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NONLINEAR LIE-TYPE DERIVATIONS OF VON NEUMANN ALGEBRAS AND RELATED TOPICS

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Abstract. Motivated by the powerful and elegant works of Miers (1971, 1973, 1978) we mainly study nonlinear Lie-type derivations of von Neumann algebras. Let $\mathcal{A}$ be a von Neumann algebra without abelian central summands of type I_1. It is shown that every nonlinear Lie n-derivation of $\mathcal{A}$ has the standard form, that is, can be expressed as a sum of an additive derivation and a central-valued mapping which annihilates each $(n-1)$th commutator of $\mathcal{A}$. Several potential research topics related to our work are also presented.

1. Introduction. Let $\mathcal{R}$ be a commutative ring with identity, $\mathcal{A}$ be a unital associative algebra over $\mathcal{R}$, and $\mathcal{Z}(\mathcal{A})$ be the center of $\mathcal{A}$. An additive mapping $\varphi\colon \mathcal{A} \to \mathcal{A}$ is called a *derivation* if

$$\varphi(xy) = \varphi(x)y + x\varphi(y) \quad \text{for all } x, y \in \mathcal{A}.$$

Let $[x, y] = xy - yx$ denote the Lie product of elements $x, y \in \mathcal{A}$. An additive mapping $\varphi\colon \mathcal{A} \to \mathcal{A}$ is called a *Lie derivation* if it is a derivation according to the Lie product, i.e.,

$$\varphi([x, y]) = [\varphi(x), y] + [x, \varphi(y)] \quad \text{for all } x, y \in \mathcal{A}.$$

A *Lie triple derivation* is an additive mapping $\varphi\colon \mathcal{A} \to \mathcal{A}$ which satisfies

$$\varphi([[x, y], z]) = [[\varphi(x), y], z] + [[x, \varphi(y)], z] + [[x, y], \varphi(z)]$$

for all $x, y, z \in \mathcal{A}$. A *nonlinear derivation* (resp. *nonlinear Lie derivation*, *nonlinear Lie triple derivation*) has the same definition with the additivity assumption omitted.

Obviously, a derivation is a Lie derivation, and a Lie derivation is a Lie triple derivation. But the converse statements are not true in general. For instance, suppose that $d\colon \mathcal{A} \to \mathcal{A}$ is a derivation and that f is a mapping from $\mathcal{A}$ into its center $\mathcal{Z}(\mathcal{A})$ such that $f([x, y]) = 0$ for all $x, y \in \mathcal{A}$. Then $\varphi = d + f$ is a Lie derivation of $\mathcal{A}$, but not necessarily a derivation. Similarly, if $d\colon \mathcal{A} \to \mathcal{A}$ is a derivation and $f\colon \mathcal{A} \to \mathcal{Z}(\mathcal{A})$ is a mapping such that

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$f([[x,y],z]) = 0$ for all $x,y,z \in \mathcal{A}$, then $\varphi = d + f$ is a Lie triple derivation of $\mathcal{A}$, but not necessarily a Lie derivation.

Mimicking the definitions of Lie derivations and Lie triple derivations, we can extend them in a more general way. Suppose that $n \geq 2$ is a fixed positive integer. Consider the sequence of polynomials

$$\begin{aligned} p_1(x_1) &= x_1, \\ p_2(x_1,x_2) &= [p_1(x_1),x_2] = [x_1,x_2], \\ p_3(x_1,x_2,x_3) &= [p_2(x_1,x_2),x_3] = [[x_1,x_2],x_3], \\ p_4(x_1,x_2,x_3,x_4) &= [p_3(x_1,x_2,x_3),x_4] = [[[x_1,x_2],x_3],x_4], \quad \ldots, \\ p_n(x_1,\ldots,x_n) &= [p_{n-1}(x_1,x_2,\ldots,x_{n-1}),x_n]. \end{aligned}$$

The polynomial $p_n(x_1,\ldots,x_n)$ is said to be an $(n-1)$*th commutator* $(n \geq 2)$. A mapping $\varphi\colon \mathcal{A} \to \mathcal{A}$ is called a *nonlinear Lie n-derivation* if

$$(\spadesuit) \qquad \varphi(p_n(x_1,\ldots,x_n)) = \sum_{i=1}^{n} p_n(x_1,\ldots,x_{i-1},\varphi(x_i),x_{i+1},\ldots,x_n)$$

for all $x_1,\ldots,x_n \in \mathcal{A}$, and a *Lie n-derivation* if φ is moreover additive. It should be stressed that nonlinear Lie n-derivations here are not necessarily additive. Lie n-derivations were introduced by Abdullaev [Ab], where the form of Lie n-derivations of a certain von Neumann algebra (or of its skew-adjoint part) was described. By definition, a Lie derivation is a Lie 2-derivation and a Lie triple derivation is a Lie 3-derivation. Moreover, we have the following fact.

