$. Then there exists $g \in C(X)$ such that $I \subseteq \bigcap_{n=1}^{\infty} (g^n)$. Moreover, I is essential in (g^n) for all n and $Z(g) = \bigcap_{i=1}^{\infty} Z(f_i)$.*

Proof. Without loss of generality, we may assume that $|f_i| \leq 1$ for all i (otherwise replace each f_i by $f_i/(1+f_i^2)$). Clearly $I \subseteq (S)$, where $S = \bigcup_{m=1}^{\infty}\{f_m^{1/3}, f_m^{1/5}, \ldots, f_m^{1/(2n+1)}, \ldots\}$. Now for each m, define

$$g_m = \sum_{n=1}^{\infty} \frac{|f_m|^{1/(2n+1)}}{2^n}.$$

Then $g_m \in C(X)$, $|g_m| \leq 1$ and for each n we have $|f_m|^{1/(2n+1)} \leq 2^n|g_m|$, i.e., $|f_m| \leq 2^{n(2n+1)}|g_m|^{2n+1} \leq |\lambda g_m^n|^2$, where $\lambda = 2^{n(2n+1)/2}$. Now by [14, 1D], we infer that each f_m is a multiple of g_m^n for all $n \geq 1$ and $Z(g_m) = Z(f_m)$ for all m. Again we define

$$g = \sum_{m=1}^{\infty} \frac{|g_m|^{1/2}}{2^m} \in C(X),$$

i.e., for each m, we have $|g_m|^{1/2} \leq 2^m g$, and again by [14, 1D], we have $g \mid g_m$ for each m. This means that for each n, m we have $g^n \mid f_m$ and it is clear that $Z(g) = \bigcap_{i=1}^{\infty} Z(f_i)$. Finally, we note that $Z(g) = \bigcap_{f\in I} Z(f)$, i.e., in view of Lemma 1.8, we have $\mathrm{Ann}(g) = \mathrm{Ann}(I)$ and therefore I is essential in (g) by Lemma 1.7. ■

The next result, whose proof is essentially in the last part of the proof of Theorem 6.1 in [13], is just a restatement of Proposition 2.4 in [21]; see also [8].

Proposition 3.2. *Let J be an ideal in $C(X)$. Then J and $\sqrt{J}$ have the same largest z-ideal.*

Proof. Since the sum of z-ideals in a proper ideal is a proper z-ideal, it suffices to show that whenever I is a z-ideal contained in $\sqrt{J}$, then $I \subseteq J$. To see this, let $f \in I$. By replacing f by $f/(1+f^2)$ we may assume that $|f| \leq 1$. Now put $g = \sum_{n=1}^{\infty} |f|^{1/n}/2^n$. Then $g \in C(X)$, $Z(g) = Z(f)$, i.e., $g \in I$. Thus $g \in \sqrt{J}$, i.e., $g^n \in J$ for some $n > 1$. Now, $2^{-n^2}|f|^{1/n^2} \leq g$, i.e., $|f| \leq (2^{n^3}g^n)^n$. This means that $g^n \mid f$, by [14, 1D.3], i.e., $f \in J$. ■

The previous fact is also true in $C(X)/C_F(X)$.

Corollary 3.3. *Let $\bar{J} = J/C_F(X)$ be an ideal in $C(X)/C_F(X)$. Then $\bar{J}$ and $\sqrt{\bar{J}}$ have the same largest z-ideal.*

Proof. First, we note that if $A/C_F(X)$ is a z-ideal in $C(X)/C_F(X)$, then A is a z-ideal in $C(X)$. To see this, let $f \in A$. If $f \in C_F(X)$, then the intersection of all maximal ideals containing f, M_f say, is contained in $C_F(X)$; a fortiori it is contained in A (as $C_F(X)$ is a z-ideal). Thus we may assume that $f \notin C_F(X)$ and consider $\bar{f} = f + C_F(X) \in A/C_F(X)$. Now let $\{P_i/C_F(X)\}_{i\in K}$ be the set of all maximal ideals in $C(X)/C_F(X)$ containing $\bar{f}$, i.e., $\bigcap_{i\in K} P_i \subseteq A$. This implies that $M_f \subseteq \bigcap_{i\in K} P_i \subseteq A$ and we are done.

