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SPACES OF MULTIPLIERS AND THEIR PREDUALS FOR THE ORDER MULTIPLICATION ON $[0,1]$. II

BY

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Abstract. Consider $I = [0,1]$ as a compact topological semigroup with max multiplication and usual topology, and let $C(I), L^p(I), 1 \le p \le \infty$, be the associated algebras. The aim of this paper is to study the spaces $\mathrm{Hom}_{C(I)}(L^r(I), L^p(I))$, $r > p$, and their preduals.

1. Introduction. The multipliers from $L^r(I)$ to $L^p(I)$, $1 \le r, p \le \infty$, where I denotes the topological semigroup $[0,1]$ with max multiplication and the usual topology, have been studied by Baker, Pym and Vasudeva [1]. The identification of multiplier spaces and their preduals from $L^r(I)$ to $L^p(I), 1 \le r \le p \le \infty$, has been carried out by Bhatnagar and Vasudeva [2]. The case when $r > p$ has evaded the authors. The present study does not close this gap completely; however, it provides a set of necessary conditions and another set of sufficient conditions for a linear operator to be a multiplier from $L^r(I)$ to $L^p(I), r > p$. As a natural outcome of the methods employed, we find that a multiplier from $L^r(I)$ to $L^p(I), r > p$, need not be bounded as was assumed in [1]. As there is substantial overlap between the arguments presented in [2] and the present note, we have tried to be as brief as possible. For details, the reader may refer to [2].

2. Preliminaries. The set $I = [0,1]$ equipped with max multiplication and usual topology is a compact topological semigroup. Let $C(I)$ and $L^p(I), 1 \le p \le \infty$, have their usual meanings. $L^p(I)$ is a left Banach $C(I)$-module under convolution $*$ defined by

$$\varphi * g(t) = \varphi(t) \int_0^t g(s)\, ds + g(t) \int_0^t \varphi(s)\, ds$$

for almost all $t \in I$, where $\varphi \in C(I)$ and $g \in L^p(I)$. We denote this left Banach $C(I)$-module by L^p_*. The adjoint action $\circ$ of an element $\varphi \in C(I)$ on $L^{p'}(I)$, $1/p + 1/p' = 1$, under which $L^{p'}(I)$ becomes a right Banach

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$C(I)$-module is defined by

$$g \circ \varphi(s) = g(s)\int_0^s \varphi(t)\,dt + \int_s^1 \varphi(t)g(t)\,dt, \qquad g \in L^{p'}(I).$$

The right Banach $C(I)$-module $L^{p'}(I)$ with the adjoint action defined above is denoted by $L_\circ^{p'}$. The above definitions can be extended to $\varphi \in L^r(I)$, $r > p$, by denseness of $C(I)$ in $L^r(I)$.

Let $M_\circ^{r,p}$ denote $\mathrm{Hom}_{C(I)}(L_*^r, L_*^p)$ and $M_*^{r,p}$ denote $Hom_{C(I)}(L_*^r, L_\circ^p)$ for $1 \le r, p \le \infty$. If

$$A_*^{r,p} = L_*^r \hat{\otimes}_{C(I)} L_*^{p'}, \qquad A_\circ^{r,p} = L_*^r \hat{\otimes}_{C(I)} L_\circ^{p'},$$

where the tensor product is the projective tensor product of Banach modules, then it follows, using a theorem of Rieffel [5], that $(A_*^{r,p})^* = M_*^{r,p}$ and $(A_\circ^{r,p})^* = M_\circ^{r,p}$.

