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ON THE SIZE OF $L(1,\chi)$ AND S. CHOWLA'S HYPOTHESIS IMPLYING THAT $L(1,\chi)>0$ FOR $s>0$ AND FOR REAL CHARACTERS χ

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Abstract. We give explicit constants κ such that if χ is a real non-principal Dirichlet character for which $L(1,\chi)\le\kappa$, then Chowla's hypothesis is not satisfied and we cannot use Chowla's method for proving that $L(s,\chi)>0$ for $s>0$. These constants are larger than the previous ones $\kappa=1-\log 2=0.306\ldots$ and $\kappa=0.367\ldots$ we obtained elsewhere.

1. Introduction. Throughout this paper, we let χ be a real non-principal Dirichlet character modulo $f>1$. Setting $\chi_0=\chi$, we define inductively the functions χ_k for $k\ge 0$ by means of

$$\chi_{k+1}(n)=\sum_{a=0}^{n}\chi_k(a).$$

Define

$$m(\chi):=\min\{k\ge 0\,;\chi_k\ge 0\}$$

if this set is non-empty and $m(\chi)=\infty$ otherwise. Since χ_1 is f-periodic and $|\chi_1(n)|\le f$ for all $n\ge 0$, by induction on $k\ge 0$ we have

$$\Gamma(s)L(s,\chi)=\int_0^\infty(1-e^{-t})^k\Big(\sum_{n\ge 1}\chi_k(n)e^{-nt}\Big)t^s\,\frac{dt}{t}\qquad(k\ge 1\text{ and }\Re(s)>0).$$

In particular, if $m(\chi)<\infty$ then $L(s,\chi)>0$ for $s>0$ (see [Cho], [CD], [CDH], [CH] and [Ros] for further results). *Chowla's Hypothesis* asserts that $m(\chi)<\infty$ for all non-principal real characters χ (see [Cho]). H. Heilbronn disproved this hypothesis (see [Heil] and the historical remarks in [BM, p. 25]). In fact, the set of real non-principal characters χ for which $m(\chi)<\infty$ has asymptotic density 0 in $\{\chi$ real and non-principal$\}$ (see [BM, Corollary] and use Proposition 1.1).

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Now, let

$$P(t,\chi) := \sum_{n=1}^{f} \chi(n)t^n \quad \text{and} \quad F(t,\chi) := \sum_{n\geq 1} \chi(n)t^n \qquad (|t|<1)$$

be the *Fekete polynomial* and the related infinite series (see [FP]).

PROPOSITION 1.1 (see [BPW, Lemma 6]). *The following assertions are equivalent:*

(i) $m(\chi) < \infty$,
(ii) $P(t,\chi) > 0$ *for* $t \in (0,1)$ (*which can be checked numerically by using* Sturm's algorithm), *and*
(iii) $F(t,\chi) > 0$ *for* $t \in (0,1)$.

If χ is odd, then χ_1 being f-periodic we have $\chi_1 \geq 0$ if and only if $\chi_1(n) \geq 0$ for $1 \leq n \leq f$. If χ is even, then $\chi_1(f-2) = -1$, but χ_2 being f-periodic we have $\chi_2 \geq 0$ if and only if $\chi_2(n) \geq 0$ for $1 \leq n \leq f$. Both these conditions can easily be checked numerically, i.e., one can easily ascertain whether $m(\chi) = 1$ for χ odd or whether $m(\chi) = 2$ for χ even, both in time $O(f^{1+\epsilon})$. However, we let the reader think about how one could for some given χ, (a) ascertain that say $\chi_{40} \geq 0$, (b) check whether $m(\chi) < \infty$ (Sturm's algorithm invoked in Proposition 1.1 is computationally useless for f not that large), and (c) compute $m(\chi)$ if this is the case. We will come back to these problems in a forthcoming paper.

Our present problem is to explain how one can sometimes ascertain that $m(\chi) = \infty$ by proving relationships between Chowla's Hypothesis and the size of $L(1,\chi)$. S. Chowla observed that $L(1,\chi) \geq 1/(1+m(\chi))$. In [Lou03] and [Lou04], we greatly improved upon S. Chowla's result by proving that $L(1,\chi) \leq 1 - \log 2 = 0.306\ldots$ implies $m(\chi) = \infty$. (By [CE] and [Ell], it follows that $m(\chi) = \infty$ for infinitely many real non-principal characters χ). In Theorem 2.1, we give a general result which enables us to obtain constants greater than $1 - \log 2$ for which this result still holds true:

THEOREM 1.2. *Let χ be a real non-principal Dirichlet character.*

(1) *If* $\chi(2) = -1$ *and* $L(1,\chi) \leq 0.373043$, *then* $m(\chi) = \infty$.
(2) *If* $\chi(2) = 0$ *and* $L(1,\chi) \leq 0.545986$, *then* $m(\chi) = \infty$.
(3) *If* $\chi(2) = +1$ *and* $L(1,\chi) \leq 0.939751$, *then* $m(\chi) = \infty$.
(4) *If* $\chi(3) = 0$ *and* $L(1,\chi) \leq 0.470215$, *then* $m(\chi) = \infty$.
(5) *If* $\chi(2) = \chi(3) = 0$ *and* $L(1,\chi) \leq 0.690830$, *then* $m(\chi) = \infty$.
(6) *If* $\chi(2) = \chi(3) = +1$ *and* $L(1,\chi) \leq 1.624353$, *then* $m(\chi) = \infty$.