Now we claim that whenever $\bar{I} = I/C_F(X)$ is a z-ideal in $\sqrt{\bar{J}}$, then $\bar{I} \subseteq \bar{J}$, and this completes the proof. To this end, by the first part we note that I is a z-ideal in $C(X)$ and from $\bar{I} \subseteq \sqrt{\bar{J}} = \sqrt{J}/C_F(X)$ we infer that $I \subseteq \sqrt{J}$, i.e., by the previous result $I \subseteq J$ and therefore $\bar{I} \subseteq \bar{J}$. ∎

REMARK 3.4. Our proof shows that the previous corollary holds in any factor ring $C(X)/I$, where I is a z-ideal in $C(X)$. We also note that if Q is a primary ideal in $C(X)$, then $\sqrt{Q} = P$ is a prime ideal (this is true in any commutative ring), i.e., $P_0 \subseteq \sqrt{Q} = P$, where P_0 is a minimal prime ideal which is a z-ideal (see [14, Theorem 14.7]). Hence the fact that $\sqrt{Q}$ and Q have the same z-ideals immediately implies that $P_0 \subseteq Q$. It is also evident that if $\sqrt{Q} = P$ is a maximal ideal, then $Q = P$, which is Corollary 2.7 in [21].

The following well known result is now an easy consequence of our Proposition 3.1 (see [12]).

COROLLARY 3.5. *Let $M = (f_1, f_2, \ldots)$ be a fixed countably generated maximal ideal in $C(X)$. Then $M = M_x$, where x is an isolated point.*

Proof. By Proposition 3.1, $M \subseteq (g)$, and as M is fixed, we have $Z(g) \neq \emptyset$, i.e., $M = (g)$. But $M = (g) = M^2 = (g^2)$. Now let $g = g^2 f$ and put $e = gf$; then clearly $e = e^2$, $M = (e)$ and the proof is complete. ∎

COROLLARY 3.6. *Suppose the set of isolated points of X is countable. Then no fixed maximal ideal of $C(X)/C_F(X)$ can be countably generated (note that an ideal $\bar{I} = I/C_F(X)$ is fixed if I is fixed).*

Proof. First we note that $C_F(X)$ is countably generated ($C_F(X) = \sum_{i=1}^{\infty} \oplus e_i C(X)$, where $e_i = e_i^2$, $Z(e_i) = X \setminus \{x_i\}$, where x_i is an isolated point in X; see [17]). Now let $\overline{M} = M/C_F(X)$ be a fixed countably generated maximal ideal in $C(X)/C_F(X)$. Then $M = \sum_{i=1}^{\infty} f_i C(X) + C_F(X)$, i.e., M is a fixed countably generated maximal ideal in $C(X)$. Hence $M = eC(X)$, where $e = e^2$, i.e., $C(X)/M \cong (1-e)C(X) \subseteq C_F(X)$. Thus $C(X) = eC(X) \oplus (1-e)C(X) \subseteq M$, which is absurd. ∎

4. Essential ideals in $C(X)$ modulo its principal ideals. In this section we investigate the essentiality of prime ideals and z-ideals of $C(X)$ modulo its principal ideals. Using these characterizations, we show that a space X is an almost P-space (with a dense set of isolated points) if and only if every factor ring of $C(X)$ modulo a principal ideal contains a nonessential z-ideal (prime ideal).

We start with an example of an essential minimal prime ideal in a factor ring of $C(X)$ modulo a principal ideal.

EXAMPLE 4.1. Suppose that $W^\star = W \cup \{\omega_1\}$, where W is the space of countable ordinals, and ω_1 is the first uncountable ordinal. Clearly $O_{\omega_1} = M_{\omega_1}$ (see [14]). Suppose that $f \in C(W^\star)$ is such that $f(1) \neq 0$ and $f(\alpha) = 0$ for every $\alpha > 1$. Then $O_{\omega_1}/(f)$ is a minimal prime ideal in $C(W^\star)/(f)$. We show that $O_{\omega_1}/(f)$ is an essential ideal in $C(W^\star)/(f)$. If $h \notin O_{\omega_1}$, then $h(\omega_1) \neq 0$ and hence there exists $\lambda \in W$ such that $h(\alpha) \neq 0$ for every $\alpha \geq \lambda$. Take $g \in O_{\omega_1}$ such that $g(\lambda+1) \neq 0$ and $g(\alpha) = 0$ for every $\alpha \neq \lambda+1$. Now $gh \in O_{\omega_1} \setminus (f)$ (for if $gh \in (f)$, then $Z(f) \subseteq Z(gh)$, but $g(\lambda+1)h(\lambda+1) \neq 0$ and $f(\lambda+1) = 0$). This shows that $O_{\omega_1}/(f)$ is an essential ideal.

If the set of isolated points in X is finite, the next result can be considered as a generalization of Theorem 2.2.