3. Description of preduals. We define an operator

$$B : L^r(I) \hat{\otimes} L^{p'}(I) \to L^{p'}(I),$$

where $1 \le r, p' < \infty$, by

$$B(f \otimes g)(s) = g(s)\int_0^s f(t)\,dt.$$

The image of B in $L^{p'}(I)$ will be called $B^{r,p}$ if $1 < r, p < \infty$, and B^∞ in L^∞ if $r = p' = \infty$. Let $I_n = [0, 2^{-n}]$ and $J_n = [2^{-n}, 2^{-n+1}]$, $n = 1, 2, \ldots$. For a measurable function φ on I, let $P_n\varphi$ denote the function $\chi_{J_n}\varphi$, $n = 1, 2, \ldots$. Define $e_n = 2^n\chi_{I_n}$, $n = 1, 2, \ldots$. Since $\int_0^1 e_n(s)\,ds = 1$ for each n, we can easily see that if $f \equiv 0$ on I_n then $B(e_n \otimes f) = f$. As $f = \sum_{n=1}^\infty P_n f$, we obtain

$$f = \sum_{n=1}^\infty B(e_n \otimes P_n f) = \sum_{n=1}^\infty e_n \circ P_n f = \sum_{n=1}^\infty e_n * P_n f.$$

Let

$$C^{r,p} = \Big\{\varphi : \varphi \text{ is measurable and } \sum_{n=1}^\infty 2^{n/r'}\|P_n\varphi\|_{p'} < \infty\Big\},$$

$$C_u^{r,p} = \{\varphi : \varphi \text{ is measurable and } \sum_{n=1}^\infty (2^{n/r'}\|P_n\varphi\|_{p'})^u < \infty\Big\},$$

where $1/u = 1/r + 1/p'$. It may be noted that for $r = p$, $C^{p,p} = C_u^{p,p}$. We first characterize B^∞.

PROPOSITION 1. *$B^\infty = \{\varphi : \varphi$ is measurable and $\varphi(s)/s$ is essentially bounded$\}$.*

Proof. If φ is measurable and $\varphi(s)/s$ is essentially bounded then φ can be written as $\varphi(s) = (\varphi(s)/s)\int_0^s 1\,dt$. Consequently, $\varphi = B(\psi)$, where $\psi(s) = 1 \otimes \varphi(s)/s$. On the other hand, if $\varphi = B(f \otimes g)$, where $f, g \in L^\infty$, then $\varphi(s) = g(s)\int_0^s f(t)\,dt$, $s \in I$, is measurable. Moreover,

$$\left|\frac{\varphi(s)}{s}\right| = \left|\frac{g(s)}{s}\int\limits_0^s f(t)\,dt\right| \le \|g\|_\infty \|f\|_\infty.$$

REMARK. It is not difficult to see that the requirement that $\varphi(s)/s$ be essentially bounded is equivalent to $\sup_n 2^n \|P_n\varphi\|_\infty < \infty$.

THEOREM 1. *For $r > p$, $C^{r,p} \subseteq B^{r,p} \subseteq C_u^{r,p}$, where $1/u = 1/r + 1/p'$.*

Proof. If $\varphi \in C^{r,p}$, then $\sum_{n=1}^\infty 2^{n/r'} \|P_n\varphi\|_{p'} < \infty$, so that $\varphi = \sum_{n=1}^\infty B(e_n \otimes P_n\varphi)$ and $\sum_{n=1}^\infty \|e_n\|_r \|P_n\varphi\|_{p'} < \infty$. Thus $\psi = \sum_{n=1}^\infty e_n \otimes P_n\varphi \in L^r \otimes L^{p'}$ and $B(\psi) = \varphi$.

To prove the other inclusion, fix r and p and let $q = 1 + r/p'$, so that $q' = 1 + p'/r$. Let $\alpha = q/r$. Then

$$\alpha \cdot \frac{1}{q} + (1-\alpha) \cdot \frac{1}{\infty} = \frac{1}{r}, \qquad \alpha \cdot \frac{1}{q'} + (1-\alpha) \cdot \frac{1}{\infty} = \frac{1}{p'}.$$