With larger constants we are more likely to be able to ascertain fast that $m(\chi) = \infty$ for a given character χ. Indeed, assume that χ is odd and primitive mod $f > 1$. The analytic class number formula yields $L(1,\chi) =$

$\pi h_f/\sqrt{f}$, where h_f is the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-f})$ of conductor f. The point is that h_f can be rigorously computed in time $O(f^{1/2+\epsilon})$ (see [Lou02]). Hence, for some odd χ's mod f we can ascertain that $m(\chi)=\infty$ in time $O(f^{1/2+\epsilon})$. The same remark applies to the case of even primitive characters.

Now, if ψ mod fd is induced by χ mod f, then

$$L(s,\psi)=L(s,\chi)\prod_{p|d}\left(1-\frac{\chi(p)}{p^s}\right)\quad(\Re(s)>0). \tag{1}$$

The *Generalized Chowla Hypothesis* asserts that for any χ there exists some ψ induced by χ such that $m(\psi)<\infty$ (see [CDH], [CH] or [Ros]), which implies $L(s,\psi)>0$ for $s>0$, and $L(s,\chi)>0$ for $s>0$, by (1). In fact, it has been conjectured that if χ is odd then there exists some ψ mod fd induced by χ such that $m(\psi)=1$, i.e. such that $\psi_1(n)\geq 0$ for $1\leq n\leq fd$. However (see [CDH]), nobody has been able to prove this hypothesis in the difficult special case that χ is the odd character mod 163 associated with the imaginary quadratic field $\mathbb{Q}(\sqrt{-163})$ of class number 1, for which $L(1,\chi)=\pi/\sqrt{163}$ is small (since $\pi/\sqrt{163}<1-\log 2$, we know beforehand that $m(\chi)=\infty$ for this character). In fact, it is because he could not check this hypothesis in that case that J. B. Rosser developed in [Ros] a completely different technique to prove that $L(s,\chi)>0$ for $s\in(0,1)$ for this character mod 163. We will explain in Section 7 how the present ideas could help us find such a d for this character, if one exists. In particular, see Proposition 6.2.

2. The main idea

THEOREM 2.1. *If $m(\chi)<\infty$, then for any $t\in(0,1)$ we have*

$$L(1,\chi)>G(t,\chi):=\sum_{n\geq 1}\frac{\chi(n)}{n}t^n=\log(1-t)+\sum_{n\geq 1}\frac{1+\chi(n)}{n}t^n.$$

Proof. By induction on $k\geq 0$, for $t\in(0,1)$, we have

$$tG'(t,\chi)=\sum_{n\geq 1}\chi(n)t^n=(1-t)^k\sum_{n\geq 1}\chi_k(n)t^n.$$

Consequently, if $m(\chi)<\infty$, then $G'(t,\chi)>0$ and $t\mapsto G(t,\chi)$ increases with $t\in(0,1)$. Since $\lim_{1^-}G(t,\chi)=L(1,\chi)$, by Abel's theorem, we obtain $L(1,\chi)>G(t,\chi)$ for any $t\in(0,1)$. ∎

We derive explicit results from Theorem 2.1. The key point is that $(1+\chi(n))/2\geq 0$ for $n\geq 1$, which yields

$$G(t,\chi)\geq\log(1-t)+\sum_{n\in E}\frac{1+\chi(n)}{n}t^n\quad(t\in(0,1)) \tag{2}$$

for any set E (finite or infinite) of positive integers. In particular, by taking $E=\{1\}$, we have $G(t,\chi)\geq f(t):=\log(1-t)+2t$. Since $f'(t)=(1-2t)/(1-t)$, we choose $t_0=1/2$ and find that $F(t_0,\chi)\geq f(t_0)=1-\log 2$. Hence, by Theorem 2.1, we have a very short proof of the following result obtained in [Lou03]:

Corollary 2.2. *If* $L(1,\chi)\leq 1-\log 2=0.306852\ldots$, *then* $m(\chi)=\infty$.

