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SOLVABILITY OF THE FUNCTIONAL EQUATION $f = (T - I)h$ FOR VECTOR-VALUED FUNCTIONS

BY

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Abstract. Let X be a reflexive Banach space and $(\Omega, \mathcal{A}, \mu)$ be a probability measure space. Let $T : M(\mu; X) \to M(\mu; X)$ be a linear operator, where $M(\mu; X)$ is the space of all X-valued strongly measurable functions on $(\Omega, \mathcal{A}, \mu)$. We assume that T is continuous in the sense that if (f_n) is a sequence in $M(\mu; X)$ and $\lim_{n\to\infty} f_n = f$ in measure for some $f \in M(\mu; X)$, then also $\lim_{n\to\infty} Tf_n = Tf$ in measure. Then we consider the functional equation $f = (T - I)h$, where $f \in M(\mu; X)$ is given. We obtain several conditions for the existence of $h \in M(\mu; X)$ satisfying $f = (T - I)h$.

1. Introduction. Let $(X, \|\cdot\|_X)$ be a reflexive Banach space and $(\Omega, \mathcal{A}, \mu)$ be a probability measure space. Let $M(\mu; X)$ denote the linear space of all X-valued strongly measurable functions on Ω under pointwise operations. Two functions f and g in $M(\mu; X)$ are not distinguished provided that $f(\omega) = g(\omega)$ for almost all $\omega \in \Omega$. We define a metric d_0 on $M(\mu; X)$ by

$$d_0(f, g) := \int_\Omega \frac{\|f(\omega) - g(\omega)\|_X}{1 + \|f(\omega) - g(\omega)\|_X}\, d\mu.$$

It is known that under this metric $M(\mu; X)$ becomes an F-space (see p. 8 of [8] for the definition of an F-space). It is easily checked that if (f_n) is a sequence in $M(\mu; X)$, then $\lim_{n\to\infty} d_0(f_n, f) = 0$ for some $f \in M(\mu; X)$ is equivalent to the convergence of f_n to f in measure as $n \to \infty$.

For $0 < p < \infty$, let $L_p(\mu; X)$ denote the set of all functions f in $M(\mu; X)$ such that $\int_\Omega \|f(\omega)\|_X^p\, d\mu < \infty$. If $0 < p < 1$, then under the metric

$$d_p(f, g) := \int_\Omega \|f(\omega) - g(\omega)\|_X^p\, d\mu \quad (= \|f - g\|_p^p)$$

$L_p(\mu; X)$ becomes an F-space, and if $1 \le p < \infty$, then under the norm

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$$\|f\|_p := \Big(\int_\Omega \|f(\omega)\|_X^p \, d\mu\Big)^{1/p}$$

$L_p(\mu; X)$ becomes a Banach space. For $p = \infty$, let $L_\infty(\mu; X)$ denote the set of all functions f in $M(\mu; X)$ such that $\|f\|_\infty := \operatorname{ess\,sup}\{\|f(\omega)\|_X : \omega \in \Omega\} < \infty$. Then $L_\infty(\mu; X)$ becomes a Banach space under the norm $\|\cdot\|_\infty$. The symbols $M(\mu)$ and $L_p(\mu)$ $(0 < p \le \infty)$ mean $M(\mu; X)$ and $L_p(\mu; X)$, respectively, for $X =$ the scalars.

Let $T : M(\mu; X) \to M(\mu; X)$ be a linear operator continuous with respect to the metric d_0, and let $f \in M(\mu; X)$ be given. We consider the solvability of the cohomology equation $f = (T - I)h$. We first prove that if $0 < r_n < 1$ and $\lim_{n\to\infty} r_n = 1$, and the series $\sum_{k=0}^\infty r_n^k T^k f$ is summable in $M(\mu; X)$ for every $n \ge 1$, then the condition

$$\sup_{n\ge1} \int_\Omega \Big\| \Big(\sum_{k=0}^\infty r_n^k T^k f\Big)(\omega)\Big\|_X d\lambda < \infty,$$

where λ is a σ-finite measure equivalent to μ, implies that there exists $h \in L_1(\lambda; X)$ such that $f = (T - I)h$. Applying this, we then extend Assani's result [2] to vector-valued functions. Next, we consider a Lamperti-type operator $T = T_{\xi,\tau}$ (i.e., T has the form $Tf(\omega) = \xi(\omega)f(\tau\omega)$ for $f \in M(\mu; X)$ and $\omega \in \Omega$, where $\xi \in M(\mu)$ and τ is a null-preserving transformation on $(\Omega, \mathcal{A}, \mu)$). We extend Krzyżewski's result [6] to vector-valued functions. Lastly, we consider a Lamperti-type operator $T = T_{\xi,\tau}$, where τ is a measure-preserving transformation on $(\Omega, \mathcal{A}, \mu)$. Under this assumption, we extend Alonso, Hong and Obaya's result [1] and the author's result [9] to vector-valued functions.

For general notions and definitions in ergodic theory we follow Krengel's book [5].