Since $B(L^q \otimes L^{q'}) \subset B^{q,q}$ and $B(L^\infty \otimes L^\infty) \subset B^\infty$, it follows by interpolation, using Calderón [3], that B maps $L^r \otimes L^{p'}$ into a suitable intermediate space between $B^{q,q}$ and B^∞. (It may be observed that $B^{q,q} = C^{q,q}$). To see this, note that $B^{q,q}$ may be regarded as a mixed L^p space, viz., $L^1(\mathbb{N}, \nu, L^{q'}(I))$, where ν is the measure on $\mathbb{N}$ assigning mass 2^{-n} to the element $\{n\}$, and B^∞ may be regarded as $L^\infty(\mathbb{N}, \nu, L^\infty(I))$. These identifications are obtained by associating with $\varphi \in B^{q,q}$ (or B^∞) the function $f : \mathbb{N} \times I \to \mathbb{C}$ given by

$$f(n,t) = 2^n \varphi\left(\frac{t+1}{2^n}\right), \qquad n \in \mathbb{N},\ t \in I. \tag{1}$$

By Calderón [3], the intermediate space with index α between $L^1(\mathbb{N}, \nu, L^{q'}(I))$ and $L^\infty(\mathbb{N}, \nu, L^\infty(I))$ is contained in $L^u(\mathbb{N}, \nu, L^v(I))$, where

$$\alpha \cdot \frac{1}{1} + (1-\alpha) \cdot \frac{1}{\infty} = \frac{1}{u}, \qquad \alpha \cdot \frac{1}{q'} + (1-\alpha) \cdot \frac{1}{\infty} = \frac{1}{v}.$$

Thus $1/u = 1/r + 1/p'$, and $v = p'$. It follows that B maps $L^r \hat{\otimes} L^{p'}$ into $L^u(\mathbb{N}, \nu, L^{p'}(I))$ and by identification (1) this corresponds to measurable functions φ on I such that $\sum_{n=1}^\infty (2^{n/r'} \|P_n\varphi\|_{p'})^u < \infty$. Thus $\varphi \in C_u^{r,p}$. This completes the proof.

We next characterize the predual $A_\circ^{r,p}$ of the multiplier space $M_\circ^{r,p}$. Let AC_u° $(u \ge 1)$ be the space of absolutely continuous functions on $[0,1]$ whose derivative belongs to $L^u(I)$ and which vanish at 1.

THEOREM 2. *For* $r \geq p$, $C^{r,p} \subseteq A_\circ^{r,p} \subseteq B^{r,p} + AC_u^\circ$, *where* $1/r + 1/p' = 1/u$.

Proof. If $\varphi \in C^{r,p}$, we can write $\varphi = \sum_{n=1}^\infty e_n \circ P_n\varphi$ and by definition of $C^{r,p}$, $\sum_{n=1}^\infty \|e_n\|_r \, \|P_n\varphi\|_{p'} < \infty$ so that $\varphi \in A_\circ^{r,p}$.

Clearly, every element of $A_\circ^{r,p}$ is a sum of the form $\varphi+\psi$, where $\varphi \in B^{r,p}$ and $\psi \in AC_u^\circ$, so that we have the required result.

THEOREM 3. (a) $A_*^{r,p}$ *is a semisimple commutative Banach algebra under convolution. It has an approximate identity. The maximal ideal space of* $A_*^{r,p}$ *is the interval* $(0,1]$ *with the interval topology.*

(b) $C^{r,p} + C^{p',r'} \subseteq A_*^{r,p} \subseteq B^{r,p} + B^{p',r'}$ *for* $1 \leq r, p \leq \infty$.

(c) $C^{p',r'} \subseteq C^{r,p}$ *if* $r \geq p'$, *and* $C^{r,p} \subseteq C^{p',r'}$ *if* $r \leq p'$,
$C_u^{p',r'} \subseteq C_u^{r,p}$ *if* $r \geq p'$, *and* $C_u^{r,p} \subseteq C_u^{p',r'}$ *if* $r \leq p'$.

(d) $C^{r,p} \subseteq A_*^{r,p} \subseteq C_u^{r,p}$ *if* $r \geq p'$, *and* $C^{p',r'} \subseteq A_*^{r,p} \subseteq C_u^{p',r'}$ *if* $r \leq p'$.

Proof. (a) $A_*^{r,p} = L^r \hat{\otimes}_{C(I)} L^{p'}$, being the projective tensor product of two Banach algebras, is a Banach algebra. For the detailed proof, consult Theorem 7 of [2].

