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COFINITENESS OF GENERALIZED LOCAL COHOMOLOGY MODULES

BY

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Abstract. Let $\mathfrak{a}$ denote an ideal of a commutative Noetherian ring R, and M and N two finitely generated R-modules with $\operatorname{pd} M < \infty$. It is shown that if either $\mathfrak{a}$ is principal, or R is complete local and $\mathfrak{a}$ is a prime ideal with $\dim R/\mathfrak{a} = 1$, then the generalized local cohomology module $H^i_{\mathfrak{a}}(M,N)$ is $\mathfrak{a}$-cofinite for all $i \geq 0$. This provides an affirmative answer to a question proposed in [13].

1. Introduction. A generalization of local cohomology functors has been given by J. Herzog in [6]. Let $\mathfrak{a}$ denote an ideal of a commutative Noetherian ring R. For each $i \geq 0$, the functor $H^i_{\mathfrak{a}}(\cdot,\cdot)$ is defined by $H^i_{\mathfrak{a}}(M,N) = \varinjlim_n \operatorname{Ext}^i_R(M/\mathfrak{a}^n M, N)$ for all R-modules M and N. Clearly, this is a generalization of the usual local cohomology functor. The study of this concept was continued in [10], [2] and [12]. Recently, there is some new interest in generalized local cohomology (see e.g. [1], [13] and [14]).

In 1969, A. Grothendieck conjectured that if $\mathfrak{a}$ is an ideal of R and N is a finitely generated R-module, then $\operatorname{Hom}_R(R/\mathfrak{a}, H^i_{\mathfrak{a}}(N))$ is finitely generated for all $i \geq 0$. R. Hartshorne provided a counter-example to this conjecture in [5]. He defined a module N to be $\mathfrak{a}$-*cofinite* if $\operatorname{Supp}_R N \subseteq V(\mathfrak{a})$ and $\operatorname{Ext}^i_R(R/\mathfrak{a}, N)$ is finitely generated for all $i \geq 0$, and he asked the following question.

Question 1.1. Let $\mathfrak{a}$ be an ideal of R, and N a finitely generated R-module. When are $H^i_{\mathfrak{a}}(N)$ $\mathfrak{a}$-cofinite for all $i \geq 0$?

Hartshorne [5, Corollaries 6.3 and 7.7] proved that if $\mathfrak{a}$ is an ideal of the complete regular local ring R and N is a finitely generated R-module, then $H^i_{\mathfrak{a}}(N)$ is $\mathfrak{a}$-cofinite in two cases:

(i) (see [5, Corollary 6.3]) $\mathfrak{a}$ is a principal ideal,
(ii) (see [5, Corollary 7.7]) $\mathfrak{a}$ is a prime ideal with $\dim R/\mathfrak{a} = 1$.

This subject was studied by several authors afterward (see e.g. [8], [4] and [15]). The best result concerning cofiniteness of local cohomology is:

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THEOREM 1.2 ([8], [4], [15]). *Let $\mathfrak{a}$ be an ideal of R, and N a finitely generated R-module. If either $\mathfrak{a}$ is principal or R is local and* $\dim R/\mathfrak{a} = 1$, *then $H^i_{\mathfrak{a}}(N)$ is $\mathfrak{a}$-cofinite for all $i \geq 0$.*

S. Yassemi [13, Question 2.7] asked whether 1.2 holds for generalized local cohomology. The main aim of this paper is to extend 1.2 to that setting. More precisely, we prove the following.

THEOREM 1.3. *Let $\mathfrak{a}$ be an ideal of the ring R. Let M and N be two finitely generated R-modules with* $\operatorname{pd} M < \infty$. *If either*

(i) $\mathfrak{a}$ *is principal*, *or*

(ii) *R is complete local and $\mathfrak{a}$ is a prime ideal with* $\dim R/\mathfrak{a} = 1$,

then $H^i_{\mathfrak{a}}(M, N)$ is $\mathfrak{a}$-cofinite for all $i \geq 0$.