3. Taking into account prime numbers p for which $\chi(p)=0$. In some cases, we can readily improve upon this result. For example, let us assume that χ ranges over the characters for which $\chi(2)=0$. Then $\chi(1)=1$, $\chi(n)=0$ for $n\geq 2$ even and $\chi(n)\geq -1$ otherwise. Hence, for $t\in(0,1)$,

$$G(t,\chi)\geq F_2(t):=\log(1-t)+2t+\sum_{m\geq 1}\frac{1}{2m}t^{2m}=2t+\frac{1}{2}\log\left(\frac{1-t}{1+t}\right).$$

More generally, assume that $\chi(p)=0$ for all the prime divisors $p\geq 2$ of a squarefree integer $d_0>1$. Then $(1+\chi(n))/n=1/n$ for $\gcd(n,d_0)>1$. Hence,

$$\log(1-t)+\sum_{\gcd(n,d_0)>1}\frac{1+\chi(n)}{n}t^n=\sum_{\delta\mid d_0}\frac{\mu(\delta)}{\delta}\log(1-t^\delta)\qquad(t\in(0,1)).$$

Since $\chi(1)=1$, using (2) with $E=\{1\}\cup\{n\geq 1\,;\,\gcd(n,d_0)>1\}$ we obtain

$$G(t,\chi)\geq F_{d_0}(t):=2t+\sum_{\delta\mid d_0}\frac{\mu(\delta)}{\delta}\log(1-t^\delta)\qquad(t\in(0,1)).\tag{3}$$

Set $P_{d_0}(t):=(1-t^d)F'_{d_0}(t)\in\mathbb{Z}[X]$. Both $F_{d_0}(t)$ and $P_{d_0}(t)$ can be computed inductively by using $F_1(t)=\log(1-t)+2t$, $P_1(t)=1-2t$,

$$F_{d_0p}(t)=F_{d_0}(t)-\frac{1}{p}F_{d_0}(t^p)+\frac{2}{p}t^p$$

and

$$P_{d_0p}(t)=\frac{1-t^{dp}}{1-t^d}P_{d_0}(t)-t^{p-1}P_{d_0}(t^p)+2(1-t^{d_0p})t^{p-1},$$

where $p\geq 2$ is a prime that does not divide $d\geq 1$.

For $d_0=2$, we obtain $P_2(t)=1-2t^2$, whose only root in $(0,1)$ is $t_0=1/\sqrt{2}$, for which $F_2(t_0)=\sqrt{2}-\log(1+\sqrt{2})=0.532839\ldots$. Hence, if 2 divides f and $L(1,\chi)\leq\sqrt{2}-\log(1+\sqrt{2})=0.532839\ldots$, then $m(\chi)=\infty$ (to be improved in Proposition 5.3).

For $d_0=6=2\cdot 3$, we obtain $P_6(t)=1-t^4-2t^6$, whose only root in $(0,1)$ is $t_0=0.810\ldots$, for which $F_6(t_0)=0.690357\ldots$ (to be improved in Proposition 5.5). More generally, we have:

d_0	t_0	$F_d(t_0)$
$30 = 2\cdot 3\cdot 5$	0.8586	0.772333...
$210 = 2\cdot 3\cdot 5\cdot 7$	0.8886	0.828093...
$2310 = 2\cdot 3\cdot 5\cdot 7\cdot 11$	0.9059	0.855750...
$30030 = 2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13$	0.9198	0.879331...
$510510 = 2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17$	0.9296	0.894912...
$9699690 = 2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19$	0.9380	0.909271...
$223092870 = 2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23$	0.9444	0.920087...
$6469693230 = 2\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot 29$	0.9491	0.927153...

In the next section, we will generalize this approach to improve upon (3) (see (9)): given two finite disjoint sets $\mathcal{P}_0$ and $\mathcal{P}_1$ of prime numbers, we want to find a lower bound on $G(t,\chi)$ where χ ranges over all the real and non-principal Dirichlet characters such that $\chi(p) = 0$ if $p \in \mathcal{P}_0$ and $\chi(p) = +1$ if $p \in \mathcal{P}_1$.

4. Taking into account any finite set of prime numbers. Fix three finite pairwise disjoint (possibly empty) sets $\mathcal{P} = \{p_k\,; 1 \le k \le m\}$, $\mathcal{P}_0$ and $\mathcal{P}_1$ of m, m_0 and m_1 prime numbers. Set $d_0 = \prod_{p\in\mathcal{P}_0} p \ge 1$. Let χ range over the real and non-principal Dirichlet characters for which $\chi(p) = 0$ if $p \in \mathcal{P}_0$ and $\chi(p) = +1$ if $p \in \mathcal{P}_1$. Fix $N \ge 1$. Set $l_N = \operatorname{lcm}\{n\,; 1 \le n \le N\}$. Let $E_N(\mathcal{P},\mathcal{P}_1)$ denote the set of those positive integers less than or equal to N whose prime divisors lie in $\mathcal{P}\cup\mathcal{P}_1$. Hence, $1 \in E_N(\mathcal{P},\mathcal{P}_1)$. Using (2) with $E_N(\mathcal{P},\mathcal{P}_1) \cup \{n \ge 1\,; \gcd(n,d_0) > 1\}$, we obtain

$$(4)\quad G(t,\chi) \ge \sum_{\delta\mid d_0} \frac{\mu(\delta)}{\delta}\log(1-t^\delta) + \sum_{n\in E_N(\mathcal{P},\mathcal{P}_1)} \frac{1+\chi(n)}{n} t^n \qquad (t \in (0,1)).$$

Now, the idea is to prove that the worst case in (4) is when χ takes on the value -1 on as many prime numbers as possible, i.e. when $\chi(p) = -1$ for $p \in \mathcal{P}$.