PROPOSITION 1.1. *If $\varphi\colon \mathcal{A} \to \mathcal{A}$ is a Lie n-derivation, then φ is a Lie $(n+k(n-1))$-derivation for every $k \in \mathbb{N}_0$.*

Proof. We will use induction on k. There is nothing to prove for $k=0$. Suppose that $k \geq 1$ and that the statement holds true for $k-1$. Set $m = n+k(n-1)$. Then

$$\begin{aligned} &\varphi(p_m(x_1,\ldots,x_m)) \\ &\ = \varphi([[\ldots[[p_{m-(n-1)}(x_1,\ldots,x_{m-(n-1)}),x_{m-(n-1)+1}],x_{m-(n-1)+2}],\ldots],x_m]). \end{aligned}$$

Since φ is a Lie n-derivation and

$$\begin{aligned} &\varphi(p_{m-(n-1)}(x_1,\ldots,x_{m-(n-1)})) \\ &\qquad = \sum_{i=1}^{m-(n-1)} p_{m-(n-1)}(x_1,\ldots,x_{i-1},\varphi(x_i),x_{i+1},\ldots,x_{m-(n-1)}), \end{aligned}$$

we get

$$\varphi(p_m(x_1,\ldots,x_m)) = \sum_{i=1}^{m} p_m(x_1,\ldots,x_{i-1},\varphi(x_i),x_{i+1},\ldots,x_m)$$

for all $x_1,\ldots,x_m \in \mathcal{A}$. ■

Remark 1.2. Of course, the converse of Proposition 1.1 is not in general true even for $n = 3$. This can be seen by the previous paragraph.

Furthermore, if $d\colon \mathcal{A} \to \mathcal{A}$ is a derivation and $f\colon \mathcal{A} \to \mathcal{Z}(\mathcal{A})$ is such that $f(p_n(x_1, \ldots, x_n)) = 0$ for all $x_1, \ldots, x_n \in \mathcal{A}$ $(n \geq 2)$, then the mapping

$$\varphi = d + f \tag{$\clubsuit$}$$

is a nonlinear Lie n-derivation of $\mathcal{A}$ $(n \geq 2)$. But it is not a nonlinear derivation of $\mathcal{A}$ in the case where f does not annihilate $\mathcal{A}$. We shall say that a nonlinear Lie n-derivation φ of $\mathcal{A}$ is *standard* if it can be expressed in the form ($\clubsuit$) $(n \geq 2)$.

Recently, there has been an increasing interest in the question of when Lie-type derivations (or nonlinear Lie-type derivations) on operator algebras are of the form ($\clubsuit$). Many authors have made essential contributions to related topics. It is Miers who initiated the study of Lie-type derivations of von Neumann algebras in [M2, M3]. He proved that every Lie derivation on a von Neumann algebra $\mathcal{A}$ is of the form ($\clubsuit$). He further observed that if $\mathcal{A}$ is a von Neumann algebra without abelian central summands of type I_1, then every Lie triple derivation also has the form ($\clubsuit$). Mathieu and Villena [MV] showed that every Lie derivation on C^*-algebras has the form ($\clubsuit$). Cheung [Che] gave a sufficient condition for every Lie derivation on a triangular algebra to be standard.