2. Results. Our main result is the following theorem.

THEOREM 1. *Let X be a reflexive Banach space and v be a strictly positive measurable function on Ω. Let T be a continuous linear operator on $M(\mu; X)$, and $f \in M(\mu; X)$. Assume that (r_n) is a sequence of positive numbers such that $0 < r_n < 1$, $\lim_{n\to\infty} r_n = 1$, and the series $\sum_{k=0}^\infty r_n^k T^k f$ is summable in $M(\mu; X)$ for every $n \ge 1$. Then the condition*

$$C := \sup_{n\ge1} \int_\Omega \Big\| \Big(\sum_{k=0}^\infty r_n^k T^k f\Big)(\omega)\Big\|_X v(\omega)\, d\mu < \infty \tag{1}$$

implies that there exists $h \in M(\mu; X)$ satisfying $f = (T - I)h$ and $\int_\Omega \|h(\omega)\|_X v(\omega)\, d\mu \le C$.

For the proof of Theorem 1 we need the following key lemma.

LEMMA 1 (cf. [6]). *Let X be a reflexive Banach space and (f_n) be a sequence in $M(\mu; X)$ such that*

$$\sup_{n\geq 1} \|f_n(\omega)\|_X < \infty \quad \textit{for almost all } \omega \in \Omega. \tag{2}$$

Then there exists a function $g \in M(\mu; X)$ satisfying

$$g(\omega) = \lim_{n\to\infty} \widetilde{f}_n(\omega) \quad \textit{for almost all } \omega \in \Omega, \tag{3}$$

where $\widetilde{f}_n$ is a function in the convex hull $\mathrm{co}(\{f_l : l \geq n\})$.

Proof. Define a nonnegative measurable function F on Ω by

$$F(\omega) = \sup_{n\geq 1} \|f_n(\omega)\|_X \quad (\omega \in \Omega),$$

and put

$$\Omega_l = \{\omega : F(\omega) \leq l\} \quad (l = 1, 2, \ldots).$$

It follows that $\Omega_1 \subset \Omega_2 \subset \cdots$ and $\Omega = \lim_{l\to\infty} \Omega_l$ (mod μ). Since $\{f_n : n \geq 1\}$ is uniformly bounded on Ω_l and since $L_2(\Omega_l; X)$ is reflexive by the reflexivity of X (cf. Corollary IV.1.2 of [3]), it follows that the set $\{f_n|_{\Omega_l} : n \geq 1\}$ is weakly sequentially compact in $L_2(\Omega_l; X)$. Thus, there exists a subsequence $(f_{n'})$ of (f_n) and a function $g_l \in L_2(\Omega_l; X)$ such that

$$f_{n'}|_{\Omega_l} \to g_l \quad \text{weakly in } L_2(\Omega_l; X) \text{ as } n' \to \infty.$$

By the diagonal argument we see that there exists a subsequence (f_{n_k}) of (f_n) and a function $g \in M(\mu; X)$ such that for each $l \geq 1$,

$$f_{n_k}|_{\Omega_l} \to g|_{\Omega_l} \quad \text{weakly in } L_2(\Omega_l; X) \text{ as } k \to \infty.$$

By Theorem 3.13 of [8], there exists $\widetilde{f}_l \in \mathrm{co}(\{f_{n_k} : k \geq l\})$ for each $l \geq 1$ such that

$$\sum_{k=l}^{\infty} \int_{\Omega_l} \|g(\omega) - \widetilde{f}_k(\omega)\|_X^2 \, d\mu < \sum_{k=l}^{\infty} 2^{-k} \leq 1 \quad (l \geq 1).$$

Hence, $\lim_{k\to\infty} \widetilde{f}_k(\omega) = g(\omega)$ for almost all $\omega \in \Omega$, and the proof is complete.

Proof of Theorem 1. Let $f_n = \sum_{k=0}^{\infty} r_n^k T^k f$ for each $n \geq 1$. Since

$$\lim_{N\to\infty} d_0\Big(\sum_{k=0}^{N} r_n^k T^k f, f_n\Big) = 0,$$

it follows that $\lim_{N\to\infty} d_0(r_n^N T^N f, 0) = 0$, i.e., $\lim_{N\to\infty} r_n^N T^N f = 0$ in

$M(\mu; X)$. This together with the continuity of T implies

$$f_n - Tf_n = \lim_{N\to\infty} \Big(\sum_{k=0}^{N} r_n^k T^k f - \sum_{k=0}^{N} r_n^k T^{k+1} f \Big)$$
$$= \lim_{N\to\infty} \Big[f - (1-r_n) T \Big(\sum_{k=0}^{N-1} r_n^k T^k f \Big) - r_n^N T^{N+1} f \Big]$$
$$= f - (1-r_n) T f_n \quad (\text{in } M(\mu; X)).$$

Here, since $\int_{\Omega} (1-r_n) \|f_n(\omega)\|_X v(\omega)\, d\mu \le (1-r_n) C \to 0$ as $n \to \infty$, we may assume without loss of generality (if necessary, choose a subsequence of (f_n)) that

(4) $$\lim_{n\to\infty} (1-r_n) f_n(\omega) = 0 \quad \text{for almost all } \omega \in \Omega.$$

Thus

$$\lim_{n\to\infty} (1-r_n) f_n = 0 \quad (\text{in } M(\mu; X)),$$

and so by the continuity of T,

$$\lim_{n\to\infty} (1-r_n) T f_n = 0 \quad (\text{in } M(\mu; X)).$$

Thus, choosing a further subsequence of (f_n) if necessary, we may again assume that