Let $\lambda_{\mathcal{P}_1}$ denote the completely multiplicative arithmetic function defined on the prime numbers by

$$(5)\qquad \lambda_{\mathcal{P}_1}(p) = \begin{cases} -1 & \text{if } p \notin \mathcal{P}_1, \\ +1 & \text{if } p \in \mathcal{P}_1. \end{cases}$$

Finally, for $(x_1,\dots,x_m) \in \{-1,0,1\}^m$, let $X_{x_1,\dots,x_m,\mathcal{P}_1}$ denote the completely multiplicative arithmetic function defined on the prime numbers by

$$X_{x_1,\dots,x_m,\mathcal{P}_1}(p) = \begin{cases} x_k & \text{if } p = p_k \in \mathcal{P}, \\ +1 & \text{if } p \in \mathcal{P}_1. \end{cases}$$

Set

$$P_N(x_1,\dots,x_m,\mathcal{P},\mathcal{P}_1;t) := l_N \sum_{n\in E_N(\mathcal{P},\mathcal{P}_1)} \frac{1+X_{x_1,\dots,x_m,\mathcal{P}_1}(n)}{n} t^n \in \mathbb{Z}[X].$$

In particular,

$$P_N(-1,\dots,-1,\mathcal{P},\mathcal{P}_1;t) = l_N \sum_{n\in E_N(\mathcal{P},\mathcal{P}_1)} \frac{1+\lambda_{\mathcal{P}_1}(n)}{n} t^n \in \mathbb{Z}[X]. \tag{6}$$

For any real and non-principal Dirichlet character χ, we can choose the x_k's so that $x_k = \chi(p_k)$ for $1 \le k \le m$, which yields

$$\sum_{n\in E_N(\mathcal{P},\mathcal{P}_1)} \frac{1+\chi(n)}{n} t^n = \frac{1}{l_N} P_N(x_1,\dots,x_m,\mathcal{P},\mathcal{P}_1;t).$$

Hence, we deduce that for any real and non-principal Dirichlet characters χ for which $\chi(p) = 0$ if $p \in \mathcal{P}_0$ and $\chi(p) = +1$ if $p \in \mathcal{P}_1$, any $t \in (0,1)$, any $m \ge 1$ and any $N \ge 1$ we have

(7)

$$G(t,\chi) \ge \sum_{\delta\mid d_0} \frac{\mu(\delta)}{\delta} \log(1-t^\delta) + \frac{1}{l_N} \min_{(x_1,\dots,x_m)\in\{-1,0,1\}^m} P_N(x_1,\dots,x_m,\mathcal{P},\mathcal{P}_1;t).$$

Now, by Theorem 2.1, we want to compute the greatest value as t ranges in $(0,1)$ of the right hand side of (7). Of course, we could perform some numerical analysis to evaluate the greatest value as t ranges in $(0,1)$ of each of the 3^m functions

$$\sum_{\delta\mid d_0} \frac{\mu(\delta)}{\delta} \log(1-t^\delta) + \frac{1}{l_N} P_N(x_1,\dots,x_m,\mathcal{P},\mathcal{P}_1;t),$$

as $(x_1,\dots,x_m)$ ranges over $\{-1,0,1\}^m$. However, for most choices of $\mathcal{P}$, $\mathcal{P}_1$ and N, Lemma 4.1 below enables us to greatly simplify this task, namely, to prove that for any $t \in (0,1)$ we have

(8)

$$\min_{(x_1,\dots,x_m)\in\{-1,0,1\}^m} P_N(x_1,\dots,x_m,\mathcal{P},\mathcal{P}_1;t) = P_{m,N}(-1,\dots,-1,\mathcal{P},\mathcal{P}_1;t).$$

For example, it readily shows that this holds true for $N = 1000$, $m = 10$, $\mathcal{P} = \{2,3,5,7,11,13,17,19,23,29\}$ and $\mathcal{P}_0 = \mathcal{P}_1 = \emptyset$.

