COLLOQUIUM MATHEMATICUM

VOL. 132 2013 NO. 1

CYCLIC MEAN-VALUE INEQUALITIES FOR THE GAMMA FUNCTION

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Abstract. We present two cyclic inequalities involving the classical Γ-function of Euler and the (unweighted) power mean

$$M_t(a,b)=\left(\frac{a^t+b^t}{2}\right)^{1/t}\quad (t\neq 0),\qquad M_0(a,b)=\sqrt{ab}\quad (a,b>0).$$

(I) Let $2\le n\in\mathbb{N}$ and $r\in\mathbb{R}$. The inequality

$$\prod_{j=1}^{n}\Gamma\left(\frac{1}{1+M_r(x_j,x_{j+1})}\right)\le\prod_{j=1}^{n}\Gamma\left(\frac{1}{1+x_j}\right)\qquad (x_{n+1}=x_1)$$

holds for all $x_j>0$ $(j=1,\dots,n)$ if and only if $r\le 0$.

(II) Let $2\le n\in\mathbb{N}$ and $s\in\mathbb{R}$. The inequality

$$\prod_{j=1}^{n}\Gamma\left(\frac{1}{1+x_j}\right)\le\prod_{j=1}^{n}\Gamma\left(\frac{1}{1+M_s(x_j,x_{j+1})}\right)\qquad (x_{n+1}=x_1)$$

is valid for all $x_j>0$ $(j=1,\dots,n)$ if and only if

$$s\ge\max_{0<x<1}P(x)=1.0309\ldots.$$

Here,

$$P(x)=2x-1+x(x-1)\frac{\psi'(x)}{\psi(x)}\quad\text{and}\quad \psi=\Gamma'/\Gamma.$$

1. Introduction. In 1954, H. S. Shapiro [12] published the following cyclic inequality in the Problem section of The American Mathematical Monthly:

$$\frac{n}{2}\le\frac{x_1}{x_2+x_3}+\frac{x_2}{x_3+x_4}+\dots+\frac{x_{n-1}}{x_n+x_1}+\frac{x_n}{x_1+x_2},\tag{1.1}$$

where $x_1,\dots,x_n$ $(n\ge 3)$ are positive real numbers. This inequality attracted the attention of numerous mathematicians, who tried to answer the question: for which n is (1.1) true? The problem is now solved: inequality (1.1) is valid for odd $n\le 23$ and for even $n\le 12$. For all other n it is false. Detailed information on the history of Shapiro's inequality can be found in

2010 *Mathematics Subject Classification*: 26D07, 33B15.

Key words and phrases: gamma function, cyclic inequalities, mean-values, convex, concave.

DOI: 10.4064/cm132-1-3

the survey paper [6]. A collection of many other cyclic inequalities is given in the recently published monograph [4]; see also [9].

Shapiro's inequality is a typical example of those cyclic inequalities presented in the literature. Indeed, most of them provide sums or products of real numbers, whereas only few cyclic inequalities involving classical special functions are known. In 1956, E. M. Wright [15] offered an elegant cyclic inequality which is valid for a whole class of functions. In fact, he presented a short proof for the following theorem.

If f is a positive function, monotone or convex on an interval I, then for $x, y, z \in I$ we have

$$0 < (x-y)(x-z)f(x) + (y-x)(y-z)f(y) + (z-x)(z-y)f(z), \tag{1.2}$$

unless $x = y = z$.

The famous gamma function of Euler, defined for positive real numbers x by

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt = \frac{1}{x} \prod_{n=1}^\infty \left\{ \left(1 + \frac{1}{n}\right)^x \left(1 + \frac{x}{n}\right)^{-1} \right\},$$

is strictly convex on $(0, \infty)$, so that we obtain, for $x, y, z > 0$,

$$0 \le (x-y)(x-z)\Gamma(x) + (y-x)(y-z)\Gamma(y) + (z-x)(z-y)\Gamma(z). \tag{1.3}$$

A refinement of (1.3) is given in [2]. It was proved that the lower bound 0 can be replaced by

$$c \cdot \left((x-y)^2 + (y-z)^2 + (z-x)^2\right)$$

with

$$c = \frac{1}{2} \min_{t>0} \Gamma(t) = 0.442\ldots.$$

The constant factor is the best possible.