(b) If $\varphi \in C^{r,p}$, then $\sum_{n=1}^\infty 2^{n/r'} \|P_n\varphi\|_{p'} < \infty$ and $\varphi = \sum_{n=1}^\infty e_n * P_n\varphi \in A_*^{r,p}$. Similarly for $C^{p',r'}$. It is clear that every element of $A_*^{r,p}$ is a sum $\varphi + \psi$, where $\varphi \in B^{r,p}$ and $\psi \in B^{p',r'}$, so that $A_*^{r,p} \subseteq B^{r,p} + B^{p',r'}$.

(c) We show that $C^{p',r'} \subseteq C^{r,p}$ if $r \geq p'$, the other proofs are similar. If $\varphi \in C^{p',r'}$, then $\sum_{n=1}^\infty 2^{n/p} \|P_n\varphi\|_r < \infty$ and

$$\sum_{n=1}^\infty 2^{n/r'} \|P_n\varphi\|_{p'} \leq \sum_{n=1}^\infty 2^{n/r'} \|P_n\varphi\|_r (2^{-n})^{1/p'-1/r} = \sum_{n=1}^\infty 2^{n/p} \|P_n\varphi\|_r < \infty$$

so that $\varphi \in C^{r,p}$.

(d) is clear in view of (c).

4. Multipliers. In this section we study the multipliers, namely, $M_*^{r,p} = \mathrm{Hom}_{C(I)}(L_*^r, L_\circ^p)$ and $M_\circ^{r,p} = \mathrm{Hom}_{C(I)}(L_*^r, L_*^p)$. We have $M_*^{r,p} = (A_*^{r,p})^*$, and we deal with the case $r \geq p'$; the other case can be obtained by identifying $A_*^{r,p}$ and $A_*^{p',r'}$. The following theorem gives us a necessary condition for a multiplier to be in $M_*^{r,p}$.

THEOREM 4. *Let* $r \geq p'$. *If* $t \in M_*^{r,p}$ *then* t *is measurable and*

$$\sup_n 2^{-n/r'} \|P_n t\|_p < \infty.$$

Proof. $M_*^{r,p} = (A_*^{r,p})^* \subseteq (C^{r,p})^*$ by Theorem 3. Therefore, if $t \in M_*^{r,p}$ then t is measurable and $\sup_n 2^{-n/r'} \|P_n t\|_p < \infty$.

The following theorem gives us a sufficient condition for a multiplier to be in $M_*^{r,p}$.

THEOREM 5. *Let $r \geq p'$. If t is measurable and satisfies*

$$\sum_{n=1}^{\infty}(2^{-n/r'}\|P_n t\|_p)^{u'} < \infty,$$

then $t \in M_*^{r,p}$.

Proof. $M_*^{r,p} = (A_*^{r,p})^* \supseteq (C_u^{r,p})^*$ by Theorem 3. Therefore if t is measurable and satisfies $\sum_{n=1}^{\infty}(2^{-n/r'}\|P_n t\|_p)^{u'} < \infty$ then $t \in M_*^{r,p}$.

Next, we look at $\mathrm{Hom}_{C(I)}(L_*^r, L_*^p)$, i.e., $(L^r \hat{\otimes}_{C(I)} L^{p'})^*$, where the action of $C(I)$ on $L^r(I)$ is by $*$ and on $L^{p'}(I)$ by $\circ$. The natural map $w : L_*^r \hat{\otimes} L_\circ^{p'} \to L^r \circ L^{p'}$ factors through $L^r \hat{\otimes}_{C(I)} L^{p'}$ (note that $C(I)$ can be replaced by another dense subalgebra contained in $L^r \cap L^{p'}$). Theorem 1 of [2] shows that the map $L_*^r \hat{\otimes}_{C(I)} L_\circ^{p'} \to L^r \circ L^{p'}$ induced by w is one-to-one. Then $L^r \circ L^{p'}$ can be identified with $\{\sum_{i=1}^{\infty} f_i \circ g_i : f_i \in L^r,\ g_i \in L^{p'}$ and $\sum_{i=1}^{\infty}\|f_i\|_r\|g_i\|_{p'} < \infty\}$, which we can identify with $L_*^r \hat{\otimes}_{C(I)} L_\circ^{p'}$, i.e., the predual of $\mathrm{Hom}_{C(I)}(L_*^r, L_*^p)$.