LEMMA 4.1. *Let* $P(t) = \sum_{k=1}^N p_k t^k$ *and* $Q(t) = \sum_{k=1}^N q_k t^k$. *Set* $\Delta_l = \sum_{k=1}^l (p_k - q_k)$. *If* $\Delta_l \ge 0$ *for* $1 \le l \le N$, *then* $P(t) \ge Q(t)$ *for* $t \in [0,1]$.

Proof. Set $\Delta_0 = 0$. Then

$$P(t) - Q(t) = \sum_{k=1}^N (\Delta_k - \Delta_{k-1}) t^k = \Big(\sum_{k=1}^{n-1} \Delta_k (t^k - t^{k+1})\Big) + \Delta_N t^N. \ \blacksquare$$

Using (7), (8) and (6) we finally obtain a general improvement on (3):

Proposition 4.2. *Fix three finite pairwise disjoints (possibly empty) sets $\mathcal{P}$, $\mathcal{P}_0$ and $\mathcal{P}_1$ of m, m_0 and m_1 prime numbers. Set $d_0 = \prod_{p\in\mathcal{P}_0} p \geq 1$. Let $\lambda_{\mathcal{P}_1}$ be as in* (5). *Fix $N \geq 1$ and set $l_N = \operatorname{lcm}\{n\,; 1 \leq n \leq N\}$. Let $E_N(\mathcal{P},\mathcal{P}_1) \ni 1$ denote the set of all positive integers $\leq N$ whose prime divisors lie in $\mathcal{P}\cup\mathcal{P}_1$.*

Assume that the following hypothesis (H) *holds true: as $(x_1,\dots,x_m)$ runs over the 3^m elements of $\{-1,0,+1\}^m$, the rational integers*

$$\Delta_l := \sum_{\substack{n=1\\ n\in E_N(\mathcal{P},\mathcal{P}_1)}}^{l} \frac{l_N}{n}(X_{x_1,\dots,x_m,\mathcal{P}_1}(n) - \lambda_{\mathcal{P}_1}(n)) \in \mathbb{Z}$$

are non-negative for $1 \leq l \leq N$.

Then, for any real and non-principal Dirichlet characters χ for which $\chi(p) = 0$ if $p \in \mathcal{P}_0$ and $\chi(p) = +1$ if $p \in \mathcal{P}_1$, we have $G(t,\chi) \geq F(t)$ for $t \in (0,1)$, where

$$F(t) = F_N(t,\mathcal{P},\mathcal{P}_0,\mathcal{P}_1) := \sum_{\delta|d_0} \frac{\mu(\delta)}{\delta}\log(1-t^\delta) + 2\sum_{n\in E}\frac{1}{n}t^n \tag{9}$$

and $E := \{n \in E_N(\mathcal{P},\mathcal{P}_1)\,; \lambda_{\mathcal{P}_1}(n) = +1\} \ni 1$.

Moreover, $(1-t^d)F'(t) \in \mathbb{Z}[t]$.

We would like to emphasize that, in order to use Proposition 4.2, we only have to check that the $3^m N$ rational integers Δ_l are not negative, which in principle could be done by hand. Moreover, if $m = 0$, i.e. if $\mathcal{P} = \emptyset$, then hypothesis (H) clearly holds true.

5. Proof of Theorem 1.2. Taking $m=8$, $\mathcal{P}=\{2,3,5,7,11,13,17,19\}$, $\mathcal{P}_0 = \mathcal{P}_1 = \emptyset$ and $N = 40$, we obtain $E = \{1,4,6,9,10,14,15,16,21,22,24,25,26,33,34,35,36,38,39,40\}$. Using Prof. Kida's UBASIC on a PC, we checked in 10 seconds that Hypothesis (H) is satisfied. Hence, we obtain

$$G(t,\chi) \geq F(t) := \log(1-t) + 2\sum_{n\in E}\frac{1}{n}t^n, \qquad t \in (0,1).$$

Choosing $t = t_0 = 0.670\ldots$, the only real root in $(0,1)$ of $P(t) := (1-t)F'(t) = -1 + 2(1-t)\sum_{n\in E} t^{n-1}$, we obtain:

Proposition 5.1. $L(1,\chi) \leq F(t_0) = 0.373043\ldots$ *implies $m(\chi) = \infty$.*

Assume that $\chi(2) = +1$. Taking $m = 7$, $\mathcal{P} = \{3,5,7,11,13,17,19\}$, $\mathcal{P}_0 = \emptyset$, $\mathcal{P}_1 = \{2\}$ and $N = 60$, we obtain $E = \{1,2,4,8,9,15,16,18,21,25,$

$30, 32, 33, 35, 36, 39, 42, 49, 50, 51, 55, 57, 60\}$ and

$$G(t,\chi) \geq F(t) := \log(1-t) + 2\sum_{n\in E} \frac{1}{n}t^n, \qquad t \in (0,1).$$