(5) $$\lim_{n\to\infty} (1-r_n) T f_n(\omega) = 0 \quad \text{for almost all } \omega \in \Omega.$$

Let then $h_n(\omega) = \|f_n(\omega)\|_X$ for $n \ge 1$. By condition (1) we have

$$\sup_{n\ge1} \int_{\Omega} h_n(\omega) v(\omega)\, d\mu = C < \infty,$$

and hence we may apply Komlós's theorem [4] to infer that there exists a subsequence (h_{n_k}) of (h_n) and a nonnegative function $H \in L_1(v d\mu)$ such that

$$H(\omega) = \lim_{N\to\infty} N^{-1} \sum_{k=1}^{N} h_{n_k}(\omega) \quad \text{for almost all } \omega \in \Omega.$$

In order to prove the theorem, we may assume without loss of generality that $n_k = k$ for all $k \ge 1$, i.e., $(h_{n_k}) = (h_k)$. Under this assumption

$$H(\omega) = \lim_{N\to\infty} N^{-1} \sum_{k=1}^{N} h_k(\omega) \quad \text{for almost all } \omega \in \Omega,$$

and thus

(6) $$\sup_{N\ge1} \Big\| N^{-1} \sum_{k=1}^{N} f_k(\omega) \Big\|_X \le \sup_{N\ge1} N^{-1} \sum_{k=1}^{N} h_k(\omega) < \infty \quad \text{for almost all } \omega \in \Omega.$$

Then, by Lemma 1 there exists a sequence $(\widetilde{F}_n)$ of functions in $M(\mu; X)$

such that

(i) $\widetilde{F}_n \in \mathrm{co}(\{N^{-1}\sum_{k=1}^N f_k : N \ge n\})$ for every $n \ge 1$, and

(ii) the limit

$$G(\omega) = \lim_{n\to\infty} \widetilde{F}_n(\omega) \tag{7}$$

exists for almost all $\omega \in \Omega$.

We then deduce by Fatou's lemma and (1) that

$$\int_\Omega \|G(\omega)\|_X v(\omega)\,d\mu \le \liminf_{n\to\infty} \int_\Omega \|\widetilde{F}_n(\omega)\|_X v(\omega)\,d\mu \le C. \tag{8}$$

On the other hand, since

$$\widetilde{F}_n \in \mathrm{co}\Big(\Big\{N^{-1}\sum_{k=1}^N f_k : N \ge n\Big\}\Big) \quad \text{and} \quad f_k - Tf_k = f - (1-r_k)Tf_k,$$

(5) implies that

$$\lim_{n\to\infty}(\widetilde{F}_n(\omega) - T\widetilde{F}_n(\omega)) = f(\omega) \quad \text{for almost all } \omega \in \Omega,$$

and hence

$$G - TG = \lim_{n\to\infty}(\widetilde{F}_n - T\widetilde{F}_n) = f \quad (\text{in } M(\mu; X)),$$

which completes the proof.

COROLLARY 1 (cf. [2]). *Let X be a reflexive Banach space and v be a strictly positive measurable function on Ω. Let $T : L_1(vd\mu; X) \to L_1(vd\mu; X)$ be a linear operator continuous with respect to the metric d_0, and assume that $f \in L_1(vd\mu; X)$. Then the condition*

$$C := \sup_{n\ge1} \int_\Omega \Big\| \sum_{k=0}^{n-1} T^k f(\omega)\Big\|_X v(\omega)\,d\mu < \infty \tag{9}$$

implies that there exists $h \in L_1(vd\mu; X)$ with $f = (T-I)h$ and $\|h\|_{L_1(vd\mu;X)} \le C$.

Proof. For $0 < r < 1$ we have (formally)

$$\sum_{k=0}^\infty r^k T^k f = (1-r)\Big(\sum_{k=0}^\infty r^k\Big)\sum_{k=0}^\infty r^k T^k f = (1-r)\sum_{k=0}^\infty r^k\Big(\sum_{j=0}^k T^j f\Big),$$

and (9) implies that $\|\sum_{j=0}^k T^j f\|_{L_1(vd\mu;X)} \le C$ for all $k \ge 0$. Thus, the series $\sum_{k=0}^\infty r^k T^k f$ is summable in $L_1(vd\mu; X)$, and hence also in $M(\mu; X)$. Furthermore, we have

$$\sup_{0<r<1} \int_\Omega \Big\|\Big(\sum_{k=0}^\infty r^k T^k f\Big)(\omega)\Big\|_X v(\omega)\,d\mu \le C.$$

Hence, the desired conclusion follows from the proof of Theorem 1.

The following example, showing that the converse implications of Theorem 1 and Corollary 1 do not hold in general, may be interesting.