THEOREM 6. *Let $r \geq p$. If $\beta \in M_\circ^{r,p}$ then β is measurable and satisfies $\sup_n 2^{-n/r'}\|P_n\beta\|_p < \infty$.*

Proof. $M_\circ^{r,p} = (A_\circ^{r,p})^* \subseteq (C^{r,p})^*$ by Theorem 2. Thus if $\beta \in M_\circ^{r,p}$ then $\beta \in (C^{r,p})^*$ and so β is measurable and satisfies $\sup_n 2^{-n/r'}\|P_n\beta\|_p < \infty$.

THEOREM 7. *Let $r \geq p$. If $\beta \in L^{u'}(I)$ has an a.e. derivative h which satisfies $\sum_{n=1}^{\infty}(2^{-n/r'}\|P_n h\|_p)^{u'} < \infty$, then $\beta \in M_\circ^{r,p}$.*

Proof. $M_\circ^{r,p} = (A_\circ^{r,p})^* \supseteq (C_u^{r,p} + AC_u^\circ)^*$ by Theorems 1 and 2. Suppose $\mu \in (C_u^{r,p} + AC_u^\circ)^*$. Then $\mu|_{AC_u^\circ} \in (AC_u^\circ)^*$ and $\mu|_{C_u^{r,p}} \in (C_u^{r,p})^*$. Note that $(AC_u^\circ)^* = L^{u'}(I)$, via the pairing

$$\langle \mu, \varphi\rangle = \int_0^1 \beta(s)f(s)\,ds, \tag{2}$$

where $\varphi(s) = \int_s^1 f(t)\,dt$ is in AC_u° and μ corresponds to $\beta \in L^{u'}(I)$. Since $\mu|_{C_u^{r,p}} \in (C_u^{r,p})^*$ it follows that $\beta \in (C_u^{r,p})^*$. For any $\varepsilon > 0$, $L^{p'}(I_\varepsilon) \subset C_u^{r,p}$ ($I_\varepsilon = [\varepsilon, 1]$) and so $\mu|_{L^{p'}(I_\varepsilon)}$ corresponds to an $L^p(I_\varepsilon)$ function h_ε such that for $\varphi \in L^{p'}(I_\varepsilon)$,

$$\langle \mu, \varphi\rangle = \int_\varepsilon^1 h_\varepsilon(s)\varphi(s)\,ds = \int_0^1 h_\varepsilon(s)\varphi(s)\,ds,$$

where φ is taken to be zero on $[0,\varepsilon)$. It is clear that the h_ε's are compatible, i.e. $h_{\varepsilon'} = h_\varepsilon$ on $[\varepsilon, 1]$ if $\varepsilon' < \varepsilon$. Moreover, for $\varphi = \int_s^1 f(t)\,dt \in L^{p'}(I_\varepsilon)$ we have

$$\langle \mu, \varphi \rangle = \int_0^1 h_\varepsilon(s) \int_s^1 f(t)\,dt\,ds = \int_0^1 f(t) \int_\varepsilon^t h_\varepsilon(s)\,ds\,dt.$$

Comparing this with (2) we get

$$\beta(t) = \int_\varepsilon^t h_\varepsilon(s)\,ds \qquad \text{for } t > \varepsilon.$$

Thus $\beta'(t) = h_\varepsilon(t)$ a.e. on $[\varepsilon, 1]$ so we have proved that there exists h measurable on (0,1] such that $h \in L^p(I_\varepsilon)$ for every $\varepsilon > 0$ and $\beta'(t) = h(t)$ a.e. on $(0, 1]$. If we take $\varphi \in B^{r,p} \subseteq C_u^{r,p}$, then

$$\langle \mu, \varphi \rangle = \int_0^1 h(t)\varphi(t)\,dt$$

exists and is finite because $\sum_{n=1}^\infty (2^{-n/r'} \|P_n h\|_p)^{u'} < \infty$. The multiplier M_β corresponding to β is given by

$$M_\beta(f)(t) = \Big[\beta(1) - \int_t^1 h(s)\,ds\Big] f(s) + h(t) \int_0^t f(s)\,ds \qquad \text{for } f \in L^r(I).$$