Here, we are concerned with another cyclic inequality involving the Γ-function:

$$\begin{aligned} \Gamma\left(\frac{1}{1+\sqrt{xy}}\right)\Gamma\left(\frac{1}{1+\sqrt{yz}}\right)\Gamma\left(\frac{1}{1+\sqrt{zx}}\right) \\ \le \Gamma\left(\frac{1}{1+x}\right)\Gamma\left(\frac{1}{1+y}\right)\Gamma\left(\frac{1}{1+z}\right). \end{aligned} \tag{1.4}$$

Many numerical examples led to the conjecture that (1.4) is valid for all positive real numbers x, y, z. On the left-hand side of (1.4) the classical geometric mean of two numbers appears. This mean-value is a member of the wellknown family of power means, defined for $a, b > 0$ and real parameters t by

$$
\begin{aligned}
&M_t(a,b) = \left(\frac{a^t + b^t}{2}\right)^{1/t} \qquad (t \neq 0),\\
&M_0(a,b) = \lim_{t\to 0} M_t(a,b) = \sqrt{ab};
\end{aligned}
\tag{1.5}
$$

see [8].

In view of (1.4) and (1.5) it is natural to look for an extension of (1.4) involving power means and to ask for all real parameters r such that

$$
\begin{aligned}
&\Gamma\left(\frac{1}{1+M_r(x,y)}\right)\Gamma\left(\frac{1}{1+M_r(y,z)}\right)\Gamma\left(\frac{1}{1+M_r(z,x)}\right)\\
&\qquad \leq \Gamma\left(\frac{1}{1+x}\right)\Gamma\left(\frac{1}{1+y}\right)\Gamma\left(\frac{1}{1+z}\right)
\end{aligned}
\tag{1.6}
$$

is valid for all $x, y, z > 0$.

In Section 3, we solve this problem. We show that (1.6) holds for all positive x, y, z if and only if $r \leq 0$. In particular, this reveals that the geometric mean inequaliy (1.4) is true.

Does there exist a converse of (1.6)? This means we ask for all real parameters s such that

$$
\begin{aligned}
&\Gamma\left(\frac{1}{1+x}\right)\Gamma\left(\frac{1}{1+y}\right)\Gamma\left(\frac{1}{1+z}\right)\\
&\qquad \leq \Gamma\left(\frac{1}{1+M_s(x,y)}\right)\Gamma\left(\frac{1}{1+M_s(y,z)}\right)\Gamma\left(\frac{1}{1+M_s(z,x)}\right)
\end{aligned}
\tag{1.7}
$$

for all $x, y, z > 0$. With regard to (1.4) it is natural that the first step to investigate (1.7) is to consider the case $s = 1$ which yields an inequality involving the arithmetic mean $M_1(x,y) = (x+y)/2$. However, this inequality is not true. A counterexample is given by setting $x = 0.3$, $y = z = 0$. Numerous computer calculations supported the conjecture that (1.7) holds with the quadratic mean $M_2(x,y) = (x^2/2+y^2/2)^{1/2}$. And, in fact, it turned out that this conjecture is true. In Section 3, we prove even more. We establish that (1.7) is valid for all $x, y, z > 0$ if and only if $s \geq 1.0309\ldots$. In order to prove our main results we need three lemmas. They are presented in the next section.

The numerical values given in this paper have been calculated via the computer program Maple 13.

2. Lemmas. Here, we provide monotonicity and concavity properties of the Γ-function, and we offer a maximum property of the psi function which is defined by $\psi = \Gamma'/\Gamma$.

Lemma 1. *Let $x, y > 0$. The function*

$$\Delta(r) = \Delta(r; x, y) = \Gamma\left(\frac{1}{1 + M_r(x, y)}\right)$$

is increasing on $\mathbb{R}$.

Proof. Let $r \in \mathbb{R}$. We have

$$0 < \frac{1}{1 + M_r(x, y)} < 1.$$

Since $r \mapsto M_r(x, y)$ is increasing on $\mathbb{R}$ (see [8]) and Γ is decreasing on $(0, 1)$, we conclude that Δ is increasing on $\mathbb{R}$. ■

Using the computational knowledge engine WolframAlpha [14] we obtain the following result.

Lemma 2. *Let*

$$\begin{aligned} P(x) &= 2x - 1 + x(x-1)\frac{\psi'(x)}{\psi(x)} \qquad (x > 0), \\ P(0) &= \lim_{x \to 0} P(x) = 0. \end{aligned} \tag{2.1}$$

Then

$$\max_{0<x<1} P(x) = 1.0309\ldots \qquad \textit{at } x = 0.9207\ldots.$$

Lemma 3. *Let $s > 0$. The function*

$$\phi(x) = \phi(s; x) = \log \Gamma\left(\frac{1}{1 + x^{1/s}}\right) \tag{2.2}$$

is concave on $(0, \infty)$ if and only if

$$s \geq \max_{0<x<1} P(x). \tag{2.3}$$

Here, P is defined in (2.1).