EXAMPLE 1. Let μ be the probability measure on the set $\mathbb{Z}$ of all integers defined by

$$\mu(\{k\}) = (1/3)2^{-|k|} \quad \text{for } k \in \mathbb{Z}.$$

Define a positive function v on $\mathbb{Z}$ by

$$v(k) = \begin{cases} 3 \cdot 2^{-k} & \text{if } k \le 0, \\ 3(k+1) \cdot 2^k & \text{if } k \ge 1. \end{cases}$$

Thus, the measure $\lambda = v d\mu$ satisfies

$$\lambda(\{k\}) = \begin{cases} 1 & \text{if } k \le 0, \\ k+1 & \text{if } k \ge 1. \end{cases}$$

Define a continuous linear operator T on $M(\mu)$ by $Tf(m) = f(m-1)$ for $m \in \mathbb{Z}$. Then, by an easy computation, the restriction of T to $L_1(vd\mu)$ is a continuous linear operator on $L_1(vd\mu)$ such that $\|T^n\|_{L_1(vd\mu)} = n+1$ for every $n \ge 0$. Let $h = \chi_{\{-1\}}$, and put $f = (T-I)h$. Then we have $f = \chi_{\{0\}} - \chi_{\{-1\}}$, and the series $\sum_{k=0}^{\infty} r^k T^k f$, where $0 < r < 1$, is summable in $L_1(vd\mu)$ and hence also in $M(\mu)$. It is easy to see that

$$\Big(\sum_{k=0}^{\infty} r^k T^k f\Big)(k) = \begin{cases} 0 & \text{if } k \le -2, \\ -1 & \text{if } k = -1, \\ r^k - r^{k+1} & \text{if } k \ge 0. \end{cases}$$

Hence,

$$\begin{aligned} \Big\| \sum_{k=0}^{\infty} r^k T^k f \Big\|_{L_1(vd\mu)} &= 1 + (1-r) \sum_{k=0}^{\infty} r^k (k+1) \\ &= 1 + (1-r)^{-1} \to \infty \quad \text{as } r \uparrow 1, \end{aligned}$$

and condition (9) does not hold either.

THEOREM 2 (cf. [6]). *Let X be a reflexive Banach space and τ be a conservative ergodic null-preserving transformation on Ω. Let $\xi \in M(\mu)$, and let $T = T_{\xi,\tau}$ be the continuous linear operator on $M(\mu; X)$ defined by $Tf(\omega) = T_{\xi,\tau} f(\omega) = \xi(\omega) f(\tau\omega)$ for $f \in M(\mu; X)$ and $\omega \in \Omega$. Then conditions* (I) *and* (II) *below satisfy* (I)$\Rightarrow$(II) *for $f \in M(\mu; X)$. If in addition $C := \sup_{n \ge 1} \|T^n\|_\infty < \infty$, then* (I) *and* (II) *are equivalent.*

(I) *There exists $A \in \mathcal{A}$ with $\mu(A) > 0$ and an absolute constant $K > 0$ such that if $\omega, \tau^n \omega \in A$ for some $n \ge 1$, then $\|S_n f(\omega)\|_X \le K$, where*

$$S_n f(\omega) := \sum_{k=0}^{n-1} T^k f(\omega) \quad \textit{for } n \ge 1.$$

(II) *There exists $h \in M(\mu; X)$ such that $f = (T-I)h$.*

Proof. (I)⇒(II). Using the equality $T^n f(\omega) = \xi(\omega)\cdots\xi(\tau^{n-1}\omega)f(\tau^n\omega)$, we first notice that for any $n, m \geq 1$ and $\omega \in \Omega$,

$$\begin{aligned} S_{n+m}f(\omega) &= S_n f(\omega) + S_m(T^n f)(\omega) \\ &= S_n f(\omega) + S_m(\xi(\cdot)\cdots\xi(\tau^{n-1}\cdot)f(\tau^n\cdot))(\omega) \\ &= S_n f(\omega) + [\xi(\omega)\cdots\xi(\tau^{n-1}\omega)]S_m f(\tau^n\omega). \end{aligned} \tag{10}$$

Next, by the conservativity and ergodicity of τ we have

$$\Omega = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\tau^{-k}A \pmod{\mu}. \tag{11}$$

Thus, it may be assumed without loss of generality that for every $\omega \in \Omega$ the set $\{n \geq 1 : \tau^n\omega \in A\}$ is infinite. Then for every $\omega \in \Omega$ there exists a strictly increasing sequence $(r_j(\omega))_{j=1}^{\infty}$ of positive integers such that

$$\{r_j(\omega) : j \geq 1\} = \{n \geq 1 : \tau^n\omega \in A\}.$$

By (10) we have

$$\begin{aligned} S_{r_j(\omega)}f(\omega) &= S_{r_1(\omega)}f(\omega) + S_{r_j(\omega)-r_1(\omega)}(T^{r_1(\omega)}f)(\omega) \\ &= S_{r_1(\omega)}f(\omega) + [\xi(\omega)\cdots\xi(\tau^{r_1(\omega)-1}\omega)]S_{r_j(\omega)-r_1(\omega)}f(\tau^{r_1(\omega)}\omega). \end{aligned}$$

Since $\tau^{r_1(\omega)}\omega$ and $\tau^{r_j(\omega)-r_1(\omega)}(\tau^{r_1(\omega)}\omega)$ belong to A, it follows from (I) that

$$\|S_{r_j(\omega)-r_1(\omega)}f(\tau^{r_1(\omega)}\omega)\|_X \leq K. \tag{12}$$

Therefore,

$$\sup_{j\geq 1}\|S_{r_j(\omega)}f(\omega)\|_X \leq \|S_{r_1(\omega)}f(\omega)\|_X + |\xi(\omega)\cdots\xi(\tau^{r_1(\omega)-1}\omega)|K < \infty. \tag{13}$$