THEOREM 4.2. *If $h \in C(X)$, then every prime ideal of $C(X)/(h)$ is an essential ideal if and only if $Z(h)$ does not contain any isolated point.*

Proof. Suppose that every prime ideal of $C(X)/(h)$ is essential and $x_0 \in Z(h)$ is an isolated point. Clearly M_{x_0} is a nonessential prime ideal in $C(X)$ and hence $M_{x_0}/(h)$ is a nonessential prime ideal of $C(X)/(h)$, a contradiction.

Conversely, suppose $Z(h)$ does not contain any isolated point and let P be a prime ideal containing h. Suppose that $f \notin P$; we must show that $(P \setminus (h)) \cap (f) \neq \emptyset$. Clearly $f \notin (h^{1/3})$, for otherwise $f \in P$. We consider two cases:

CASE 1: $Z(h) \subseteq Z(f)$. Since $h^{1/3} \in P$, we have $h^{1/3}f \in P$ but $h^{1/3}f \notin (h)$; for otherwise there exists $k \in C(X)$ such that $h^{1/3}f = hk$, and hence $f(x) = k(x)h^{2/3}(x)$ for every $x \notin Z(h)$. Now $Z(h) \subseteq Z(f)$ implies that $f = kh^{2/3}$, so $f \in (h^{1/3})$, a contradiction. This implies that $fh^{1/3} \in (P \setminus (h)) \cap (f)$.

CASE 2: $Z(h) \not\subseteq Z(f)$. There exists $x_0 \in Z(h)$ such that $x_0 \notin Z(f)$.

First, suppose that $x_0 \in \operatorname{int} Z(h)$. Since $f(x_0) \neq 0$, there is an open set G containing x_0 such that $f(x) \neq 0$ for every $x \in G$ (as x_0 is not an isolated point, G is infinite). We may assume that $G \subseteq \operatorname{int} Z(h)$. Since $Z(P)$ is at most a singleton, we may choose $x_0 \neq y \in G$ such that $y \notin Z(P)$. Now, there exists $g \in P$ such that $g(y) \neq 0$. Hence $fg \in P$ but $fg \notin (h)$, for $h(y) = 0$ but $f(y)g(y) \neq 0$, i.e., $fg \in (P \setminus (h)) \cap (f)$.

Next, suppose that $x_0 \notin \operatorname{int} Z(h)$; then x_0 is in the boundary $\partial Z(h)$ of $Z(h)$. Clearly $h^{1/3}f \in P$. We show that $h^{1/3}f \notin (h)$. If $h^{1/3}f \in (h)$, then there exists $k \in C(X)$ such that $h^{1/3}f = hk$. Thus $f(x) = k(x)h^{2/3}(x)$ for every $x \notin Z(h)$. Since $x_0 \in \partial Z(h)$, we may consider a net $x_\alpha \notin Z(h)$ such that $x_\alpha \to x_0$. Since $h, k, f \in C(X)$, we have $h^{2/3}(x_\alpha) \to h^{2/3}(x_0) = 0$ and hence $f(x_\alpha) \to 0 = f(x_0)$, a contradiction. Therefore $h^{1/3}f \in (P \setminus (h)) \cap (f)$. ∎

REMARK 4.3. If $Z(h)$ consists entirely of isolated points, then some prime ideals of $C(X)/(h)$ might be essential. If $Z(h)$ is a finite set of isolated points, then $Z(h)$ is compact and hence by Lemma 4.10 of [14], h belongs to no free ideal. Hence the set of prime ideals containing h is the set $\{M_x : x \in Z(h)\}$. Since every M_x, $x \in Z(h)$, is a nonessential maximal ideal in $C(X)$, $C(X)/(h)$ has no essential ideal. Now suppose that $Z(h)$ is infinite and every $x \in Z(h)$ is an isolated point. Again, fixed prime ideals of $C(X)$ containing h are nonessential, i.e., for every fixed prime ideal P containing h, $P/(h)$ is nonessential in $C(X)/(h)$. But if P is a free prime ideal in $C(X)$ containing h, then $P/(h)$ is essential in $C(X)/(h)$. To see this, let $f \notin P$. Since $Z(h)$ is open, $Z(h) \subseteq Z(f)$ implies that $f \in (h) \subseteq P$, a contradiction. So, we assume $Z(h) \nsubseteq Z(f)$ and take $x \in Z(h) \setminus Z(f)$. Since P is free, there exists $g \in P$ such that $g(x) \neq 0$. Now $fg \in P \cap (f)$ but $fg \notin (h)$, for $f(x)g(x) \neq 0$ and $h(x) = 0$, i.e., $fg \in (P \setminus (h)) \cap (f)$ and we are done.