All rings considered in this paper are assumed to be commutative Noetherian with identity. Our terminology follows the textbook [3].

2. Cofiniteness results. Let $\mathfrak{a}$ denote an ideal of a ring R. The *generalized local cohomology* is defined by

$$H^i_{\mathfrak{a}}(M, N) = \varinjlim_n \operatorname{Ext}^i_R(M/\mathfrak{a}^n M, N)$$

for all R-modules M and N. Note that this is in fact a generalization of the usual local cohomology, because if $M = R$, then $H^i_{\mathfrak{a}}(R, N) = H^i_{\mathfrak{a}}(N)$.

DEFINITION 2.1. Let M be an R-module. The *generalized ideal transform functor* with respect to an ideal $\mathfrak{a}$ of R is defined by

$$D_{\mathfrak{a}}(M, \cdot) = \varinjlim_n \operatorname{Hom}_R(\mathfrak{a}^n M, \cdot).$$

Let $R^i D_{\mathfrak{a}}(M, \cdot)$ denote the ith right derived functor of $D_{\mathfrak{a}}(M, \cdot)$. One can check that there is a natural isomorphism $R^i D_{\mathfrak{a}}(M, \cdot) \cong \varinjlim_n \operatorname{Ext}^i_R(\mathfrak{a}^n M, \cdot)$. Thus, by considering the Ext long exact sequences induced by the short exact sequences

$$0 \to \mathfrak{a}^n M \to M \to M/\mathfrak{a}^n M \to 0 \quad (n \in \mathbb{N}),$$

we can deduce the following lemma.

LEMMA 2.2. *Let M be an R-module. For any R-module N, there is an exact sequence*

$$0 \to H^0_{\mathfrak{a}}(M, N) \to \operatorname{Hom}_R(M, N) \to D_{\mathfrak{a}}(M, N) \to H^1_{\mathfrak{a}}(M, N) \to \cdots$$
$$\to \cdots \to H^i_{\mathfrak{a}}(M, N) \to \operatorname{Ext}^i_R(M, N) \to R^i D_{\mathfrak{a}}(M, N) \to H^{i+1}_{\mathfrak{a}}(M, N) \to \cdots .$$

Moreover, if M has finite projective dimension, then there is a natural isomorphism $H^{i+1}_{\mathfrak{a}}(M, N) \cong R^i D_{\mathfrak{a}}(M, N)$ for all $i \geq \operatorname{pd} M + 1$.

Let $\mathfrak{a}$ be an ideal of R, and M an R-module. We say that M is $\mathfrak{a}$-*cofinite* if $\operatorname{Supp}_R M \subseteq V(\mathfrak{a})$ and $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is finitely generated for all $i \geq 0$.

LEMMA 2.3. *Suppose M, N are two R-modules and $\mathfrak{a}$ an ideal of R. If M is finitely generated, then* $\operatorname{Supp}_R H^i_{\mathfrak{a}}(M,N) \subseteq V(\mathfrak{a})$ *for all $i \geq 0$.*

Proof. Let $\mathfrak{p}$ be a prime ideal of R. It follows from [9, Theorem 9.50] that

$$\operatorname{Ext}^i_R(M/\mathfrak{a}^n M, N)_{\mathfrak{p}} \cong \operatorname{Ext}^i_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}/(\mathfrak{a}^n R_{\mathfrak{p}})M_{\mathfrak{p}}, N_{\mathfrak{p}})$$

for all $i \geq 0$. On the other hand, it is well known that the tensor product preserves direct limits (see e.g. [9, Corollary 2.20]). Thus

$$\begin{aligned} H^i_{\mathfrak{a}}(M,N)_{\mathfrak{p}} &\cong R_{\mathfrak{p}} \otimes_R \varinjlim_n \operatorname{Ext}^i_R(M/\mathfrak{a}^n M, N) \\ &\cong \varinjlim_n \operatorname{Ext}^i_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}/(\mathfrak{a}^n R_{\mathfrak{p}})M_{\mathfrak{p}}, N_{\mathfrak{p}}) \cong H^i_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}). \end{aligned}$$