Choosing $t = t_0 = 0.859\ldots$, the only real root in $(0,1)$ of $P(t) := (1-t)F'(t) = -1 + 2(1-t)\sum_{n\in E} t^{n-1}$, we obtain:

Proposition 5.2. *$\chi(2) = +1$ and $L(1,\chi) \leq F(t_0) = 0.939751\ldots$ imply $m(\chi) = \infty$.*

Assume that $\chi(2)=0$. Taking $m=7$, $\mathcal{P}=\{3,5,7,11,13,17,19\}$, $\mathcal{P}_0=\{2\}$, $\mathcal{P}_1 = \emptyset$ and $N = 60$, we obtain $E = \{1, 9, 15, 21, 25, 33, 35, 39, 49, 51, 55, 57\}$ and

$$G(t,\chi) \geq F(t) := \frac{1}{2}\log\left(\frac{(1-t)^2}{1-t^2}\right) + 2\sum_{n\in E} \frac{1}{n}t^n, \qquad t \in (0,1).$$

Choosing $t = t_0 = 0.741\ldots$, the only real root in $(0,1)$ of $P(t) := (1-t^2)F'(t) = -1 + 2(1-t^2)\sum_{n\in E} t^{n-1}$, we obtain:

Proposition 5.3. *$\chi(2) = 0$ and $L(1,\chi) \leq F(t_0) = 0.545986\ldots$ imply $m(\chi) = \infty$.*

Assume that $\chi(3) = 0$. Taking $m = 9$, $\mathcal{P} = \{2, 5, 7, 11, 13, 17, 19, 23, 29\}$, $\mathcal{P}_0 = \{3\}$, $\mathcal{P}_1 = \emptyset$ and $N = 60$, we obtain $E = \{1, 4, 10, 14, 16, 22, 25, 26, 34, 35, 38, 46, 49, 55, 58\}$ and

$$G(t,\chi) \geq F(t) := \frac{1}{3}\log\left(\frac{(1-t)^3}{1-t^3}\right) + 2\sum_{n\in E} \frac{1}{n}t^n, \qquad t \in (0,1).$$

Choosing $t = t_0 = 0.762\ldots$, the only real root in $(0,1)$ of $P(t) := (1-t^3)F'(t) = -1 - t + 2(1-t^3)\sum_{n\in E} t^{n-1}$, we obtain:

Proposition 5.4. *$\chi(3) = 0$ and $L(1,\chi) \leq F(t_0) = 0.470215\ldots$ imply $m(\chi) = \infty$.*

Assume that $\chi(2) = \chi(3) = 0$. Taking $m = 4$, $\mathcal{P} = \{5, 7, 11, 13\}$, $\mathcal{P}_0 = \{2, 3\}$, $\mathcal{P}_1 = \emptyset$ and $N = 65$, we obtain $E = \{1, 25, 35, 49, 55, 65\}$ and

$$G(t,\chi) \geq F(t) := \frac{1}{6}\log\left(\frac{(1-t)^6(1-t^6)}{(1-t^2)^3(1-t^3)^2}\right) + 2\sum_{n\in E} \frac{1}{n}t^n, \qquad t \in (0,1).$$

Choosing $t = t_0 = 0.812\ldots$, the only real root in $(0,1)$ of $P(t) := (1-t^6)F'(t) = -1 - t^4 + 2(1-t^6)\sum_{n\in E} t^{n-1}$, we obtain:

Proposition 5.5. *$\chi(2) = \chi(3) = 0$ and $L(1,\chi) \leq F(t_0) = 0.690830\ldots$ imply $m(\chi) = \infty$.*

Finally, assume that $\chi(2) = \chi(3) = +1$. Taking $m = 0$, $\mathcal{P} = \mathcal{P}_0 = \emptyset$, $\mathcal{P}_1=\{2,3\}$ and $N=27$, we obtain $E=\{1,2,3,4,6,8,9,12,16,18,24,25,27\}$

and

$$G(t,\chi) \geq F(t) := \log(1-t) + 2\sum_{n\in E}\frac{1}{n}t^n, \qquad t\in(0,1).$$

Choosing $t = t_0 = 0.928\ldots$, the only real root in $(0,1)$ of $P(t) := (1-t)F'(t) = -1 + 2(1-t)\sum_{n\in E} t^{n-1}$, we obtain:

PROPOSITION 5.6. *$\chi(2)=\chi(3)=+1$ and $L(1,\chi)\leq F(t_0)=1.624353\ldots$ imply $m(\chi)=\infty$.*

6. On the Generalized Chowla Hypothesis. For χ a character mod f and $d\geq 1$, let $\chi^{(d)}$ be the character mod fd induced by χ. We say that the *Generalized Chowla Hypothesis* holds true for χ if there exists $d\geq 1$ such that $m(\chi^{(d)})<\infty$, in which case $d_\chi\geq 1$ denotes the least such $d\geq 1$. Otherwise, we set $d_\chi = 0$. Hence, $d_\chi = 1$ if and only if $m(\chi)<\infty$. Let also $2\leq p_1(\chi) < p_2(\chi) < \cdots$ be the sorted prime numbers in the set $\{p\,;p\geq 2 \text{ prime and } \chi(p)=-1\}$. Set $D_0=1$ and $D_t = D_t(\chi) = \prod_{k=1}^t p_k(\chi)$ for $t\geq 1$. Set $D_\chi = D_{t_\chi}$ where $t_\chi := \min\{t\geq 0\,; m(\chi^{(D_t)})<\infty\}$ if this set is not empty. Otherwise, set $D_\chi = 0$.