To prove the next theorem we need the following lemma.

Lemma 4.4. *Suppose that $h \in C(X)$ and h^λ is defined for the positive real number $\lambda < 1$. Then $(h^\lambda)/(h)$ is an essential ideal in $C(X)/(h)$ if and only if* $\operatorname{int} Z(h) = \emptyset$.

Proof. If $\operatorname{int} Z(h) \neq \emptyset$, then $\operatorname{int} Z(h^\lambda) \neq \emptyset$ and by Proposition 1.5, (h^λ) is not essential in $C(X)$, hence $(h^\lambda)/(h)$ is not essential in $C(X)/(h)$.

Conversely, suppose that $\operatorname{int} Z(h) = \emptyset$. We will show that $(h^{(n-1)/n})/(h)$ is an essential ideal in $C(X)/(h)$, where n is any odd integer. To see this, let $f \notin (h^{(n-1)/n})$ and consider two cases:

Case 1: $Z(h) \subseteq Z(f)$. First, suppose that for all integers $1 \leq k \leq n-2$ we have $h^{k/n} \nmid f$. Clearly $h^{(n-1)/n} f \in (h^{(n-1)/n})$. But $h^{n-1/n} f \notin (h)$, for otherwise there exists $g \in C(X)$ such that $h^{(n-1)/n} f = hg$ and hence $f(x) = h^{1/n}(x)g(x)$ for every $x \notin Z(h)$. Now $Z(h) \subseteq Z(f)$ implies that $f = h^{1/n} g$, a contradiction. This shows that $h^{(n-1)/n} f \in (h^{(n-1)/n}) \setminus (h)$. Next suppose that there exists $1 \leq k \leq n-2$ such that $h^{k/n} \mid f$ and let k be the largest integer with this property. Thus $h^{(k+1)/n} \nmid f$. We show that $h^{(n-k-1)/n} f \in (h^{(n-1)/n}) \setminus (h)$. Since $h^{k/n} \mid f$, there exists $g \in C(X)$ such that $f = h^{k/n} g$ and hence $h^{(n-k-1)/n} f = h^{(n-k-1)/n} h^{k/n} g = h^{(n-1)/n} g$. If $h^{(n-k-1)/n} f \in (h)$, then there exists $g \in C(X)$ such that $f(x) = h^{(k+1)/n}(x) g(x)$ for every $x \notin Z(h)$. But $Z(h) \subseteq Z(f)$ implies that f is a multiple of $h^{(k+1)/n}$, a contradiction. Thus $h^{(n-k-1)/n} f \in ((h^{(n-1)/n}) \setminus (h)) \cap (f)$. In any case, we have shown that $(h^{(n-1)/n})/(h)$ is an essential ideal.

Case 2: $Z(h) \nsubseteq Z(f)$. Let $x_0 \in Z(h) \setminus Z(f)$. Since $\operatorname{int} Z(h) = \emptyset$, x_0 is contained in the boundary $\partial Z(h)$ of $Z(h)$. It is enough to show that $h^{(n-1)/n} f \notin (h)$. If $h^{(n-1)/n} f \in (h)$, then there exists $g \in C(X)$ such that $h^{(n-1)/n} f = gh$ and hence $f(x) = h^{1/n}(x) g(x)$ for every $x \notin Z(h)$. Since $x_0 \in \partial Z(h)$, there exists a net $x_\alpha \notin Z(h)$ such that $x_\alpha \to x_0$. Therefore

$f(x_\alpha) = h^{1/n}(x_\alpha)g(x_\alpha) \to 0$ and this means that $f(x_0) = 0$, a contradiction. Thus $(h^{(n-1)/n})/(h)$ is an essential ideal in $C(X)/(h)$.

Now for every $0 < \lambda < 1$, there exists an odd integer n such that $\lambda < (n-1)/n$. Since $(h^{(n-1)/n})/(h) \subseteq (h^\lambda)/(h)$ and $(h^{(n-1)/n})/(h)$ is an essential ideal in $C(X)/(h)$, a fortiori $(h^\lambda)/(h)$ is essential too. ∎

REMARK 4.5. Note that Lemma 4.4 does not imply that any principal ideal of $C(X)/(h)$ with $\operatorname{int} Z(h) = \emptyset$ is an essential ideal. For example, let $X = (-\infty, 0) \cup (0, \infty)$, $h, k \in C(X)$, $Z(h) = \{-1\}$ and $Z(k) = \{1\}$. Clearly $\operatorname{int} Z(hk) = \emptyset$ but we show that $(h)/(hk)$ is not an essential ideal in $C(X)/(kh)$. Choose $f \in C(X)$ such that $Z(f) = (0, \infty)$, hence $f \notin (h)$. Let $g \in C(X)$ be such that $fg \in (h)$. Then there exists $t \in C(X)$ such that $fg = th$. Since $Z(f) \cap Z(h) = \emptyset$, we have $Z(f) \subseteq Z(t)$. Now $Z(k) \subseteq \operatorname{int} Z(f) \subseteq \operatorname{int} Z(t)$ and [14, 1D] implies that t is a multiple of k and hence $fg \in (hk)$, i.e., $(h)/(hk)$ is not essential.