This shows that $\operatorname{Supp}_R H^i_{\mathfrak{a}}(M,N) \subseteq V(\mathfrak{a})$, as required. ■

LEMMA 2.4. (i) *If M is a finitely generated R-module such that* $\operatorname{Supp}_R M \subseteq V(\mathfrak{a})$, *then M is $\mathfrak{a}$-cofinite.*

(ii) *Let $0 \to L \to M \to N \to 0$ be an exact sequence of R-modules. Whenever two of L, M or N are $\mathfrak{a}$-cofinite, then so is the third.*

Proof. (i) Since M is finitely generated, it follows that $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is finitely generated for all $i \geq 0$. Hence M is $\mathfrak{a}$-cofinite, by definition.

(ii) This is well known and can be deduced easily by considering the long exact sequence

$$\cdots \to \operatorname{Ext}^i_R(R/\mathfrak{a}, L) \to \operatorname{Ext}^i_R(R/\mathfrak{a}, M) \to \operatorname{Ext}^i_R(R/\mathfrak{a}, N) \to \operatorname{Ext}^{i+1}_R(R/\mathfrak{a}, L) \to \cdots. \quad ■$$

LEMMA 2.5. *Let $\mathfrak{a} = Ra$ be a principal ideal of R, and M and N two finitely generated R-modules. Let $\operatorname{Hom}_R(M,N)_a$ denote the localization of $\operatorname{Hom}_R(M,N)$ with respect to the multiplicative closed subset $\{a^i : i \geq 0\}$ of R. Then*

(i) *there is a natural isomorphism* $D_{\mathfrak{a}}(M,N) \cong \operatorname{Hom}_R(M,N)_a$,

(ii) $H^1_{\mathfrak{a}}(M,N)$ *is $\mathfrak{a}$-cofinite.*

Proof. (i) If a is nilpotent, then it is clear that both $D_{\mathfrak{a}}(M,N)$ and $\operatorname{Hom}_R(M,N)_a$ will vanish. Hence, we may and do assume that a is not nilpotent. For all $i, j \in \mathbb{N}$ with $j \geq i$, let $\pi_{ij} : \operatorname{Hom}_R(a^i M, N) \to \operatorname{Hom}_R(a^j M, N)$ be defined by $\pi_{ij}(f) = f|_{a^j M}$ for $f \in \operatorname{Hom}_R(a^i M, N)$. Also, denote the natural map $\operatorname{Hom}_R(a^i M, N) \to D_{\mathfrak{a}}(M,N)$ by π_i. Recall that we defined $D_{\mathfrak{a}}(M,N)$ as the direct limit of the direct system $(\operatorname{Hom}_R(a^i M, N), \pi_{ij})_{i,j \in \mathbb{N}}$.

Now define $\psi_i : \operatorname{Hom}_R(a^i M, N) \to (\operatorname{Hom}_R(M,N))_a$ by $\psi_i(f) = f\lambda_i/a^i$, where $\lambda_i : M \to a^i M$ is defined by $\lambda_i(m) = a^i m$ for $m \in M$. Clearly $\{\psi_i\}_{i \in \mathbb{N}}$

is a morphism of direct systems. Assume $\psi : D_{\mathfrak{a}}(M,N) \to (\operatorname{Hom}_R(M,N))_a$ is the homomorphism induced by $\{\psi_i\}_{i\in\mathbb{N}}$. Thus for each $g \in D_{\mathfrak{a}}(M,N)$, we have $\psi(g) = \psi_i(f)$, where $i \in \mathbb{N}$ and $f \in \operatorname{Hom}_R(a^iM,N)$ are such that $\pi_i(f) = g$.