Hence, putting

$$f_n(\omega) := n^{-1}\sum_{j=1}^{n} S_{r_j(\omega)}f(\omega) \quad (n \geq 1), \tag{14}$$

we get

$$\sup_{n\geq 1}\|f_n\|_X < \infty \quad \text{for every } \omega \in \Omega, \tag{15}$$

so that by Lemma 1 there exists a strongly measurable X-valued function g on Ω such that

$$g(\omega) = \lim_{n\to\infty}\widetilde{f}_n(\omega) \quad \text{for almost all } \omega \in \Omega, \tag{16}$$

where $\widetilde{f}_n \in \mathrm{co}(\{f_k : k \geq n\})$. Thus, $\widetilde{f}_n$ has the form

$$\widetilde{f}_n = \sum_{j=1}^{j(n)} C_{n,j}f_{k_j},$$

where

$$C_{n,j} > 0, \quad \sum_{j=1}^{j(n)} C_{n,j} = 1, \quad k_j \geq n \quad \text{for } 1 \leq j \leq j(n).$$

And from (16) we may assume without loss of generality that

$$g(\omega) = \lim_{n\to\infty} \widetilde{f}_n(\omega) \quad \text{for all } \omega \in \Omega. \tag{17}$$

CASE 1: $r_1(\omega) = 1$. Then, by the definition of $r_j(\omega)$ it follows that $r_j(\tau\omega) = r_{j+1}(\omega) - 1$ for every $j \geq 1$. Thus, using the equality

$$S_n f(\omega) - \xi(\omega) S_{n-1} f(\tau\omega) = f(\omega) \quad \text{for } n \geq 2 \text{ and } \omega \in \Omega, \tag{18}$$

we get

$$\sum_{j=2}^{n} S_{r_j(\omega)} f(\omega) - \xi(\omega) \sum_{j=1}^{n-1} S_{r_j(\tau\omega)} f(\tau\omega) = (n-1) f(\omega).$$

From this, together with the fact that $S_{r_1(\omega)} f(\omega) = S_1 f(\omega) = f(\omega)$, we have (cf. (14))

$$\begin{aligned} f_n(\omega) - \xi(\omega) f_n(\tau\omega) &= \frac{n-1}{n} f(\omega) + \frac{1}{n} f(\omega) - \frac{1}{n} \xi(\omega) S_{r_n(\tau\omega)} f(\tau\omega) \\ &= f(\omega) - \frac{1}{n} \xi(\omega) S_{r_n(\tau\omega)} f(\tau\omega), \end{aligned}$$

and by (13),

$$\lim_{n\to\infty} n^{-1} |\xi(\omega)| \cdot \|S_{r_n(\tau\omega)} f(\tau\omega)\|_X = 0.$$

Consequently,

$$(I-T)g(\omega) = g(\omega) - \xi(\omega) g(\tau\omega) = \lim_{n\to\infty} [\widetilde{f}_n(\omega) - \xi(\omega) \widetilde{f}_n(\tau\omega)] = f(\omega).$$

CASE 2: $r_1(\omega) \geq 2$. Then we have $r_j(\tau\omega) = r_j(\omega) - 1$, so that by (18),

$$\sum_{j=1}^{n} S_{r_j(\omega)} f(\omega) - \xi(\omega) \sum_{j=1}^{n} S_{r_j(\tau\omega)} f(\tau\omega) = n f(\omega).$$

Thus it follows that $f_n(\omega) - \xi(\omega) f_n(\tau\omega) = f(\omega)$ for every $n \geq 1$, and

$$(I-T)g(\omega) = g(\omega) - \xi(\omega) g(\tau\omega) = \lim_{n\to\infty} [\widetilde{f}_n(\omega) - \xi(\omega) \widetilde{f}_n(\tau\omega)] = f(\omega).$$

This completes the proof of (I)⇒(II).

To prove the second half of Theorem 2, assume that $C := \sup_{n\geq 1} \|T^n\|_\infty < \infty$, and that (II) holds. Then there exists a constant $M > 0$ such that the set $A = \{\omega : \|h(\omega)\|_X \leq M\}$ satisfies $\mu(A) > 0$. Since $f = Th - h$, we may assume without loss of generality that $f(\omega) = Th(\omega) - h(\omega)$ for all $\omega \in \Omega$. Then, for all $n \geq 1$ and $\omega \in \Omega$ we have

$$S_n f(\omega) = T^n h(\omega) - h(\omega) = \xi(\omega) \cdots \xi(\tau^{n-1}\omega) h(\tau^n\omega) - h(\omega).$$

Since $\|T^n\|_\infty = \|\xi(\cdot)\xi(\tau\cdot)\cdots\xi(\tau^{n-1}\cdot)\|_\infty \le C$ for all $n \ge 1$ by hypothesis, we may assume without loss of generality that $|\xi(\omega)\xi(\tau\omega)\cdots\xi(\tau^{n-1}\omega)| \le C$ for all $n \ge 1$ and $\omega \in \Omega$. Then

$$\|S_n f(\omega)\|_X \le C\|h(\tau^n\omega)\|_X + \|h(\omega)\|_X \quad \text{for all } n \ge 1 \text{ and } \omega \in \Omega,$$

and so $\omega, \tau^n\omega \in A$ for some $n \ge 1$ implies that $\|S_n f(\omega)\|_X \le CM + M$. Therefore, (I) holds with $K := CM + M$, and the proof is complete.