THEOREM 4.6. *Every z-ideal of $C(X)/(h)$ is essential if and only if $\operatorname{int} Z(h) = \emptyset$ (i.e., the principal ideal (h) is an essential ideal).*

Proof. If $\operatorname{int} Z(h) = \emptyset$ and $E/(h)$ is a z-ideal in $C(X)/(h)$, then $h^{1/3} \in E$. Now by Lemma 4.4, $(h^{1/3})/(h)$ is essential and $(h^{1/3})/(h) \subseteq E/(h)$ implies that $E/(h)$ is also an essential ideal in $C(X)/(h)$. Conversely, suppose $\operatorname{int} Z(h) \neq \emptyset$. Then $M_{Z(h)} = M_h$ is a z-ideal in $C(X)$ containing h which is not essential by Proposition 1.5 (note that $M_h/(h)$ is clearly a z-ideal). Now, $M_h/(h)$ is not an essential ideal in $C(X)/(h)$, a contradiction. ∎

COROLLARY 4.7.

(a) *X is an almost P-space with a dense set of isolated points if and only if every factor ring of $C(X)$ modulo a principal ideal contains a nonessential prime ideal.*
(b) *X is an almost P-space if and only if every factor ring of $C(X)$ modulo a principal ideal contains a nonessential z-ideal.*

5. Ue-rings. A ring is a *Ue-ring* (resp. *Uem-ring*) if it has a unique proper essential (essential maximal) ideal (see [16] for more details). In [16, Corollary 32], it is observed that $C(X)$ is never a Ue-ring. Similarly, one can show that $C(X)/C_F(X)$ is also never a Ue-ring, for otherwise its socle must be maximal, but the socle of $C(X)/C_F(X)$ is zero (see Theorem 2.5(2)). This means that $C(X)/C_F(X)$ becomes a field, which is impossible, for $C(X)/C_F(X)$ cannot even be a domain (note that $C_F(X)$ is never a prime ideal; see [11, Proposition 1.2] and [4, Proposition 2.5]). The proof of the fact that $C(X)$ is never a Ue-ring can be given in various ways; for example, the above proof in fact shows that. For another proof see [16, Corollary 32]. The following is still a different one which is more elementary and yields a more general result. Before giving this proof, we record the next lemma whose proof is very elementary. Incidentally, this can be considered as a generalization of Proposition 35 in [16].

LEMMA 5.1. *Let R be a duo ring* (*i.e., every one-sided ideal in R is two-sided*) *with only a finite number of essential ideals. Then each prime ideal in R must be maximal.*

Proof. Suppose P is a nonmaximal prime ideal in R. Clearly, R/P is a duo domain, i.e., each nonzero ideal I/P in R/P is essential, i.e., I is essential in R. But it is trivial to see that each duo domain which is not a division ring has infinitely many ideals (for any nonunit u in such a ring take the ideals (u^n), $n = 1, 2, \ldots$), i.e., R must have infinitely many essential ideals, which is absurd. ∎

PROPOSITION 5.2. *Neither $C(X)$ nor $C(X)/C_F(X)$ has only a finite number of essential ideals.*

Proof. Suppose, on the contrary, that any of the above rings contains only a finite number of essential ideals. Then by the previous lemma, prime ideals in both rings are maximal, i.e., both rings are regular (reduced rings whose prime ideals are maximal, are regular). But all maximal ideals of $C(X)/C_F(X)$ are essential (see Theorem 2.2 or [11, Proposition 1.2]). This means that by our assumption, $C(X)/C_F(X)$ has only a finite number of maximal ideals, $M_1/C_F(X), \ldots, M_n/C_F(X)$ say, and $C_F(X) = \bigcap_{i=1}^n M_i$. Now by the Chinese remainder theorem,

$$\frac{C(X)}{C_F(X)} \cong \frac{C(X)}{M_1} \oplus \cdots \oplus \frac{C(X)}{M_n},$$

i.e., $C(X)/C_F(X)$ must be its own socle, which is absurd (see Theorem 2.5(2)). ∎

In [16, Proposition 31], it is shown that X is the one-point compactification of a discrete space if and only if $C(X)$ is a Uem-ring. As for $C(X)/C_F(X)$, we have the following immediate result whose proof is left to the reader (see also [16, Proposition 9]).