Lu and his students studied Lie-type derivations of various operator algebras in their systematic works [Lu1, Lu2, Lu3, LuL1, LuL2]. The operator algebras involved include the algebras of bounded linear operators, CSL algebras, $\mathcal{J}$-subspace lattice algebras, nest algebras on Hilbert spaces, reflexive algebras. Roughly speaking, every Lie derivation or Lie triple derivation on these operator algebras has the form ($\clubsuit$). Ji and Wang [JLZ, JW] obtained the same result for Lie triple derivations of TUHF algebras and for multiplicative Lie triple derivations of triangular algebras. Sun and Ma [SM] proved the same for Lie triple derivations of the nest algebra $\operatorname{Alg}\mathcal{N}$, where $\mathcal{N}$ is a nontrivial nest on an arbitrary Banach space $\mathcal{X}$. Benkovič and Eremita [BeEr] addressed the question of when all nonlinear Lie n-derivations of a triangular ring $\mathcal{T}$ have the form ($\clubsuit$). Their main result applies to the classical examples of triangular rings: nest algebras and (block) upper triangular matrix rings. Bai and Du [BD] studied nonlinear Lie derivations of von Neumann algebras and got the following result. Let $\mathcal{A}$ be a von Neumann algebra without abelian central summands of type I_1. Let $\varphi : \mathcal{A} \to \mathcal{A}$ be a nonlinear Lie derivation of $\mathcal{A}$. Then φ is of the standard form ($\clubsuit$). That is, $\varphi = d + f$, where d is an additive derivation of $\mathcal{A}$ and f is a mapping of $\mathcal{A}$ into its center $\mathcal{Z}(\mathcal{A})$ which annihilates each commutator of $\mathcal{A}$.

2. Preliminaries. Throughout this paper, we always assume that $\mathcal{A}$ is a von Neumann algebra without abelian central summands of type I_1. We denote the center of $\mathcal{A}$ by $\mathcal{Z}(\mathcal{A})$. If $A = A^* \in \mathcal{A}$, the *central core* of A, denoted by $\underline{A}$, is defined to be $\sup\{S \in \mathcal{Z}(\mathcal{A}) \mid S = S^* \leq A\}$. Clearly, the central core of a projection P is the largest central projection contained in P. For arbitrary $A \in \mathcal{A}$, the *central carrier* of A, denoted by $\overline{A}$, is the intersection of all central projections P such that $PA = A$.

Let P and Q be nonzero orthogonal projections in $\mathcal{A}$ with $P + Q = I$, $\overline{P} = \overline{Q} = I$ and $\underline{P} = \underline{Q} = 0$ (see [M1] and the last paragraph of p. 58 in [M3]). Let $\mathcal{A}_{11} = \{PAP \mid A \in \mathcal{A}\}$, $\mathcal{A}_{12} = \{PAQ \mid A \in \mathcal{A}\}$, $\mathcal{A}_{21} = \{QAP \mid A \in \mathcal{A}\}$, $\mathcal{A}_{22} = \{QAQ \mid A \in \mathcal{A}\}$. Then we usually write $\mathcal{A}$ as $\mathcal{A} = \mathcal{A}_{11} + \mathcal{A}_{12} + \mathcal{A}_{21} + \mathcal{A}_{22}$. Let us list several basic facts which will be used.

Lemma 2.1. *Let $\mathcal{A}$ be a von Neumann algebra without abelian central summands of type I_1.*

(1) ([M3, Lemma 1]) *For any $i, j, k \in \{1, 2\}$, if $A_{ij} \in \mathcal{A}_{ij}$ and $A_{ij}X = 0$ for all $X \in \mathcal{A}_{jk}$, then $A_{ij} = 0$.*
(2) ([M1, Lemma 5]) *If $A \in \mathcal{A}$ commutes with PXQ and QXP for all $X \in \mathcal{A}$, then A commutes with PXP and QXQ for all $X \in \mathcal{A}$, and hence $A \in \mathcal{Z}(\mathcal{A})$.*
(3) ([M1, Lemma 14]) $\mathcal{A}_{ii} \cap \mathcal{Z}(\mathcal{A}) = \{0\}$ $(i = 1, 2)$.
(4) ([BM, Lemma 5]) *If $C \in \mathcal{Z}(\mathcal{A})$ such that $C\mathcal{A} \subseteq \mathcal{Z}(\mathcal{A})$, then $C = 0$.*

Let us recall the classical Kleinecke–Shirokov theorem, which plays an important role in our proofs.