We show that ψ is an isomorphism. First, we show that ψ is injective. Suppose $\psi(g) = 0$ for some $g \in D_{\mathfrak{a}}(M,N)$. There are $i \in \mathbb{N}$ and $f \in \operatorname{Hom}_R(a^iM,N)$ such that $g = \pi_i(f)$. Hence

$$\psi(g) = \psi_i(f) = f\lambda_i/a^i = 0.$$

Hence there is $t \in \mathbb{N}$ such that $a^t(f\lambda_i) = 0$. Set $j = i + t$. Then $\pi_{ij}(f) = 0$ and so

$$g = \pi_i(f) = \pi_j(\pi_{ij}(f)) = 0.$$

Next, we show that ψ is surjective. Let $x_1,\dots,x_t$ be a set of generators of M. Let $l \in (\operatorname{Hom}_R(M,N))_a$. Then there are $h \in \operatorname{Hom}_R(M,N)$ and $c \in \mathbb{N}$ such that $l = h/a^c$. Since N is a Noetherian R-module, there exists an integer $e \geq c$ such that $(0 :_N a^e) = (0 :_N a^{e+j})$ for all $j \geq 0$. Define $f \in \operatorname{Hom}_R(a^{2e}M,N)$ by $f(a^{2e}x) = a^{2e-c}h(x)$ for $x \in M$. If $a^{2e}x = a^{2e}x'$ for some x and x' in M, then $h(x - x') \in (0 :_N a^{2e})$. Hence $a^{2e-c}h(x) = a^{2e-c}h(x')$. Therefore f is well defined. Set $g = \pi_{2e}(f)$. Then

$$\psi(g) = \psi_{2e}(f) = f\lambda_{2e}/a^{2e} = h/a^c = l.$$

Thus ψ is surjective.

(ii) Let $\psi : D_{\mathfrak{a}}(M,N) \to (\operatorname{Hom}_R(M,N))_a$ be as above. By part (i), [3, Theorem 2.2.4(i)] and 2.2 we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 \to & H^0_{\mathfrak{a}}(M,N) & \to \operatorname{Hom}_R(M,N) & \xrightarrow{f} & D_a(M,N) & \to & H^1_{\mathfrak{a}}(M,N) & \xrightarrow{g} & \operatorname{Ext}^1_R(M,N) \\
 & & \downarrow \mathrm{id} & & \downarrow \psi & & & & \\
0 \to & \Gamma_{\mathfrak{a}}(\operatorname{Hom}_R(M,N)) & \to \operatorname{Hom}_R(M,N) & \xrightarrow{h} & (\operatorname{Hom}_R(M,N))_a & \to & H^1_{\mathfrak{a}}(\operatorname{Hom}_R(M,N)) & \to & 0
\end{array}$$

Let K be the kernel of g. We have $K \cong \operatorname{coker} f$ and $H^1_{\mathfrak{a}}(\operatorname{Hom}_R(M,N)) \cong \operatorname{coker} h$. The map ψ induces an isomorphism $\psi^* : \operatorname{coker} f \to \operatorname{coker} h$ defined by $\psi^*(x + \operatorname{im} f) = \psi(x) + \operatorname{im} h$ for $x + \operatorname{im} f \in \operatorname{coker} f$. Hence $K \cong H^1_{\mathfrak{a}}(\operatorname{Hom}_R(M,N))$. Therefore K is $\mathfrak{a}$-cofinite, by 1.2. Now consider the exact sequence

$$0 \to K \to H^1_{\mathfrak{a}}(M,N) \to \operatorname{im} g \to 0.$$

Since $\operatorname{Ext}^1_R(M,N)$ is finitely generated, it follows by 2.3 and 2.4(i) that $\operatorname{im} g$ is $\mathfrak{a}$-cofinite. Thus $H^1_{\mathfrak{a}}(M,N)$ is $\mathfrak{a}$-cofinite, by 2.4(ii). ■

