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ON METRICS OF CHARACTERISTIC ZERO

BY

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Abstract. We introduce and study the concept of characteristic for metrics. It turns out that metrizable spaces endowed with an L^*-operator which admit a metric of characteristic zero play an important role in the theory of fixed points. We prove the existence of such spaces among infinite-dimensional linear topological spaces.

1. Introduction. For a given metric (or pseudometric) space (X, ρ) let the *characteristic* of the metric (resp. pseudometric) ρ be defined as follows:

$$\chi(\rho) := \inf\{\operatorname{ord}(P) \cdot \rho(P) : P \in \mathcal{U}_X\},$$

where $\mathcal{U}_X$ is the family of all open point finite coverings of X, $\rho(P)$ is the diameter of the covering P, i.e.,

$$\rho(P) := \sup\{\operatorname{diam}(A) : A \in P\},$$

and $\operatorname{ord}(P)$ means the *order* of the covering P, i.e.,

$$\operatorname{ord}(P) := \sup\{|\{A \in P : x \in A\}| : x \in X\}.$$

The symbol $|C|$ stands for the cardinality of the set C and we set $\chi(\rho) = \infty$ in case $\operatorname{ord}(P)$ or $\rho(P)$ is infinite. In this paper, we are concerned primarily with the following general problem.

Problem 1. *Let X be a metrizable space. Does there exist a metric $\rho : X \times X \to [0, \infty)$ of characteristic zero, $\chi(\rho) = 0$, that is compatible with the topology of X?*

It is well known that any metrizable space X admits a metric ρ that is compatible with the topology of X and such that $\operatorname{diam} X \le 1$. Hence any metrizable space admits a metric that is compatible with the topology of the space and of characteristic not greater than 1. Let us notice that if X has finite covering dimension, $\dim X < \infty$, then $\chi(\rho) = 0$ for each metric ρ compatible with the topology of X. The Hilbert cube I^∞, i.e., the Cartesian product of countably many segments $[0, 1]$ with the product Tikhonov topology is an instance of an infinite-dimensional metrizable space

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that has a metric of characteristic zero and compatible with the topology of I^∞, namely the standard metric

$$\rho(x,y) := \sum_{n=1}^{\infty} \frac{|x_n - y_n|}{2^n}, \qquad x = (x_1, x_2, \ldots),\ y = (y_1, y_2, \ldots).$$

A variation of Problem 1 in the setting of linear topological spaces is of particular interest for us. Let us recall that a *subnorm* on a linear topological space E satisfies the following conditions:

$$\|x\| = 0 \ \Leftrightarrow\ x = 0, \qquad \|x+y\| \le \|x\| + \|y\|,$$
$$\|tx\| \le \|x\| \quad \text{for each } t,\ |t| \le 1.$$

PROBLEM 2. *Let X be a convex compact subset of a metrizable linear space E. Is there a subnorm $\|\cdot\| : E \to [0,\infty)$ such that the metric $\rho(x,y) := \|x - y\|$ for $x, y \in X$ has characteristic zero and is compatible with the topology of X?*

Our motivation and interest in studying metrizable spaces admitting metrics of characteristic zero stems from the fact that they play an important role in the theory of fixed points. To wit, let us recall the concept of an L^*-operator (introduced in [5]).

For a given non-empty set X, let $[X]^{<\omega}$ denote the set of all finite subsets of X and let $\exp(X)$ denote the family non-empty subsets of X. An *L^*-operator* on X is any map $\Lambda : [X]^{<\omega} \to \exp(X)$ that satisfies the following condition:

(∗) *For each open covering $\{U_a : a \in A\}$ of X, where $A \in [X]^{<\omega}$, there exists $B \subset A$ such that $\Lambda(B) \cap \bigcap\{U_b : b \in B\} \neq \emptyset$.*

The convex hull operator constitutes a basic example of an L^*-operator on any linear topological space (cf. [5], [4]). More examples can be found in [4], [5].

An L^*-operator $\Lambda : [X]^{<\omega} \to \exp(X)$ is said to be *uniformly continuous* with respect to a metric ρ if there exists a real $c \ge 1$ such that

$$\forall_{A \in [X]^{<\omega}}\ \forall_{\epsilon > 0}\ \forall_{x \in X} \quad A \subset B(x,\epsilon) \ \Rightarrow\ \Lambda(A) \subset B(x, \epsilon c).$$

An L^*-operator $\Lambda : [X]^{<\omega} \to \exp(X)$ is said to be *weakly uniformly continuous* with respect to ρ if

$$\forall_{A \in [X]^{<\omega}}\ \forall_{\epsilon > 0}\ \forall_{x \in X} \quad A \subset B(x,\epsilon) \ \Rightarrow\ \Lambda(A) \subset B(x, \epsilon |A|),$$

where $B(x,\epsilon) := \{y \in X : \rho(x,y) < \epsilon\}$.