Lemma 2.2 ([K, Sh]). *Let a, b be elements of a Banach algebra $\mathcal{B}$ such that $[[a, b], b] = 0$. Then $[a, b]$ is quasi-nilpotent.*

The main result of this paper is

Theorem 2.3. *Let $\mathcal{A}$ be a von Neumann algebra without abelian central summands of type I_1. Let $\varphi\colon \mathcal{A} \to \mathcal{A}$ be a nonlinear Lie n-derivation of $\mathcal{A}$. Then φ is of the standard form (♣). That is, $\varphi = d + f$, where d is an additive derivation of $\mathcal{A}$ and f is a central-valued mapping which annihilates each $(n-1)$th commutator of $\mathcal{A}$.*

As direct consequences of Theorem 2.3 we have

Corollary 2.4 ([BD, Main Theorem]). *Let $\mathcal{A}$ be a von Neumann algebra without abelian central summands of type I_1. Let $\varphi\colon \mathcal{A} \to \mathcal{A}$ be a nonlinear Lie 2-derivation of $\mathcal{A}$. Then φ is of the standard form (♣). That is, $\varphi = d + f$, where d is an additive derivation of $\mathcal{A}$ and f is a central-valued mapping which annihilates each commutator of $\mathcal{A}$.*

COROLLARY 2.5 ([M3, Theorem 1]). *Let $\mathcal{A}$ be a von Neumann algebra without abelian central summands of type I_1. Let $\varphi\colon \mathcal{A} \to \mathcal{A}$ be a nonlinear Lie 3-derivation of $\mathcal{A}$. Then φ is of the standard form (♣). That is, $\varphi = d+f$, where d is an additive derivation of $\mathcal{A}$ and f is a central-valued mapping which annihilates each second commutator of $\mathcal{A}$.*

REMARK 2.6. Corollary 2.5 means that the proof of [M3] can work as is even when the Lie triple derivation φ is not necessarily additive.

3. Proof of the main result. Before proving our main theorem we need some lemmas. Let $\mathcal{A}$ be a von Neumann algebra without abelian central summands of type I_1 and φ be a nonlinear Lie n-derivation of $\mathcal{A}$. It is clear that every Lie derivation is a Lie n-derivation for $n \geq 3$. Without loss of generality we always assume $n \geq 3$ below.

LEMMA 3.1. *If $[A, B] \in \mathcal{Z}(\mathcal{A})$ for $A, B \in \mathcal{A}$, then $[\varphi(A), B]+[A, \varphi(B)] \in \mathcal{Z}(\mathcal{A})$.*

Proof. Note that

$$\varphi(0) = \varphi(p_n(0, \ldots, 0)) = 0.$$

If $[A, B] \in \mathcal{Z}(\mathcal{A})$ for $A, B \in \mathcal{A}$, then $p_n(A, B, A_1, \ldots, A_{n-2}) = 0$ for all $A_1, \ldots, A_{n-2} \in \mathcal{A}$. Applying φ to this polynomial yields

$$[\cdots[[\varphi(A), B] + [A, \varphi(B)], A_1], \ldots, A_{n-2}] = 0$$

for all $A_1, \ldots, A_{n-2} \in \mathcal{A}$. Therefore $[\cdots[[\varphi(A), B]+[A, \varphi(B)], A_1], \ldots, A_{n-3}] \in \mathcal{Z}(\mathcal{A})$ for all $A_1, \ldots, A_{n-3} \in \mathcal{A}$. On the other hand, by Lemma 2.2 we know that $[\cdots[[\varphi(A), B] + [A, \varphi(B)], A_1], \ldots, A_{n-3}]$ is quasi-nilpotent for all $A_1, \ldots, A_{n-3} \in \mathcal{A}$, so it is zero. A direct recursive procedure shows that $[[\varphi(A), B]+[A, \varphi(B)], A_1] = 0$ for all $A_1 \in \mathcal{A}$. That is, $[\varphi(A), B]+[A, \varphi(B)] \in \mathcal{Z}(\mathcal{A})$. ∎

For later proofs we state an equivalent definition of a nonlinear Lie n-derivation. Define a recursive sequence of polynomials by letting

$$\begin{aligned} q_1(x_1) &= x_1, \\ q_2(x_1, x_2) &= [x_2, q_1(x_1)] = [x_2, x_1], \quad \ldots, \\ q_n(x_1, \ldots, x_n) &= [x_n, q_{n-1}(x_1, \ldots, x_{n-1})]. \end{aligned}$$

Then the definition of a nonlinear Lie n-derivation in this setting becomes

$$\varphi(q_n(x_1, \ldots, x_n)) = \sum_{i=1}^{n} q_n(x_1, \ldots, x_{i-1}, \varphi(x_i), x_{i+1}, \ldots, x_n) \tag{1}$$

for all $x_1, \ldots, x_n \in \mathcal{A}$. Conversely, a mapping of $\mathcal{A}$ satisfying (1) is a nonlinear Lie n-derivation.