Proof. Differentiation yields

$$-\frac{s^2(1 + x^{1/s})^2 x^{2-1/s}}{\psi((1 + x^{1/s})^{-1})}\phi''(x) = P((1 + x^{1/s})^{-1}) - s. \tag{2.4}$$

If $s \geq \max_{0<x<1} P(x)$, then

$$s \geq P((1 + x^{1/s})^{-1}) \qquad \text{for } x > 0. \tag{2.5}$$

Using (2.4) and (2.5) gives $\phi''(x) \leq 0$.

Conversely, if ϕ'' is nonpositive on $(0, \infty)$, then (2.4) implies that (2.5) holds. It follows that

$$s \geq P(x) \qquad \text{for } x \in (0, 1).$$

This leads to (2.3). ■

3. Main results. We are now ready to determine all $r \in \mathbb{R}$ such that (1.6) holds for all $x, y, z > 0$. Actually, we prove an n-dimensional analogue of (1.6).

THEOREM 1. *Let $2 \le n \in \mathbb{N}$ and $r \in \mathbb{R}$. The inequality*

$$\prod_{j=1}^{n} \Gamma\left(\frac{1}{1 + M_r(x_j, x_{j+1})}\right) \le \prod_{j=1}^{n} \Gamma\left(\frac{1}{1 + x_j}\right) \qquad (x_{n+1} = x_1) \tag{3.1}$$

holds for all positive real numbers $x_1, \ldots, x_n$ if and only if $r \le 0$.

Proof. We assume (for a contradiction) that there exists a parameter $r > 0$ such that (3.1) is valid for all positive numbers $x_1, \ldots, x_n$. Then we set $x_1 = x > 0$ and $x_2 = \cdots = x_n = y > 0$. This yields

$$\Gamma\left(\frac{1}{1 + M_r(x, y)}\right)^2 \le \Gamma\left(\frac{1}{1 + x}\right)\Gamma\left(\frac{1}{1 + y}\right).$$

We let y tend to 0 and obtain

$$\Gamma\left(\frac{1}{1 + \delta x}\right)^2 \le \Gamma\left(\frac{1}{1 + x}\right) \quad \text{with } \delta = 2^{-1/r}.$$

Next, we multiply by $(1 + \delta x)^{-2}$ and make use of the recurrence formula $t\Gamma(t) = \Gamma(1 + t)$. It follows that

$$\Gamma\left(1 + \frac{1}{1 + \delta x}\right)^2 \le \frac{1 + x}{(1 + \delta x)^2}\Gamma\left(1 + \frac{1}{1 + x}\right).$$

We let x tend to ∞ and obtain $1 \le 0$. This contradiction leads to $r \le 0$.

Next, we show that (3.1) is valid if $r \le 0$. Applying Lemma 1 reveals that it suffices to prove (3.1) for $r = 0$. For $x \in \mathbb{R}$ we define

$$A_1(x) = \log \Gamma\left(\frac{1}{1 + e^x}\right).$$

Then we get

$$A_1''(x) = y(1 - y)A_2(y) \tag{3.2}$$

with

$$A_2(y) = (1 - 2y)\psi(y) + y(1 - y)\psi'(y) \quad \text{and} \quad y = (1 + e^x)^{-1} \in (0, 1).$$

Since $\psi' > 0$ on $(0, \infty)$ (see [1]), we conclude that $A_2(y) > 0$ if $1/2 \le y < 1$. The representation

$$A_2(y) = (1 - 2y)\psi(y + 1) + y(1 - y)\psi'(y + 1) + 1 \tag{3.3}$$

reveals that A_2 is positive on $(x_0 - 1, 1/2)$, too. Here, $\psi(x_0) = 0$. And, if $0 < y \le x_0 - 1$, then (3.3) gives

$$A_2(y) \ge \psi(1) + 1 = 0.42\ldots.$$

Using (3.2) implies that A_1 is strictly convex on $\mathbb{R}$. Thus, for $a, b \in \mathbb{R}$ we get

$$\Gamma\Big(\frac{1}{1+e^{(a+b)/2}}\Big) \le \Big[\Gamma\Big(\frac{1}{1+e^a}\Big)\Gamma\Big(\frac{1}{1+e^b}\Big)\Big]^{1/2}.$$

Equality holds if and only if $a = b$.

Next, we set $a = \log x_j$, $b = \log x_{j+1}$ $(j = 1, \dots, n)$ and multiply. This leads to

$$\begin{aligned}\prod_{j=1}^{n} \Gamma\Big(\frac{1}{1+M_0(x_j, x_{j+1})}\Big) &\le \prod_{j=1}^{n}\Big[\Gamma\Big(\frac{1}{1+x_j}\Big)\Gamma\Big(\frac{1}{1+x_{j+1}}\Big)\Big]^{1/2} \\ &= \prod_{j=1}^{n} \Gamma\Big(\frac{1}{1+x_j}\Big) \qquad (x_{n+1} = x_1),\end{aligned}$$

with equality if and only if $x_1 = \cdots = x_n$. ∎

Remark 1. Since A_1 is positive and convex on $\mathbb{R}$, we conclude from Wright's inequality (1.2) that

$$1 < \Gamma\Big(\frac{1}{1+e^x}\Big)^{(x-y)(x-z)} \Gamma\Big(\frac{1}{1+e^y}\Big)^{(y-x)(y-z)} \Gamma\Big(\frac{1}{1+e^z}\Big)^{(z-x)(z-y)}$$

is valid for all real numbers x, y, z except $x = y = z$.