Moreover, define $B(A,\eta) := \bigcup\{B(a,\eta) : a \in A\}$.

Let us verify that if $X \subset E$ is a convex subset of a metric linear space (E,ρ) with the metric ρ induced by a norm (resp. subnorm) $\|\cdot\| : E \to [0,\infty)$, $\rho(x,y) := \|x - y\|$, then the L^*-operator $\operatorname{conv} : [X]^{<\omega} \to \exp(X)$ is uniformly continuous (resp. weakly uniformly continuous) with respect to ρ.

Indeed, let $A \subset B(x,\epsilon)$, $A = \{x_0,\dots,x_n\}$, $z \in \operatorname{conv} A$, $z = \sum_{i=0}^n t_i x_i$, where $\sum_{i=0}^n t_i = 1$, $t_i \ge 0$. Then

$$\|z-x\| = \Big\|\sum_{i=0}^n t_i(x_i-x)\Big\| \le \sum_{i=0}^n \|t_i(x_i-x)\|.$$

If $\|\cdot\|$ is a norm, we get

$$\|z-x\| \le \sum_{i=0}^n t_i\|x_i-x\| < \epsilon \sum_{i=0}^n t_i = \epsilon.$$

If $\|\cdot\|$ is only a subnorm,

$$\|z-x\| = \sum_{i=0}^n \|t_i(x_i-x)\| \le \sum_{i=0}^n \|x_i-x\| < \epsilon\cdot(n+1) = \epsilon\cdot|A|.$$

Theorem 1 (A Brouwer–Schauder type theorem). *Let X be a metrizable space and Λ an L^*-operator on X. Suppose $g : X \to Y$ is a continuous map into a compact subspace Y of X, and ρ is a metric on X compatible with the topology. If the L^*-operator Λ is either uniformly continuous with respect to ρ, or weakly uniformly continuous and $\chi(\rho) = 0$ on Y, then g has a fixed point.*

Proof. Suppose g has no fixed point. Then $\epsilon := \inf\{\rho(x,g(x)) : x \in X\} > 0$. Choose an open finite covering P of Y, $P = \{U_x : x \in A\}$, $A \subset Y$, $x \in U_x$, such that $\rho(P) < \epsilon/(2c)$ and also $\operatorname{ord}(P)\cdot\rho(P) < \epsilon/2$ if Λ is weakly uniformly continuous with respect to ρ.

Applying the property $(*)$ to the covering $\{g^{-1}(U_x) : x \in A\}$ of X one can find a subset $B \subset A$ and a point $d \in X$ such that

$$d \in \Lambda(B) \cap \bigcap_{b\in B} g^{-1}(U_b).$$

Then $g(d) \in \bigcap_{b\in B} U_b$, and since $b \in U_b$ we get

$$B \subset \bigcup_{b\in B} U_b \subset B(g(d), 2\rho(P)).$$

We see that in the case when Λ is uniformly continuous,

$$d \in \Lambda(B) \subset B(g(d), 2c\rho(P)) \subset B(g(d),\epsilon),$$

and if Λ is weakly uniformly continuous then

$$d \in \Lambda(B) \subset B(g(d), 2|B|\rho(P)) \subset B(g(d),\epsilon),$$

contradicting $\rho(d,g(d)) \ge \epsilon$ in both cases. ■

Corollary. *If $g : X \to Y$ is a continuous map from a metrizable space X with an L^*-operator Λ which is weakly uniformly continuous with respect to some compatible metric on Y, where $Y \subset X$ is a compact subspace of finite dimension, then g has a fixed point.*

2. Constructing metrics of characteristic zero. A covering P of a set X is a *star-refinement* of a covering Q if for each $U \in P$ there is a set $V \in Q$ such that $\mathrm{st}(U, P) \subset V$, where $\mathrm{st}(U, P) := \bigcup\{W \in P : U \cap W \neq \emptyset\}$. The following lemma is of great importance for us. The idea of the proof is very old and it is due to Birkhoff, Kakutani and Tukey. Some clues can be found in Rudin [6] and Engelking [2].