PROPOSITION 1. *Let τ be an invertible null-preserving transformation on Ω, and $\xi \in M(\mu)$. Let $T = T_{\xi,\tau}$ be as in Theorem 2. Then the following conditions are equivalent*:

(I) *The restriction of T to $L_\infty(\mu; X)$ is an invertible operator on $L_\infty(\mu; X)$ such that $C := \sup\{\|T^n\|_\infty : n \in \mathbb{Z}\} < \infty$.*
(II) *There exists $\zeta \in L_\infty^+(\mu)$, with $1/\zeta \in L_\infty^+(\mu)$, such that $|\xi(\omega)| = \zeta(\tau\omega)/\zeta(\omega)$ for almost all $\omega \in \Omega$.*

Proof. (I)⇒(II). Since the restriction of T to $L_\infty(\mu; X)$ is an invertible operator on $L_\infty(\mu; X)$ by hypothesis, it follows that $|\xi(\omega)| > 0$ for almost all $\omega \in \Omega$, and for every $n \ge 1$ we have

$$\text{(19)} \qquad \begin{cases} T^n f(\omega) = \xi(\omega)\cdots\xi(\tau^{n-1}\omega)f(\tau^n\omega), \\ T^{-n} f(\omega) = \dfrac{1}{\xi(\tau^{-1}\omega)\cdots\xi(\tau^{-n}\omega)}\, f(\tau^{-n}\omega). \end{cases}$$

Thus, by the inequalities $\|T^n\|_\infty \le C$ and $\|T^{-n}\|_\infty \le C$,

$$|\xi(\omega)\cdots\xi(\tau^{n-1}\omega)| \le C, \quad \frac{1}{|\xi(\tau^{-1}\omega)\cdots\xi(\tau^{-n}\omega)|} \le C \quad \text{for almost all } \omega \in \Omega,$$

and since τ is invertible,

$$\frac{1}{|\xi(\omega)\cdots\xi(\tau^{n-1}\omega)|} \le C \quad \text{for almost all } \omega \in \Omega \ (n \ge 1).$$

It follows that

$$\text{(20)} \quad -\log C \le \sum_{j=0}^{n-1} \log|\xi(\tau^j\omega)| \le \log C \quad \text{for almost all } \omega \in \Omega \ (n \ge 1).$$

Now, we apply Corollary 6 of [7] to infer that there exists $g \in L_\infty(\mu)$ such that $\log|\xi(\omega)| = g(\tau\omega) - g(\omega)$ for almost all $\omega \in \Omega$. Since $|\xi(\omega)| = e^{g(\tau\omega)}/e^{g(\omega)}$, the function $\zeta(\omega) := e^{g(\omega)}$ ($\omega \in \Omega$) satisfies $|\xi(\omega)| = \zeta(\tau\omega)/\zeta(\omega)$ for almost all $\omega \in \Omega$, and furthermore by the fact that $g \in L_\infty(\mu)$ we have $\zeta, 1/\zeta \in L_\infty^+(\mu)$.

(II)⇒(I). Condition (II) implies that

$$\begin{aligned}\|1/\zeta\|_\infty^{-1}\|\zeta\|_\infty^{-1} &\le |\xi(\omega)| = \zeta(\tau\omega)/\zeta(\omega) \\ &\le \|\zeta\|_\infty\|1/\zeta\|_\infty \quad \text{for almost all } \omega \in \Omega.\end{aligned}$$

It follows that the restriction of T to $L_\infty(\mu;X)$ is an invertible operator on $L_\infty(\mu;X)$, and for every $n\in\mathbb{Z}$ and $f\in L_\infty(\mu;X)$ we have

$$\|T^n f(\omega)\|_X = \frac{\zeta(\tau^n\omega)}{\zeta(\omega)}\,\|f(\tau^n\omega)\|_X \quad \text{for almost all } \omega\in\Omega.$$

Thus

$$\|T^n\|_\infty \le \|\zeta\|_\infty \|1/\zeta\|_\infty \quad (n\in\mathbb{Z}),$$

and this completes the proof.

From now on we restrict ourselves to the case where τ is a measure-preserving transformation in order to discuss the solvability problem in $L_p(\mu;X)$, with $0<p\le\infty$.

PROPOSITION 2. *Let τ be an invertible measure-preserving transformation on Ω, and $\xi\in M(\mu)$. Let $T=T_{\xi,\tau}$ be as in Theorem 2. If $0<p<\infty$, then the following conditions are equivalent*:

- (I) *The restriction of T to $L_p(\mu;X)$ is an invertible operator on $L_p(\mu;X)$ such that $\sup\{\|T^n\|_p : n\in\mathbb{Z}\}<\infty$, where $\|T^n\|_p := \sup\{\|T^n f\|_p : \|f\|_p=1,\ f\in L_p(\mu;X)\}$.*
- (II) *There exists $\zeta\in L_\infty^+(\mu)$, with $1/\zeta\in L_\infty^+(\mu)$, such that $|\xi(\omega)| = \zeta(\tau\omega)/\zeta(\omega)$ for almost all $\omega\in\Omega$.*

Proof. (I)⇒(II). Since the restriction of T to $L_p(\mu;X)$ is an invertible operator on $L_p(\mu;X)$ by hypothesis, it follows as above that $|\xi(\omega)|>0$ for almost all $\omega\in\Omega$. Hence, (19) holds for every $n\ge1$, and since τ is measure-preserving, we then have

$$\|T^n\|_p = \|\xi(\cdot)\cdots\xi(\tau^{n-1}\cdot)\|_\infty, \qquad \|T^{-n}\|_p = \left\|\frac{1}{\xi(\tau^{-1}\cdot)\cdots\xi(\tau^{-n}\cdot)}\right\|_\infty.$$

Thus, we can apply the proof of (I)⇒(II) of Proposition 1 to obtain the present implication.