PROPOSITION 5.3. *$C(X)$ is a Uem-ring if and only if $C(X)/C_F(X)$ is a Uem-ring. Moreover*, *in this case $C(X)/C_F(X)$ is a local ring.*

Next, it is natural to ask when $C(X)$ has only a finite number of essential maximal ideals. The following settles this question.

PROPOSITION 5.4. *Either $C(X)$ or $C(X)/C_F(X)$ contains only a finite number of essential maximal ideals if and only if X is a compact space with only a finite number of nonisolated points. Moreover*, *in this case*, *$C(X)/C_F(X)$ is a semilocal ring* (*i.e., it has only a finite number of maximal ideals*).

Proof. We first note that for each $p \in \beta X \setminus X$, we have a free maximal ideal M^p, which is essential. Thus if any of the above rings has only a finite number of essential maximal ideals, the set $\beta X \setminus X$ cannot be infinite. This means that βX must coincide with the realcompactification of X, for otherwise $\beta X \setminus X$ is an infinite set (see [14, 9D.2]). Now we have $C(X) \cong C(\beta X)$, i.e., $C(X) = C^\star(X)$ and therefore X is pseudocompact. Finally, it remains to show that X is compact, for it is evident that X contains only a finite number of nonisolated points. To see this, we note that no infinite closed subset of X can consist entirely of isolated points, for this would contradict the fact that X is pseudocompact (see also [14, 1.21]). Thus any neighbourhood of the set of nonisolated points is a cofinite subset, i.e., X is compact. The converse is clear. ∎

REMARK 5.5. Either $C(X)$ or $C(X)/C_F(X)$ contains only a countable number of essential maximal ideals if and only if X is compact and contains at most a countable number of nonisolated points. To see this, we note that if $\beta X \neq \upsilon X$, then $\beta X \setminus X$ is uncountable; see [14, 9D.2], which is not possible, i.e., $C(X) \cong C(\beta X)$, i.e., X is pseudocompact. Clearly, X is Lindelöf, so it must be compact (see [10, 3.11.1]). The converse is trivial. This has an interesting consequence, namely being compact and having only a countable number of nonisolated points is an algebraic property (i.e., it is preserved under the isomorphism $C(X) \cong C(Y)$).

Finally, we observe trivially that whenever $Y \subseteq X$ is a clopen subset of X and $I = \{f \in C(X) : Y \subseteq Z(f)\}$, then $C(X)/I \cong C(Y)$. This immediately implies that if the set of isolated points of X is finite, then $C(X)/C_F(X) \cong C(Y)$ where Y is the set of nonisolated points. Therefore in this case $C(X)/C_F(X)$ has all the algebraic properties of $C(X)$. In particular, the Jacobson radical of $C(X)/C_F(X)$ is zero. But this is not true in general, for as we have already observed in this section, when X is the one-point compactification of a discrete space, then $C(X)/C_F(X)$ is a local Uem-ring, i.e., its Jacobson radical is a nonzero maximal ideal. Motivated by this fact, we conclude this section by characterizing topological spaces X such that $J(C(X)/C_F(X)) = (0)$.

Before proceeding, let us remind the reader that under the isomorphism $C(X)/C_F(X) \cong C(\upsilon X)/C_F(\upsilon X)$, the Jacobson radical of $C(X)/C_F(X)$ is sent to the Jacobson radical of $C(\upsilon X)/C_F(\upsilon X)$, i.e., $J(C(X)/C_F(X)) \cong J(C(\upsilon X)/C_F(\upsilon X))$. This means that, regarding the aforementioned problem, whenever necessary we may assume that $X = \upsilon X$. In the following we use the fact that whenever X is realcompact, then the family $C_K(X)$ of all functions with compact support (see [14, 4D]) is in fact the intersection of all the free maximal ideals in $C(X)$ (see [14, Theorem 8.19]), i.e., $J(C(X)/C_F(X)) = (C_K(X) \cap M_Y)/C_F(X)$, where Y is the set of nonisolated points of X and $M_Y = \bigcap_{x\in Y} M_x$. We should emphasize that in the following results X is not assumed to be realcompact.