LEMMA 6.1. *Let ϕ be a non-principal character. Let $p\geq 2$ be a prime.*

(1) *If $\phi(p)=0$, then $m(\phi)=m(\phi^{(p)})$.*
(2) *If $m(\phi)=\infty$ and $\phi(p)\neq -1$, then $m(\phi^{(p)})=\infty$.*
(3) *If $m(\phi)<\infty$ and $\phi(p)\neq +1$, then $m(\phi^{(p)})<\infty$.*

Proof. Assume that $\phi(p)=0$. Then $\phi(n)=\phi^{(p)}(n)$ for any $n\geq 1$. Hence $\phi_k = \phi_k^{(p)}$ for any $k\geq 1$, and $m(\phi)=m(\phi^{(p)})$.

Since $G(0,\phi)=0$ and $G'(0,\phi)=1$, if $m(\phi)=\infty$ then we can define $t_\phi = \min\{t\in(0,1)\,; G(t,\phi)=0\}$. We have $G(t,\phi)>0$ for $t\in(0,t_\phi)$. Since $\phi^{(p)}(n)=\phi(n)$ if p does not divide n and $\phi^{(p)}(n) = 0 = \phi(n)-\phi(n)$ if p divides n, we obtain

$$G(t,\phi^{(p)}) = \sum_{n\geq 1}\phi(n)t^n - \sum_{n\geq 1 \text{ and } p\mid n}\phi(n)t^n = G(t,\phi) - \phi(p)G(t^p,\phi).$$

If $m(\phi)=\infty$ and $\phi(p)\neq -1$, then $G(t_\phi,\phi)=0$ and $G(t_\phi,\phi^{(p)})\leq G(t_\phi,\phi)\leq 0$, hence $m(\phi^{(p)})=\infty$. If $m(\phi)<\infty$ and $\phi(p)\neq +1$, then $G(t,\phi)>0$ for $t\in(0,1)$ and $G(t,\phi^{(p)})\geq G(t,\phi)>0$ for $t\in(0,1)$, hence $m(\phi^{(p)})<\infty$. ∎

PROPOSITION 6.2. *If $d_\chi>1$, then d_χ is squarefree and such that $p\mid d_\chi$ implies $\chi(p)=-1$. Moreover, the Generalized Chowla Hypothesis holds true for χ if and only there exists $t\geq 0$ such that $m(\chi^{(D_t)})<\infty$.*

Proof. Let p be any prime divisor of $d_\chi>1$, write $d_\chi = dp$ and set $\phi=\chi^{(d)}$. Hence $\phi^{(p)}=\chi^{(d_\chi)}$. If p divides d, then $\phi(p)=0$ and $m(\chi^{(d)}) = m(\chi^{(d_\chi)})<\infty$ (by Lemma 6.1(1)) and $1\leq d<d_\chi$, a contradiction. Hence,

d_χ is squarefree, which implies $\phi(p) = \chi(p)$. If $\chi(p) \neq -1$, then $\phi(p) \neq -1$ and $m(\chi^{(d)}) < \infty$ (by Lemma 6.1(2)) and $1 \leq d < d_\chi$, a contradiction. Finally, by Lemma 6.1(3), if $d_\chi \neq 0$ then $m(\chi^{D_t}) < \infty$ as soon as d_χ divides D_t, i.e. as soon as t is large enough. ∎

Hence, we have $d_\chi \neq 0 \Leftrightarrow D_\chi \neq 0$ and $d_\chi \leq D_\chi$. For the real and odd character χ mod $f = 43$ we have $\chi(2) = \chi(3) = -1$, $m(\chi) = \infty$, $m(\chi^{(2)}) < \infty$ and $m(\chi^{(3)}) < \infty$ (use Sturm's algorithm). Hence, $d_\chi = 2$ but $m(\chi^{(3)}) < \infty$. This example shows that if $d_\chi \neq 0$, then we cannot expect d_χ to have the nice property that $m(\chi^{(d)}) < \infty$ if and only if $d_\chi \mid d$, even when d is restricted to be a squarefree integer such that $p \mid d_\chi$ implies $\chi(p) = -1$.