(II)⇒(I). The proof is the same as that of (II)⇒(I) of Proposition 1, and we omit the details.

REMARK 1. It follows from the above propositions that if τ is an invertible measure-preserving transformation on Ω, then $\sup\{\|T^n\|_\infty : n\in\mathbb{Z}\} = \sup\{\|T^n\|_p : n\in\mathbb{Z}\}$ for every p with $0<p<\infty$.

THEOREM 3 (cf. [1], [10]). *Let X be a reflexive Banach space and τ be an invertible measure-preserving transformation on Ω. Let $\xi\in M(\mu)$, and $T=T_{\xi,\tau}$ be as in Theorem 2. Assume that $0<p<\infty$, and that the restriction of T to $L_p(\mu;X)$ is an invertible operator on $L_p(\mu;X)$ such that $\sup\{\|T^n\|_p : n\in\mathbb{Z}\}<\infty$. Then the following conditions are equivalent for $f\in M(\mu;X)$*:

(I) *There exists* $A \in \mathcal{A}$ *with* $\mu(A) > 0$ *and an absolute constant* $K > 0$ *such that*

(i) *if* $\omega, \tau^n\omega \in A$ *for some* $n \geq 1$, *then* $\|S_n f(\omega)\|_X \leq K$,

(ii) $\liminf_{n\to\infty} n^{-1} \sum_{j=0}^{n-1} \int_A \|S_j f(\omega)\|_X^p \, d\mu < \infty$.

(II) *There exists* $h \in L_p(\mu; X)$ *such that* $f = (T - I)h$.

Proof. (I)⇒(II). By Proposition 2 there exists $\zeta \in L_\infty^+$, with $1/\zeta \in L_\infty^+(\mu)$, such that $|\xi(\omega)| = \zeta(\tau\omega)/\zeta(\omega)$ for almost all $\omega \in \Omega$. Thus, there exists a constant $D > 0$ such that for every $n \geq 1$,

$$D^{-1} \leq |\xi(\omega)\xi(\tau\omega)\cdots\xi(\tau^{n-1}\omega)| \leq D \qquad \text{for almost all } \omega \in \Omega. \tag{21}$$

Furthermore, by Theorem 2 there exists $h \in M(\mu; X)$ such that $f = (T-I)h$. Hence, $S_j f = T^j h - h$ for every $j \geq 1$, and

$$\|h(\omega) + S_j f(\omega)\|_X = \|T^j h(\omega)\|_X = |\xi(\omega)\cdots\xi(\tau^{j-1}\omega)| \cdot \|h(\tau^j\omega)\|_X \qquad \text{for almost all } \omega \in \Omega.$$

Thus, by (21) we have

$$\|h(\tau^j\omega)\|_X^p \leq D^p \|h(\omega) + S_j f(\omega)\|_X^p \qquad \text{for almost all } \omega \in \Omega. \tag{22}$$

Now, define a nonnegative measurable function F on Ω by

$$F(\omega) = \liminf_{n\to\infty} n^{-1} \sum_{j=1}^{n} \|S_j f(\omega)\|_X^p \qquad (\omega \in \Omega).$$

Then, by Fatou's lemma,

$$\int_A F(\omega)\, d\mu \leq \liminf_{n\to\infty} n^{-1} \sum_{j=1}^{n} \int_A \|S_j f(\omega)\|_X^p \, d\mu < \infty,$$

so that $F(\omega) < \infty$ for almost all $\omega \in A$. Since the function

$$G(\omega) = \liminf_{n\to\infty} n^{-1} \sum_{j=1}^{n} \|h(\omega) + S_j f(\omega)\|_X^p \qquad (\omega \in \Omega)$$

satisfies $G(\omega) < \infty$ whenever $F(\omega) < \infty$, it follows from (22) that

$$\liminf_{n\to\infty} n^{-1} \sum_{j=1}^{n} \|h(\tau^j\omega)\|_X^p \leq D^p G(\omega) < \infty \qquad \text{for almost all } \omega \in A. \tag{23}$$

On the other hand, since τ is ergodic and measure-preserving by hypothesis, the Birkhoff pointwise ergodic theorem implies that

$$\lim_{n\to\infty} n^{-1} \sum_{j=1}^{n} \|h(\tau^j\omega)\|_X^p = \int_\Omega \|h(\omega)\|_X^p \, d\mu \qquad \text{for almost all } \omega \in \Omega.$$

Therefore, by (23) we have $h \in L_p(\mu; X)$.