LEMMA. *Let $\{P_n : n \in \mathbb{N}\}$ be a sequence of arbitrary coverings of a set X such that P_{n+1} is a star-refinement of P_n for each $n \in \mathbb{N}$. Let $\varphi : \mathbb{N} \to \mathbb{N}$ be an increasing function and let $\rho : X \times X \to [0, \infty)$ be defined as follows:*

$$\rho(x, y) := \inf\Big\{\sum_{i=1}^{m} [x_{i-1}, x_i] : x_0 = x,\, x_m = y,\, x_i \in X,\, m = 1, 2, \ldots\Big\},$$

where $[x_{i-1}, x_i] := 2^{-\varphi(n)}$ if n is the smallest number with $x_{i-1} \notin \mathrm{st}(x_i, P_{n+1})$, and $[x_{i-1}, x_i] = 0$ if $x_{i-1} \in \mathrm{st}(x_i, P_n)$ for each $n \in \mathbb{N}$.

Then ρ is a pseudometric such that $\mathrm{st}(x, P_n) \subset \overline{B}(x, 2^{-\varphi(n)})$. Moreover, if $\varphi(n) \le n$, then $B(x, 2^{-n}) \subset \mathrm{st}(x, P_n)$ for each $x \in X$. If $\mathrm{ord}(P_n) \le \varphi(n) < \infty$, then ρ is of characteristic zero.

Proof. Since $y \in \mathrm{st}(x, P_n)$ if and only if $x \in \mathrm{st}(y, P_n)$, we infer that $\rho(x, y) = \rho(y, x)$. By the definition of ρ we get immediately $\rho(x, x) = 0$, $\rho(x, y) \le \rho(x, z) + \rho(z, y)$ and $\mathrm{st}(x, P_n) \subset \overline{B}(x, 2^{-\varphi(n)})$. The last inclusion implies that $\rho(P_n) \le 2^{-\varphi(n)}$. Consequently, $\mathrm{ord}(P_n) \cdot \rho(P_n) \le \varphi(n) \cdot 2^{-\varphi(n)}$ whenever $\mathrm{ord}\, P_n \le \varphi(n)$.

To finish the proof, it remains to show that in the case $\varphi(n) = n$, the inclusion $B(x, 2^{-n}) \subset \mathrm{st}(x, P_n)$ holds. To do this, it suffices to prove by induction on m that for each n, $\sum_{i=1}^{m} [x_{i-1}, x_i] < 2^{-n}$ implies $x_m \in \mathrm{st}(x_0, P_n)$.

It is clear that the implication holds for $m = 1$. Let $m \ge 2$ and assume that $\sum_{i=1}^{k} [x_{i-1}, x_i] < 2^{-n}$. Then $[x_0, x_1] < 2^{-(n+1)}$ or $[x_{m-1}, x_m] < 2^{-(n+1)}$; we may assume that $[x_0, x_1] < 2^{-(n+1)}$. Let $k \le m - 1$ be the largest index with $\sum_{i=1}^{k} [x_{i-1}, x_i] < 2^{-(n+1)}$. Then $\sum_{i=1}^{k+1} [x_{i-1}, x_i] \ge 2^{-(n+1)}$ and $\sum_{i=k+1}^{m} [x_{i-1}, x_i] < 2^{-(n+1)}$. We consider two cases: $k < m-1$ and $k = m-1$.

If $k < m - 1$, then by the inductive hypothesis $x_0 \in \mathrm{st}(x_k, P_{n+1})$, $x_{k+1} \in \mathrm{st}(x_m, P_{n+1})$ and by assumption $x_{k+1} \in \mathrm{st}(x_k, P_{n+1})$ (because $[x_{i-1}, x_i] < 2^{-(n+1)}$). This means that there are three sets $U_1, U_2, U_3 \in P_{n+1}$ such that $x_0, x_k \in U_1$, $x_k, x_{k+1} \in U_2$ and $x_{k+1}, x_m \in U_3$. Since P_{n+1} is a star-refinement of P_n, there is $U \in P_n$ such that $U_1 \cup U_2 \cup U_3 \subset U$. Then $x_0, x_m \in U$, i.e. $x_m \in \mathrm{st}(x_0, P_n)$.

If $k = m-1$, then similarly $x_0 \in \mathrm{st}(x_{m-1}, P_{n+1})$ and $x_{m-1} \in \mathrm{st}(x_m, P_{m+1})$ by the inductive hypothesis. There are $U_1, U_2 \in P_{n+1}$ with $x_0, x_{m-1} \in U_1$ and $x_{m-1}, x_m \in U_2$ and so $x_0, x_m \in U_1 \cup U_2 \subset U$ for some $U \in P_n$. Again, $x_0 \in \mathrm{st}(x_m, P_n)$. ∎

We are ready to give instances of metrics compatible with the topology of a space and of characteristic zero.