THEOREM 5.6. $J(C(X)/C_F(X)) = (0)$ *if and only if every compact subset of X contains at most a finite number of isolated points of X.*

Proof. Let D be the set of isolated points of X and $Y = \upsilon X\setminus D$. First, we assume that each compact subset of X has finite intersection with D. Now let $f^\upsilon \in C_K(\upsilon X)\cap M_Y$ ($f\mapsto f^\upsilon$ is the isomorphism of $C(X)$ onto $C(\upsilon X)$, $f^\upsilon|_X = f$; see [14, Remarks 8.8]). Then $X\setminus Z(f)\subseteq \upsilon X\setminus Z(f^\upsilon)\subseteq D$ implies that $\mathrm{cl}_X(X\setminus Z(f))\subseteq \mathrm{cl}_{\upsilon X}(\upsilon X\setminus Z(f^\upsilon))$, hence $\mathrm{cl}_X(X\setminus Z(f))$ is compact and therefore $X\setminus Z(f) = \mathrm{cl}(X\setminus Z(f))\cap D$ is finite, so $f\in C_F(X)$ and $f^\upsilon \in C_F(\upsilon X)$. This implies that $J(C(X)/C_F(X))$ as well as $J(C(\upsilon X)/C_F(\upsilon X))$ are zero, for

$$(*)\qquad J\left(\frac{C(\upsilon X)}{C_F(\upsilon X)}\right) = \frac{C_K(\upsilon X)\cap M_Y}{C_F(\upsilon X)} = (0).$$

Conversely, suppose that $J(C(X)/C_F(X)) = (0)$ and C is a compact subset of X such that $\{x_1, x_2, \ldots\}\subseteq C\cap D$ is infinite. By complete regularity of υX, we can define $f^\upsilon\in C(\upsilon X)$ such that $\upsilon X\setminus Z(f^\upsilon) = \{x_1, x_2, \ldots\}$ $\subseteq C$, i.e., $\mathrm{cl}_{\upsilon X}(\upsilon X\setminus Z(f^\upsilon))\subseteq C$ (note that C is also compact in υX). We immediately infer that $\mathrm{cl}_{\upsilon X}(\upsilon X\setminus Z(f^\upsilon))$ is compact, i.e., $f^\upsilon\in C_K(\upsilon X)\cap M_Y$, where Y is the set of nonisolated points of υX. This implies that $f^\upsilon\in C_F(\upsilon X)$ by $(*)$, which is absurd (note that $\upsilon X\setminus Z(f^\upsilon)$ must be finite). ∎

The following corollary is now immediate.

COROLLARY 5.7. *Let X be a compact space. Then* $J(C(X)/C_F(X))$ $= (0)$ *if and only if the set of isolated points of X is finite.*

PROPOSITION 5.8. $J(C(X)/C_F(X)) = (0)$ *if and only if every F_σ-set consisting of isolated points, with compact closure, is finite.*

Proof. If $J(C(X)/C_F(X)) = (0)$, then we invoke Theorem 5.6. Conversely, let $f^\upsilon\in C_K(\upsilon X)\cap M_Y$, where Y is the set of nonisolated points of υX. Clearly, $B = \upsilon X\setminus Z(f^\upsilon)$ and $A = X\setminus Z(f)\subseteq B$ are both F_σ-sets consisting of isolated points, i.e., $\mathrm{cl}_X A\subseteq \mathrm{cl}_{\upsilon X} B$ and therefore $\mathrm{cl}_X A$ is compact in X. Hence A is finite, i.e., $f\in C_F(X)$ and $f^\upsilon\in C_F(\upsilon X)$. Now, we infer immediately that $J(C(X)/C_F(X)) = (0)$, by $(*)$. ∎

COROLLARY 5.9. *Let X be first countable. Then* $J(C(X)/C_F(X)) = (0)$ *if and only if the set of isolated points of X is closed.*

Proof. If the set of isolated points, D say, of X is closed, then we invoke Theorem 5.6. Conversely, suppose that $J(C(X)/C_F(X)) = (0)$ and $x\in \mathrm{cl}\,D\setminus D$. Since X is first countable, there exists a sequence $\{x_n\}$ in D such that $x_n\to x$. Now, by complete regularity of υX there exists $f^\upsilon\in C(\upsilon X)$ such that $\upsilon X\setminus Z(f^\upsilon) = \{x_1, x_2, \ldots\}$ and therefore $\mathrm{cl}_{\upsilon X}(\upsilon X\setminus Z(f^\upsilon)) =$

$\{x_1, x_2, \ldots\} \cup \{x\}$ is compact. This means that $f^{\upsilon} \in C_K(\upsilon X) \cap M_Y$, where Y is the set of nonisolated points of υX, and therefore $f^{\upsilon} \in C_F(\upsilon X)$, by $(*)$. This is the required contradiction, for $f^{\upsilon} \in C_F(\upsilon X)$ means that $\upsilon X \setminus Z(f^{\upsilon})$ is finite. ∎

COROLLARY 5.10. *Let X be a nondiscrete P-space. Then either X is not first countable or the set of isolated points of X is not dense.*

Next, applying Corollary 5.9 or the previous result, and the fact that a discrete space is realcompact if and only if its cardinal is nonmeasurable (see [14, Theorem 12.2]), we have the following interesting fact.