LEMMA 2.6. *Let* $\mathfrak{a}$ *denote an ideal of the ring* R, *and* N *an* $\mathfrak{a}$-*cofinite* R-*module. Suppose that for any finitely generated* R-*module* M *with* $\operatorname{pd} M < \infty$, $\operatorname{Hom}_R(M,N)$ (*resp.* $M \otimes_R N$) *is* $\mathfrak{a}$-*cofinite. Then* $\operatorname{Ext}^i_R(M,N)$ (*resp.*

$\operatorname{Tor}_i^R(M,N))$ *is* $\mathfrak{a}$*-cofinite for all finitely generated* R*-modules* M *with* $\operatorname{pd} M < \infty$ *and all* $i \geq 0$*.*

Proof. We prove only the $\mathfrak{a}$-cofiniteness of $\operatorname{Ext}_R^i(M,N)$ for $i \geq 0$; the proof of the other part is similar. The proof proceeds by induction on $t = \operatorname{pd} M$. For $t = 0$, the claim holds by assumption. Now, suppose $t > 0$. There is a short exact sequence

$$0 \to K \to R^n \to M \to 0.$$

From this sequence, we deduce the exact sequence

$$0 \to \operatorname{Hom}_R(M,N) \to \operatorname{Hom}_R(R^n,N) \to \operatorname{Hom}_R(K,N) \to \operatorname{Ext}_R^1(M,N) \to 0,$$

and the isomorphisms $\operatorname{Ext}_R^{i+1}(M,N) \cong \operatorname{Ext}_R^i(K,N)$ for all $i \geq 1$. Thus from induction hypothesis, $\operatorname{Ext}_R^{i+1}(M,N)$ is $\mathfrak{a}$-cofinite for all $i \geq 1$. Note that $\operatorname{pd} K < t$. Also, by using the above exact sequence, one can check easily that $\operatorname{Ext}_R^1(M,N)$ is $\mathfrak{a}$-cofinite. Therefore, the claim follows by induction. ∎

LEMMA 2.7. *Let* $\mathfrak{a}$ *denote an ideal of the ring* R*. Let* M *and* N *be two finitely generated* R*-modules with* $\operatorname{pd} M < \infty$*. If either*

(i) $\mathfrak{a}$ *is principal*, *or*

(ii) R *is complete local and* $\mathfrak{a}$ *is a prime ideal with* $\dim R/\mathfrak{a} = 1$,

then $\operatorname{Ext}_R^p(M, H_\mathfrak{a}^q(N))$ *is* $\mathfrak{a}$*-cofinite for all* $p, q \geq 0$

Proof. First, we consider the case that $\mathfrak{a}$ is principal. By [9, Theorem 11.38], there is a Grothendieck spectral sequence

$$E_2^{p,q} := \operatorname{Ext}_R^p(M, H_\mathfrak{a}^q(N)) \underset{p}{\Longrightarrow} H_\mathfrak{a}^{p+q}(M,N).$$

We have $E_2^{p,q} = 0$ for $q \neq 0, 1$, because, by [3, Theorem 3.3.1], $H_\mathfrak{a}^q(N) = 0$ for all $q > 1$. Since $E_2^{p,0}$ is finitely generated and $\operatorname{Supp}_R E_2^{p,0} \subseteq V(\mathfrak{a})$, it follows by 2.4(i) that $E_2^{p,0}$ is $\mathfrak{a}$-cofinite. Therefore, it is enough to show that $E_2^{p,1}$ is $\mathfrak{a}$-cofinite for all $p \geq 0$. By [9, Corollary 11.44], we have an exact sequence

$$0 \to E_2^{1,0} \to H_\mathfrak{a}^1(M,N) \xrightarrow{g} E_2^{0,1} \to E_2^{2,0} \xrightarrow{f} H_\mathfrak{a}^2(M,N).$$