If the pseudometric described in the Lemma turns out to be a metric, it will be called a *metric given by the development* $\{P_n : n \in \mathbb{N}\}$.

In this part we shall show some constructions of metrics compatible with given topology.

A family $\{P_n : n \in \mathbb{N}\}$ of open coverings of a topological T_1 space X is said to be a *metrizable development* if P_{n+1} is a star-refinement of P_n for each $n \in \mathbb{N}$, and the family $\{\mathrm{st}(x, P_n) : n \in \mathbb{N}\}$ is a local base at x for each $x \in X$.

The development $\{P_n : n \in \mathbb{N}\}$ has *characteristic* r, $0 \le r \le \infty$, if

$$\lim_{n\to\infty} \mathrm{ord}(P_n) \cdot \delta(P_n) = r.$$

For every covering P of X let

$$\mathrm{ord}^*(P) := \min\{n \in \mathbb{N} : P \text{ has an open finite refinement } Q \text{ of order } \le n\}.$$

The following corollary results from the above Lemma. It can also be thought of as a metrizability theorem of the Bing, Moore, Aleksandrov, and Urysohn type.

THEOREM 2. *Let* $\{P_n : n \in \mathbb{N}\}$ *be a metrizable development of a topological space* X*. Then the metric* $\rho : X \times X \to \mathbb{R}$ *as defined in the Lemma is compatible with the topology, and* $B(x, 2^{-n}) \subset \mathrm{st}(x, P_n) \subset \overline{B}(x, 2^{-\varphi(n)})$ *for each* $x \in X$*. If* $\lim_{n\to\infty} \mathrm{ord}^*(P_n)/2^n = 0$*, then* ρ *is of characteristic zero.*

If a linear space E is metrizable, then there is a local base $\{O_n : n \in \mathbb{N}\}$ at 0 satisfying the following conditions:

1. $tO_n \subset O_n$ for each $n \in \mathbb{N}$ and $|t| \le 1$,
2. $3O_{n+1} \subset O_n$, and
3. $\{0\} = \bigcap\{O_n : n \in \mathbb{N}\}$.

A local base $\{O_n : n \in \mathbb{N}\}$ at 0 satisfying conditions 1–3 will be called a *locally metrizable development* at 0. It induces a metrizable development $\{P_n : n \in \mathbb{N}\}$ of E by setting $P_n := \{O_n + x : x \in E\}$.

Let us notice that the metric as defined in the Lemma has the following properties: $\rho(x+a, y+a) = \rho(x, y)$ and $\rho(tx, ty) \le \rho(x, y)$ for each $t \in [-1, 1]$.

Indeed, we have $[x, y] \le 2^{-n}$ if and only if $x - y \notin O_{n+1}$, and since $tx - ty = t(x - y)$ we get $[tx, ty] \le [x, y]$ for each $t \in [-1, 1]$.

Let $\mathrm{ord}^*(O_n) := \mathrm{ord}^*(P_n)$. Thus we get the following

THEOREM 3. *If* $\{O_n : n \in \mathbb{N}\}$ *is a locally metrizable development of a linear space* E*, then the function* $\|\cdot\| : E \to [0, \infty)$*,* $\|x\| := \rho(0, x)$ *given by* *a metric induced by the development is a subnorm such that*

$$B(0,2^{-n}) \subset O_n \subset \overline{B}(0,2^{-\varphi(n)}).$$

If, in addition, $\lim_{n\to\infty} \operatorname{ord}^*(O_n)/2^n = 0$, *then* $\chi(\rho) = 0$.

A result in [3] suggests the following theorem. For the convenience of the reader we provide a proof.

THEOREM 4. *Any linear metric space* (E,ρ) *of infinite dimension with a metric* $\rho(x,y) = \|x-y\|$ *given by a subnorm* $\|\cdot\|$ *contains a closed convex subset* C *of infinite dimension such that the metric* $\rho : C \times C \to [0,\infty)$ *restricted to* C *has characteristic zero.*

Proof. We shall define by induction a sequence of affinely independent points $a_0, a_1, \ldots \in E$, a sequence of families P_n, $n \in \mathbb{N}$, of open sets, and a sequence of positive real numbers $\delta_1 > \delta_2 > \cdots > 0$, $\delta_i < 2^{-i}$ such that:

(1) $\delta(P_{n+1}) < 2^{-n}$ and $\operatorname{ord}(P_n) \le n$ for each $n \in \mathbb{N}$,
(2) $C_n := \operatorname{conv}\{a_0,\ldots,a_n\} \subset \bigcup P_n \subset B(C_{n-1},\delta_n) \subset B(C_{n-1},2\delta_n) \subset \bigcup P_{n-1}$.