LEMMA 3.2. *There exists $A_0 \in \mathcal{A}$ such that $\varphi(P) - [P, A_0] \in \mathcal{Z}(\mathcal{A})$.*

Proof. It is easy to check that for any $A_{12} \in \mathcal{A}_{12}$,

$$A_{12} = q_n(A_{12}, P, \ldots, P) = [P, [P, \ldots, [P, [P, A_{12}]] \cdots]].$$

Applying φ to this polynomial we get

$$\begin{aligned}(2) \qquad \varphi(A_{12}) = {} & [\varphi(P), A_{12}] + [P, [\varphi(P), A_{12}]] + \cdots \\ & + [P, [P, \ldots, [P, [\varphi(P), A_{12}]] \cdots]] + [P, [P, \ldots, [P, [P, \varphi(A_{12})]] \cdots]].\end{aligned}$$

Note the relations

$$P([P, [\varphi(P), A_{12}]])Q = P(P[\varphi(P), A_{12}] - [\varphi(P), A_{12}]P)Q = P[\varphi(P), A_{12}]Q$$

and

$$P[P, \varphi(A_{12})]Q = P(P\varphi(A_{12}) - \varphi(A_{12})P)Q = P\varphi(A_{12})Q.$$

Multiplying by P and Q from the left and the right in (2) respectively, we have

$$P\varphi(A_{12})Q = (n-1)P[\varphi(P), A_{12}]Q + P\varphi(A_{12})Q \quad \text{for all } A_{12} \in \mathcal{A}_{12}.$$

Therefore $P[\varphi(P), A_{12}]Q = 0$ for all $A_{12} \in \mathcal{A}_{12}$. That is,

$$(3) \qquad A_{12}\varphi(P)Q = P\varphi(P)A_{12} \quad \text{for all } A_{12} \in \mathcal{A}_{12}.$$

Similarly, by using $A_{21} = p_n(A_{21}, P, P, \ldots, P)$, we obtain

$$(4) \qquad A_{21}\varphi(P)P = Q\varphi(P)A_{21} \quad \text{for all } A_{21} \in \mathcal{A}_{12}.$$

The relations (3) and (4) imply that

$$[P\varphi(P)P + Q\varphi(P)Q, A_{12}] = [P\varphi(P)P + Q\varphi(P)Q, A_{21}] = 0$$

for all $A_{12} \in \mathcal{A}_{12}$ and $A_{21} \in \mathcal{A}_{21}$. It follows from Lemma 2.1(2) that $P\varphi(P)P + Q\varphi(P)Q \in \mathcal{Z}(\mathcal{A})$. Let us write $A_0 = P\varphi(P)Q - Q\varphi(P)P$. Then $\varphi(P) - [P, A_0] = P\varphi(P)P + Q\varphi(P)Q \in \mathcal{Z}(\mathcal{A})$, which is the desired result. ∎

Let A_0 be as in Lemma 3.2. The mapping defined by $A \mapsto [A, A_0]$ (for $A \in \mathcal{A}$) is an inner derivation of $\mathcal{A}$. Without loss of generality, we may assume that $\varphi(P) \in \mathcal{Z}(\mathcal{A})$.

LEMMA 3.3. *For every $A \in \mathcal{A}_{ij}$ $(1 \le i \ne j \le 2)$, we have $\varphi(A) \in \mathcal{A}_{ij}$.*

Proof. We only consider the case of $i = 1, j = 2$, since the other case $(i = 2, j = 1)$ can be treated similarly. If $A \in \mathcal{A}_{12}$, then $A = [P, [P, \ldots, [P, A] \cdots]]$. Suppose that $\varphi(A) = \sum_{1 \le i \ne j \le 2} A_{ij}$, where $A_{ij} \in \mathcal{A}_{ij}$. Since $\varphi(P) \in \mathcal{Z}(\mathcal{A})$, we get

$$\varphi(A) = [P, [P, \ldots, [P, \varphi(A)] \cdots]] = A_{12} + (-1)^{n-1}A_{21}.$$

Therefore it is sufficient to show $A_{21} = 0$.