Indeed,

$$\begin{aligned} \langle M_\beta(f), g \rangle &= \langle \beta, f \circ g \rangle \\ &= \int_0^1 \Big\{ h(t)g(t) \int_0^t f(s)\,ds + g(t)f(t)\beta(t) \Big\}\,dt, \\ &= \int_0^1 g(t) \Big\{ \Big(\beta(1) - \int_t^1 h(s)\,ds\Big) f(t) + h(t) \int_0^t f(s)\,ds \Big\}\,dt. \end{aligned}$$

REMARKS. (i) For $r = p$ $(u = 1)$, $\beta \in L^\infty(I)$, and consequently $x \mapsto \int_x^1 h(t)\,dt$ is bounded.

(ii) The condition that $h(s) \int_0^s f(t)\,dt$ is in $L^p(I)$ for every $f \in L^r(I)$ is equivalent to $\sum_{n=1}^\infty (2^{-n/r'} \|P_n h\|_p)^{u'} < \infty$. To see this, let $g \in L^{p'}$. If $\widehat{f}(x)$ denotes $\int_0^x f(y)\,dy$ for $f \in L^r$, then

$$\begin{aligned} |\langle h\widehat{f}, g \rangle| &= \Big| \int_0^1 (h\widehat{f})(x) g(x)\,dx \Big| \le \sum_n \|P_n h\|_p \|P_n(g\widehat{f})\|_{p'} \\ &\le \Big[\sum_{n=1}^\infty (2^{-n/r'} \|P_n h\|_p)^{u'} \Big]^{1/u'} \Big[\sum_{n=1}^\infty (2^{n/r'} \|P_n(g\widehat{f})\|_{p'})^u \Big]^{1/u} < \infty, \end{aligned}$$

since $g\widehat{f} \in B^{r,p} \subseteq C_u^{r,p}$. Thus $h\widehat{f} \in L^p$ for all $f \in L^r$.

Conversely, suppose $h\widehat{f} \in L^p$ for all $f \in L^r$. Since

$$\Big[\sum_{n=1}^{\infty}(2^{-n/r'}\|P_n h\|_p)^{u'}\Big]^{1/u'} = \sup_{\varphi\in C_u^{r,p}} \frac{\|h\varphi\|_1}{\|\varphi\|},$$

and

$$\|\varphi\|_{p'} \le \|\varphi\| \quad \text{for } \varphi \in C_u^{r,p},$$

we get

$$\|h\varphi\|_1 \le \|(x+1)^{1/r'}h\|_p\|(x+1)^{-1/r'}\varphi\|_{p'} \le \|(x+1)^{1/r'}h\|_p\|\varphi\| < \infty,$$

since $f(x) = (x+1)^{-1/r} \in L^r$. Thus $[\sum_{n=1}^{\infty}(2^{-n/r'}\|P_n h\|_p)^{u'}]^{1/u'} < \infty$. Here $\|\varphi\|$ stands for $[\sum_{n=1}^{\infty}(2^{n/r'}\|P_n\varphi\|_{p'})^{u}]^{1/u}$.

(iii) In [1] the fact that a multiplier T from a Banach algebra A to itself gives rise to a bounded continuous function was used heavily. It is natural to expect that if T is a multiplier from a Banach algebra A to a Banach algebra B, where $A \subset B$, $A \ne B$, then T need not give rise to a bounded continuous function. See Larsen [4, Theorem 1.2.2]. This is corroborated by the fact that $\beta(x) = \log x$, $x \in (0,1]$, gives a multiplier from $L^r(I)$ to $L^p(I)$, $r > p$ $(u > 1)$, whereas $t \mapsto \int_t^1 (1/s)\,ds$ is not bounded.

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