Inductive construction.

STEP 0. Choose $a_0 \in E\setminus\{0\}$ and define $C_0 := \{a_0\}$ and $P_0 := \{B(a_0,1)\}$.

STEP $n+1$. Assume that we have already defined affinely independent points $a_0,\ldots,a_n$, families $P_0,\ldots,P_n$ of open sets and positive real numbers $\delta_1,\ldots,\delta_n$ satisfying (1) and (2).

Since C_n is compact, there exists δ_{n+1} with $0 < \delta_{n+1} < \delta_n$ and $\delta_{n+1} \le 2^{-(n+1)}$ such that

$$C_n \subset B(C_n,\delta_{n+1}) \subset B(C_n,2\delta_{n+1}) \subset \bigcup P_n.$$

Choose a point $a_{n+1} \in B(C_n,\delta_{n+1}) \setminus \operatorname{span} C_n$. The points $a_0,\ldots,a_{n+1}$ are affinely independent. Note that

$$C_{n+1} := \operatorname{conv}\{a_0,\ldots,a_{n+1}\} \subset B(C_n,\delta_{n+1}).$$

To see this, fix $x \in C_{n+1}$. Then

$$x = \sum_{i=0}^{n+1} t_i a_i, \qquad \sum_{i=0}^{n+1} t_i = 1,\ t_i \ge 0.$$

Choose $b \in C_n$ such that $\|a_{n+1} - b\| < \delta_{n+1}$ and put

$$y := \sum_{i=0}^{n} t_i a_i + t_{n+1} b.$$

Then $y \in C_n$ and

$$\|x-y\| = \|t_{n+1}(a_{n+1}-b)\| \le \|a_{n+1}-b\| < \delta_{n+1}.$$

This yields $x \in B(C_n, \delta_{n+1})$. Since dim $C_{n+1} = n+1$, according to theorems on shrinkings and swellings of families of sets (see [1, Theorems 1.7.8 and 3.1.2]), one can find a family P_{n+1} of open sets in E such that

$$\delta(P_{n+1}) < \delta_{n+1}, \quad \operatorname{ord}(P_{n+1}) \le n+2, \quad C_{n+1} \subset \bigcup P_{n+1} \subset B(C_n, \delta_{n+1}).$$

This completes the inductive construction. Now, let us put

$$C := \overline{\bigcup_{n=0}^{\infty} C_n}.$$

Note that

$$C \subset \bigcap_{n=0}^{\infty} \overline{B(C_n, \delta_{n+1})},$$

because $\bigcup_{n=0}^{\infty} C_n \subset \bigcap_{n=0}^{\infty} \overline{B(C_n, \delta_{n+1})}$. Thus from (1) and (2) we infer that $C \subset \bigcup P_n$ for each $n \in \mathbb{N}$. The sequence $\{P_n|C : n \in \mathbb{N}\}$ of families P_n restricted to the set C satisfies $\lim_{n\to\infty} \operatorname{ord}(P_n) \cdot \delta(P_n) = 0$. ∎

Theorem 5. *For any $0 \le r \le \infty$ there exists on the Hilbert cube a metrizable development of characteristic r.*

Proof. Let E be the linear space of all bounded real sequences $x = (x_0, x_1, x_2, \dots)$ with the norm

$$\|x\| := \sup\{|x_n| : n \in \mathbb{N}\}$$

and the induced metric $\rho(x, y) := \|x - y\|$.

Let $0 \le r \le \infty$. Choose a strongly decreasing sequence $r_0 > r_1 > r_2 > \dots > 0$ with $\lim_{n\to\infty} n \cdot r_n = 0$. The Cartesian product $C := \prod\{J_n : n \in \mathbb{N}\}$ of the segments $J_n = [0, r_n]$ is a subspace of E homeomorphic to the Hilbert cube. Since the cube $I^n := I_0 \times \dots \times I_{n-1}$ has covering dimension n and may be considered as a subspace of C by the embedding $(x_0, \dots, x_n) \mapsto (x_0, \dots, x_n, 0, 0, \dots)$, there exists a metrizable development $\{P_n : n \in \mathbb{N}\}$ of C such that $\operatorname{ord} P_n = n$ and $\rho(P_n) = r_n$. It is clear that the characteristic of the development $\{P_n : n \in \mathbb{N}\}$ is r. ∎

Problem 3. *Does there exist a metrizable development of characteristic greater than zero such that the metric described in the Lemma and induced by the development has characteristic zero?*

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