7. A computational challenge. Let χ be the real and odd Dirichlet character mod 163. Hence, $\chi(p) = -1$ for $p \leq 37$ a prime number. Set $E_\chi := \{d \geq 2\,; d \text{ squarefree and } p \mid d \text{ implies } \chi(p) = -1\}$.

We have $L(1,\chi) = \pi/\sqrt{163} = 0.246068\ldots$. Hence, $m(\chi) = \infty$, by Corollary 2.2.

The challenges are (i) to computationally prove that $d_\chi \neq 0$, and (ii) to find either d_χ or D_χ.

We have not found any of these two invariants, but we want to present the reader who would like to tackle their determination with some ideas to speed up his computation: we explain how one can easily get rid of many d's when d ranges over the positive integers less than or equal to a prescribed upper bound B.

7.1. Speeding up the search for d_χ. Let B be given. Set $E(B) := \{d \geq 2\,; d \leq B \text{ and } d \text{ squarefree}\}$ and let d range in $E_\chi(B) := \{d \in E_\chi\,; d \leq B\}$. If $m(\chi^{(d)}) < \infty$, where $d \in E_\chi$, then $L(1,\chi^{(d)}) = \psi(d)L(1,\chi) > 0.373043$, where

$$\psi(d) := \prod_{p|d}(1 + p^{-1}).$$

Hence, we must have $\psi(d) > 0.373043\sqrt{163}/\pi = 1.516012\ldots$. In particular, d cannot be a prime number. Set $E'_\chi(B) := \{n \in E_\chi(B)\,; \psi(d) > 1.516012\}$. Now, noticing that $\chi(2) = -1$, we can consider two cases:

$$\psi(d) \begin{cases} 0.373043\sqrt{163}/\pi > 1.516012 & \text{if } \gcd(d,2) = 1, \\ 0.545986\sqrt{163}/\pi > 2.218837 & \text{if } \gcd(d,2) = 1, \end{cases}$$

and let $E''_\chi(B)$ denote the set of the d's in $E_\chi(B)$ that satisfy these conditions.

Finally, noticing that $\chi(2)=\chi(3)=-1$, we can consider four cases:

$$\psi(d)\begin{cases} 0.373043\sqrt{163}/\pi > 1.516012 & \text{if } \gcd(d,6)=1,\\ 0.545986\sqrt{163}/\pi > 2.218837 & \text{if } \gcd(d,6)=2,\\ 0.470215\sqrt{163}/\pi > 1.910910 & \text{if } \gcd(d,6)=3,\\ 0.690830\sqrt{163}/\pi > 2.807469 & \text{if } \gcd(d,6)=6,\end{cases}$$

and let $E_\chi'''(B)$ denote the set of the d's in $E_\chi(B)$ that satisfy these conditions.

If we had extended our range of computation in Theorem 1.2, we could have dealt with a finer distinction of cases. The following table shows that the finer our distinction of cases, the shorter our list of d's to test to be able to compute d_χ:

B	10^3	10^4	10^5	10^6	10^7	10^8
$\#E(B)$	607	6082	60793	607925	6079290	60792693
$\#E_\chi(B)$	387	3205	27806	250290	2298910	21386754
$\#E'_\chi(B)$	122	947	8453	72324	655508	6070111
$\#E''_\chi(B)$	42	289	2442	20924	187151	1717406
$\#E'''_\chi(B)$	1	20	139	1055	8785	76003

7.2. On the size of D_χ. As for D_χ, notice that $m(\chi^{(D_t)})<\infty$ if and only if $t\ge t_\chi$, by Lemma 6.1(3). Since $\chi(p)=-1$ for $p\le 29$ a prime, the $D_t=D_t(\chi)$'s for $3\le t\le 10$ are listed in the first column of the table of Section 3. Since

$$L(1,\chi^{(D_8)})=\psi(D_8)L(1,\chi)=\frac{165888\pi}{46189\sqrt{163}}=0.883756\ldots<0.909271,$$

we have $m(\chi^{(D_8)})=\infty$, by Section 3. Hence, $t_\chi\ge 9$ and D_9 divides D_χ. Moreover, $L(1,\chi^{(D_9)})=\psi(D_9)L(1,\chi)=\frac{3981312\pi}{1062347\sqrt{163}}=0.922180\ldots$ is not less than 0.920087. However, when applying (3) with $d=D_9$ to the character $\chi^{(D_9)}$ we may use the fact that $\chi^{(D_9)}(41)=\chi(41)=+1$ to add a term $2t^{41}/41$ to the right hand side of (3), which enables us to replace the $F_d(t_0)=0.920087$ of the eighth line of this table by the larger value 0.924760. Hence, $m(\chi^{(D_9)})=\infty$, $t_\chi\ge 10$ and $D_{10}=D_{10}(\chi)=6469693230$ divides D_χ. Notice that $f:=163D_{10}\approx 10^{12}$.

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