COROLLARY 5.11. *The cardinality of a discrete space X is nonmeasurable if and only if υX is first countable.*

REMARK 5.12. If W is the space of countable ordinals, ω_1 is the first uncountable ordinal number and $W^{\star} = W \cup \{\omega_1\}$ is the one-point compactification of W (see [14, p. 74]), then by the previous results we have $J(C(W)/C_F(W)) \neq (0) \neq J(C(W^{\star})/C_F(W^{\star}))$.

Acknowledgments. The authors would like to thank the referee for a careful reading of this article.

REFERENCES

[1] F. Azarpanah, *On almost P-spaces*, Far East J. Math. Sci. Special Volume (2000), 121–132.

[2] —, *Essential ideals in $C(X)$*, Period. Math. Hungar. 31 (1995), 105–112.

[3] —, *Intersection of essential ideals in $C(X)$*, Proc. Amer. Math. Soc. 125 (1997), 2149–2154.

[4] —, *Algebraic properties of some compact spaces*, Real Anal. Exchange 25 (2000), 317–328.

[5] F. Azarpanah and O. A. S. Karamzadeh, *Algebraic characterization of some disconnected spaces*, Inst. J. Pure Appl. Math. 12 (2002), 155–168.

[6] F. Azarpanah, O. A. S. Karamzadeh and A. Rezai Aliabad, *On z°-ideals in $C(X)$*, Fund. Math. 160 (1999), 15–25.

[7] F. Azarpanah and M. Karavan, *On nonregular ideals and z°-ideals in $C(X)$*, Czechoslovak Math. J. 55 (2005), 397–407.

[8] F. Azarpanah and R. Mohamadian, *$\sqrt{z}$-ideals and $\sqrt{z^{\circ}}$-ideals in $C(X)$*, Acta Math. Sin. (Engl. Ser.) 23 (2007), 989–996.

[9] J. Dauns and L. Fuchs, *Infinite Goldie dimensions*, J. Algebra 115 (1988), 297–302.

[10] R. Engelking, *General Topology*, PWN–Polish Sci. Publ., 1977.

[11] A. A. Estaji and O. A. S. Karamzadeh, *On $C(X)$ modulo its socle*, Comm. Algebra 31 (2003), 1561–1571.

[12] L. Gillman, *Countably generated ideals in rings of continuous functions*, Proc. Amer. Math. Soc. 11 (1961), 660–666.

[13] —, *Convex pseudoprime ideals in $C(X)$*, in: General Topology and Applications (Middletown, CT, 1988), Lecture Notes in Pure Appl. Math. 123, Dekker, 1990, 87–95.

[14] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Springer, 1976.
[15] K. R. Goodearl and R. B. Warfield, Jr., *An Introduction to Noncommutative Noetherian Rings*, Cambridge Univ. Press, 1989.
[16] O. A. S. Karamzadeh, M. Motamedi and S. M. Shahrtash, *On rings with a unique proper essential right ideal*, Fund. Math. 183 (2004), 229–244.
[17] O. A. S. Karamzadeh and M. Rostami, *On the intrinsic topology and some related ideals of* $C(X)$, Proc. Amer. Math. Soc. 93 (1985), 179–184.
[18] K. Kunen, *Set Theory*, North-Holland, 1980.
[19] J. Lambek, *Lectures on Rings and Modules*, Blaisdell, 1966.
[20] R. Levy, *Almost P-spaces*, Canad. J. Math. 2 (1977), 284–288.
[21] M. A. Mulero, *Algebraic properties of rings of continuous functions*, Fund. Math. 149 (1996), 55–66.
[22] A. I. Veksler, P'*-points*, P'*-sets*, P'*-spaces. A new class of order-continuous measures and functionals*, Soviet Math. Dokl. 14 (1973), 1445–1450.
[23] S. Willard, *General Topology*, Addison-Wesley, Reading, MA, 1970.

Department of Mathematics
Chamran University
Ahvaz, Iran
E-mail: azarpanah@ipm.ir
karamzadeh@ipm.ir

Received 20 September 2006;
revised 6 July 2007 (4828)