(II)⇒(I). Since $\sup_{n\ge 1}\|T^n\|_\infty = \sup_{n\ge 1}\|T^n\|_p < \infty$, this implication follows immediately from Theorem 2, and the proof is complete.

THEOREM 4 (cf. [9]). *Let X be a reflexive Banach space and τ be an invertible measure-preserving transformation on Ω. Let $\xi \in M(\mu)$, and $T = T_{\xi,\tau}$ be as in Theorem 2. Assume that the restriction of T to $L_\infty(\mu; X)$ is an invertible operator on $L_\infty(\mu; X)$ such that* $\sup\{\|T^n\|_\infty : n \in \mathbb{Z}\} < \infty$. *Then the following conditions are equivalent for $f \in M(\mu; X)$:*

(I) *There exists $A \in \mathcal{A}$ with $\mu(A) > 0$ and an absolute constant $K > 0$ such that*

(i) *if $\omega, \tau^n\omega \in A$ for some $n \ge 1$, then $\|S_n f(\omega)\|_X \le K$,*

(ii) $\liminf_{n\to\infty} n^{-1}\sum_{j=1}^n \|\chi_A \cdot S_j f\|_\infty < \infty$.

(II) *There exists $h \in L_\infty(\mu; X)$ such that $f = (T - I)h$.*

Proof. (I)⇒(II). By Proposition 1 there exists a constant $D > 0$ such that for every $n \ge 1$ we have $D^{-1} \le |\xi(\omega)\xi(\tau\omega)\cdots\xi(\tau^{n-1}\omega)| \le D$ for almost all $\omega \in \Omega$. And by Theorem 2 there exists $h \in M(\mu; X)$ such that $f = (T - I)h$. Thus, $h + S_j f = T^j h$ for every $j \ge 1$, and we deduce by (19) applied to h in place of f that

$$\|h(\omega)\|_X + \|S_j f(\omega)\|_X \ge \|T^j h(\omega)\|_X \ge D^{-1}\|h(\tau^j\omega)\|_X \quad \text{for almost all } \omega \in \Omega.$$

It follows that

$$D\Big(\|\chi_A \cdot h\|_\infty + n^{-1}\sum_{j=1}^n \|\chi_A \cdot (S_j f)\|_\infty\Big) \ge n^{-1}\sum_{j=1}^n \|\chi_A \cdot (h\circ\tau^j)\|_\infty$$
$$= n^{-1}\sum_{j=1}^n \|(\chi_A\circ\tau^{-j})\cdot h\|_\infty \ge \Big\|\Big(n^{-1}\sum_{j=1}^n \chi_A\circ\tau^{-j}\Big)\cdot h\Big\|_\infty.$$

Here, considering the set $A \cap \{\omega : \|h(\omega)\|_X \le N\}$ for a sufficiently large $N > 0$ instead of A (if necessary), we may assume from the start that $\chi_A \cdot h \in L_\infty(\mu; X)$. Then we find by condition (ii) of (I) that

$$\liminf_{n\to\infty} \Big\|\Big(n^{-1}\sum_{j=1}^n \chi_A\circ\tau^{-j}\Big)\cdot h\Big\|_\infty < \infty. \tag{24}$$

On the other hand, by the Birkhoff pointwise ergodic theorem we have

$$\lim_{n\to\infty} n^{-1}\sum_{j=1}^n \chi_A(\tau^{-j}\omega) = \mu(A) > 0 \quad \text{for almost all } \omega \in \Omega.$$

Hence, (24) implies that $h \in L_\infty(\mu; X)$.

(II)⇒(I). This follows immediately from Theorem 2, and hence the proof is complete.

REMARK 2. One may wonder whether condition (i) of (I) can be omitted in Theorems 3 and 4. The author thinks that this is not known even if $X =$ the scalars. On the other hand, if $X =$ the scalars and $\xi \equiv 1$ on Ω, then it is known that condition (i) of (I) can be omitted in Theorems 3 and 4. See [1], [9] and [10].

REFERENCES

[1] A. I. Alonso, J. Hong and R. Obaya, *Absolutely continuous dynamics and real coboundary cocycles in L^p-spaces*, $0 < p < \infty$, Studia Math. 138 (2000), 121–134.

[2] I. Assani, *A note on the equation $y = (I - T)x$ in L^1*, Illinois J. Math. 43 (1999), 540–541.

[3] J. Diestel and J. J. Uhl, Jr., *Vector Measures*, Amer. Math. Soc., Providence, 1977.

[4] J. Komlós, *A generalization of a problem of Steinhaus*, Acta Math. Acad. Sci. Hungar. 18 (1967), 217–229.

[5] U. Krengel, *Ergodic Theorems*, de Gruyter, Berlin, 1985.

[6] K. Krzyżewski, *A note on a generalized cohomology equation*, Colloq. Math. 84/85 (2000), part 2, 279–283.

[7] M. Lin and R. Sine, *Ergodic theory and the functional equation $(I - T)x = y$*, J. Operator Theory 10 (1983), 153–166.

[8] W. Rudin, *Functional Analysis*, McGraw-Hill, New York, 1973.

[9] R. Sato, *A remark on real coboundary cocycles in L^∞-space*, Proc. Amer. Math. Soc. 131 (2003), 231–233.

[10] —, *On solvability of the cohomology equation in function spaces*, Studia Math. 156 (2003), 277–293.

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