Remark 2. A theorem of M. Petrović [10] states that if a function f is strictly convex on $[0, \infty)$, then for $x, y > 0$ we have

$$f(x) + f(y) < f(0) + f(x+y).$$

Since A_1 and $\exp(A_1)$ are strictly convex on $\mathbb{R}$ with $A_1(0) = \log\sqrt{\pi}$, we obtain, for $x, y \in (0,1)$ or $x, y \in (1, \infty)$,

$$\Gamma\Big(\frac{1}{1+x}\Big)\Gamma\Big(\frac{1}{1+y}\Big) < \sqrt{\pi}\,\Gamma\Big(\frac{1}{1+xy}\Big),$$
$$\Gamma\Big(\frac{1}{1+x}\Big) + \Gamma\Big(\frac{1}{1+y}\Big) < \sqrt{\pi} + \Gamma\Big(\frac{1}{1+xy}\Big).$$

In both cases, the constant $\sqrt{\pi}$ is the best possible.

Our second theorem provides a counterpart of inequality (3.1).

Theorem 2. *Let $2 \le n \in \mathbb{N}$ and $s \in \mathbb{R}$. The inequality*

$$\prod_{j=1}^{n} \Gamma\Big(\frac{1}{1+x_j}\Big) \le \prod_{j=1}^{n} \Gamma\Big(\frac{1}{1+M_s(x_j, x_{j+1})}\Big) \qquad (x_{n+1} = x_1) \tag{3.4}$$

holds for all positive real numbers $x_1, \dots, x_n$ if and only if

$$s \ge \max_{0<x<1} P(x) = 1.0309\dots. \tag{3.5}$$

Here, P is the function defined in (2.1).

Proof. First, we suppose that (3.4) is valid for all positive numbers $x_1, \ldots, x_n$. From Theorem 1 we conclude that $s > 0$. Setting $x_1 = x$ and $x_2 = \cdots = x_n = y$ gives

$$\Gamma\left(\frac{1}{1+x}\right)\Gamma\left(\frac{1}{1+y}\right) \leq \Gamma\left(\frac{1}{1+M_s(x,y)}\right)^2.$$

Putting $x = a^{1/s}$ and $y = b^{1/s}$ leads to

$$\frac{\phi(s;a)+\phi(s;b)}{2} \leq \phi\left(s;\frac{a+b}{2}\right), \tag{3.6}$$

where ϕ is defined in (2.2). This implies that $x \mapsto \phi(s;x)$ is concave on $(0,\infty)$. Applying Lemmas 2 and 3 yields

$$s \geq \max_{0<x<1} P(x) = 1.0309\ldots.$$

Next, we assume that (3.5) holds. Then $x \mapsto \phi(s;x)$ is concave on $(0,\infty)$, so that (3.6) is valid for $a, b > 0$. Setting $a = x_j^s$ and $b = x_{j+1}^s$ gives

$$\left[\Gamma\left(\frac{1}{1+x_j}\right)\Gamma\left(\frac{1}{1+x_{j+1}}\right)\right]^{1/2} \leq \Gamma\left(\frac{1}{1+M_s(x_j,x_{j+1})}\right).$$

This leads to

$$\begin{aligned}\prod_{j=1}^{n}\Gamma\left(\frac{1}{1+x_j}\right) &= \prod_{j=1}^{n}\left[\Gamma\left(\frac{1}{1+x_j}\right)\Gamma\left(\frac{1}{1+x_{j+1}}\right)\right]^{1/2}\\ &\leq \prod_{j=1}^{n}\Gamma\left(\frac{1}{1+M_s(x_j,x_{j+1})}\right) \qquad (x_{n+1}=x_1).\end{aligned}$$

The proof of Theorem 2 is complete. ∎

Remark 3. Recently, numerous papers appeared providing various inequalities for the gamma function and its relatives, like, for example, the incomplete gamma function, the polygamma functions, and the beta function. We refer to Sándor's detailed bibliography [11]. The main properties of the Γ-function are collected, for instance, in [1] and [3]. Interesting historical comments on this subject are given in [5], [7], and [13].

Acknowledgements. I thank the referee for helpful comments.

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Received 23 February 2013;
revised 17 May 2013 (5883)