By 2.5(ii), $H_\mathfrak{a}^1(M,N)$ is $\mathfrak{a}$-cofinite. Thus, it turns out that $\operatorname{im} g$ is $\mathfrak{a}$-cofinite, by 2.4(ii). From the exact sequence

$$0 \to \operatorname{im} g \to E_2^{0,1} \to \ker f \to 0,$$

we deduce that $E_2^{0,1}$ is $\mathfrak{a}$-cofinite. Note that $\ker f$ is a finitely generated R-module. Therefore 2.6 implies that $E_2^{p,1}$ is $\mathfrak{a}$-cofinite for all $p \geq 0$, because $H_\mathfrak{a}^1(N)$ is $\mathfrak{a}$-cofinite by 1.2.

Now suppose that R is a complete local ring and $\mathfrak{a}$ a prime ideal of R with $\dim R/\mathfrak{a} = 1$. In view of 2.6, it suffices to show that $\operatorname{Hom}_R(M, H_\mathfrak{a}^q(N))$

is $\mathfrak{a}$-cofinite for all finitely generated R-modules M with $\operatorname{pd} M < \infty$. We prove this claim by induction on $\operatorname{pd} M = t$. The case $t = 0$, is clear by 1.2. Now assume that $t > 0$ and consider the exact sequence

$$0 \to K \to R^n \to M \to 0.$$

It follows that $\operatorname{pd} K \leq t - 1$. This short exact sequence yields the exact sequence

$$0 \to \operatorname{Hom}_R(M, H^q_{\mathfrak{a}}(N)) \to \operatorname{Hom}_R(R^n, H^q_{\mathfrak{a}}(N)) \xrightarrow{f} \operatorname{Hom}_R(K, H^q_{\mathfrak{a}}(N)).$$

Since by [4, Theorem 2] the subcategory of $\mathfrak{a}$-cofinite R-modules is abelian, it follows that $\ker f \cong \operatorname{Hom}_R(M, H^q_{\mathfrak{a}}(N))$ is $\mathfrak{a}$-cofinite. Therefore the claim follows by induction. ∎

THEOREM 2.8. *Let $\mathfrak{a}$ denote a principal ideal of the ring R. Let M and N be two finitely generated R-modules with* $\operatorname{pd} M < \infty$. *Then $H^p_{\mathfrak{a}}(M, N)$ is $\mathfrak{a}$-cofinite for all $p \geq 0$.*

Proof. By [9, Theorem 11.38], there is a Grothendieck spectral sequence

$$E_2^{p,q} := \operatorname{Ext}^p_R(M, H^q_{\mathfrak{a}}(N)) \underset{p}{\Longrightarrow} H^{p+q}_{\mathfrak{a}}(M, N).$$

This implies the following exact sequence in view of [11, Ex. 5.2.2] (note that $E_2^{p,q} = 0$ for $q \neq 0, 1$):

$$\to E_2^{p,0} \xrightarrow{f} H^p_{\mathfrak{a}}(M, N) \xrightarrow{d} E_2^{p-1,1} \to E_2^{p+1,0} \xrightarrow{g} H^{p+1}_{\mathfrak{a}}(M, N) \to \cdots .$$

Now, $\operatorname{im} f$ is a quotient of $E_2^{p,0}$ and so is finitely generated. Hence $\operatorname{im} f$ is $\mathfrak{a}$-cofinite, by 2.3 and 2.4(i). Also, $\ker g$ is $\mathfrak{a}$-cofinite by the same reason. By considering the short exact sequence

$$0 \to \operatorname{im} d \to E_2^{p-1,1} \to \ker g \to 0,$$

we deduce that $\operatorname{im} d$ is $\mathfrak{a}$-cofinite. Note that $E_2^{p-1,1}$ is $\mathfrak{a}$-cofinite by 2.7(i). Now from the short exact sequence

$$0 \to \operatorname{im} f \to H^p_{\mathfrak{a}}(M, N) \to \operatorname{im} d \to 0,$$

we deduce that $H^p_{\mathfrak{a}}(M, N)$ is $\mathfrak{a}$-cofinite for all $p \geq 0$. ∎

THEOREM 2.9. *Let $\mathfrak{p}$ denote a prime ideal of the complete local ring $(R, \mathfrak{m})$ with* $\dim R/\mathfrak{p} = 1$, *and M, N two finitely generated R-modules with* $\operatorname{pd} M < \infty$. *Then $H^i_{\mathfrak{p}}(M, N)$ is $\mathfrak{p}$-cofinite for all $i \geq 0$.*

Proof. There is a spectral sequence

$$E_2^{p,q} := \operatorname{Ext}^p_R(M, H^q_{\mathfrak{p}}(N)) \underset{p}{\Longrightarrow} H^{p+q}_{\mathfrak{p}}(M, N) = E^n.$$

It follows from 2.7(ii) that $E_2^{p,q}$ is $\mathfrak{p}$-cofinite for all p, q. By considering the sequence

$$\cdots \to E_2^{p-2,q+1} \xrightarrow{d_2^{p-2,q+1}} E_2^{p,q} \xrightarrow{d_2^{p,q}} E_2^{p+2,q-1} \to \cdots ,$$

we deduce that $\operatorname{im} d_2^{p-2,q+1}$ and $\ker d_2^{p,q}$ are $\mathfrak{p}$-cofinite, by [4, Theorem 2]. Hence $E_3^{p,q} = \ker d_2^{p,q}/\operatorname{im} d_2^{p-2,q+1}$ is $\mathfrak{p}$-cofinite. By imitating this argument we find that $E_r^{p,q} = \ker d_{r-1}^{p,q}/\operatorname{im} d_{r-1}^{p-r+1,q+r-2}$ is $\mathfrak{p}$-cofinite for all $r > 0$ and so $E_\infty^{p,q}$ is $\mathfrak{p}$-cofinite for all $p, q \geq 0$. There is a filtration

$$E^n = E_0^n \supseteq \cdots \supseteq E_p^n \supseteq \cdots \supseteq E_n^n \supseteq E_{n+1}^n = 0,$$

such that $E_p^n/E_{p+1}^n \cong E_\infty^{p,n-p}$. Thus E_n^n is $\mathfrak{p}$-cofinite. Now, by applying 2.4(ii) repeatedly to the short exact sequences

$$0 \to E_{p+1}^n \to E_p^n \to E_\infty^{p,n-p} \to 0, \qquad p = 0, 1, \ldots, n-1,$$

we deduce that E^n is $\mathfrak{p}$-cofinite, as required. ∎

Many results concerning local cohomology in positive prime characteristic can be extended to generalized local cohomology. In particular the main results of [7] also hold for generalized local cohomology.

THEOREM 2.10. *Let $(R, \mathfrak{m})$ be a regular local ring of characteristic $p > 0$, and $\mathfrak{a}$ an ideal of R. Let $\mathfrak{p}$ be a prime ideal of R, and M a finitely generated R-module. Then*

(i) $\mu^i(\mathfrak{p}, H_{\mathfrak{a}}^j(M, R)) \leq \mu^i(\mathfrak{p}, \operatorname{Ext}_R^j(M/\mathfrak{a}M, R))$ *for all $j \geq 0$. In particular $\mu^i(\mathfrak{p}, H_{\mathfrak{a}}^j(M, R))$ is finite for all $j \geq 0$ and all $i \geq 0$.*

(ii) $\operatorname{Ass}_R(H_{\mathfrak{a}}^j(M, R)) \subseteq \operatorname{Ass}_R(\operatorname{Ext}_R^j(M/\mathfrak{a}M, R))$ *and so* $\operatorname{Ass}_R(H_{\mathfrak{a}}^j(M, R))$ *is finite for all $j \geq 0$.*

Proof. The proof is a straightforward adaptation of the proof of [7, Theorem 2.1 and Corollary 2.3]. ∎

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