For any $B \in \mathcal{A}_{12}$, we have $[B, A] = 0$. Lemma 3.1 yields

$$C = [\varphi(B), A] + [B, \varphi(A)] = [\varphi(B), A] + (-1)^{n-1}[B, A_{21}] \in \mathcal{Z}(\mathcal{A}).$$

Applying φ to the polynomial

$$q_n(A, P, \dots, P, B) = [B, [P, [P, \dots, [P, A] \cdots]]] = 0,$$

we obtain

$$\begin{aligned}[\varphi(B), A] + [B, A_{12} + (-1)^{n-2}A_{21}] &= [\varphi(B), A] + (-1)^{n-2}[B, A_{21}] \\ &= C - (-1)^{n-1}[B, A_{21}] + (-1)^{n-2}[B, A_{21}] \\ &= C + (-1)^{n-2}2[B, A_{21}] = 0\end{aligned}$$

for all $B \in \mathcal{A}_{12}$. Thus $[A_{21}, B] = (-1)^{n-2}\frac{1}{2}C \in \mathcal{Z}(\mathcal{A})$ for all $B \in \mathcal{A}_{12}$. By Lemma 2.2 we know that $[A_{21}, B]$ is quasi-nilpotent for all $B \in \mathcal{A}_{12}$. Hence $[A_{21}, B] = A_{21}B - BA_{21} = 0$. This implies that $A_{21}B = 0$ for all $B \in \mathcal{A}_{12}$. So $A_{21} = 0$ by Lemma 2.1(1). ∎

LEMMA 3.4. *$\varphi(Q) \in \mathcal{Z}(\mathcal{A})$.*

Proof. For any $A_{12} \in \mathcal{A}_{12}$, we have

$$A_{12} = \begin{cases} p_n(A_{12}, P, \dots, P, Q) = [[\cdots[[A_{12}, P], P], \dots, P], Q] & \text{if } n \text{ is even,} \\ p_n(P, A_{12}, P, \dots, P, Q) = [[\cdots[[P, A_{12}], P], \dots, P], Q] & \text{if } n \text{ is odd.} \end{cases}$$

In both cases we get

$$\varphi(A_{12}) = [[\cdots[\varphi(A_{12}), P], \dots, P], Q] + [A_{12}, \varphi(Q)] = \varphi(A_{12}) + [A_{12}, \varphi(Q)],$$

since $\varphi(A_{12}) \in \mathcal{A}_{12}$. Thus $[A_{12}, \varphi(Q)] = 0$ for all $A_{12} \in \mathcal{A}_{12}$. Similarly, $[A_{21}, \varphi(Q)] = 0$ for all $A_{21} \in \mathcal{A}_{21}$. Then Lemma 2.1(2) yields $\varphi(Q) \in \mathcal{Z}(\mathcal{A})$. ∎

LEMMA 3.5. *If $A \in \mathcal{A}_{ii}$, then $\varphi(A) \in \mathcal{A}_{ii} + \mathcal{Z}(\mathcal{A})$ ($i = 1, 2$).*

Proof. For each $A \in \mathcal{A}_{11}$, we assume that $\varphi(A) = \sum_{1\le i,j\le 2} A_{ij}$, where $A_{ij} \in \mathcal{A}_{ij}$. Since $p_n(A, P, \dots, P) = [\cdots[[A, P], P], \dots, P] = 0$, we obtain

$$[\cdots[[\varphi(A), P], P], \dots, P] = A_{21} + (-1)^{n-1}A_{12} = 0.$$

This shows that $A_{12} = A_{21} = 0$. Hence $\varphi(A) = A_{11} + A_{22}$. Similarly, $\varphi(B) = B_{11} + B_{22}$ for all $B \in \mathcal{A}_{22}$, where $B_{ii} \in \mathcal{A}_{ii}$ ($i = 1, 2$). Since $[A, B] = 0$, we have

$$C = [\varphi(A), B] + [A, \varphi(B)] = [A_{22}, B] + [A, B_{11}] \in \mathcal{Z}(\mathcal{A})$$

by Lemma 3.1. This implies that $[A_{22}, B] = QC \in Q\mathcal{Z}(\mathcal{A}) = \mathcal{Z}(\mathcal{A}_{22})$. Thus $[A_{22}, B]$ is central quasi-nilpotent in $\mathcal{A}_{22}$ and so is zero. Therefore $A_{22} \in \mathcal{Z}(\mathcal{A}_{22})$. There exists $D \in \mathcal{Z}(\mathcal{A})$ such that $A_{22} = QD = (I - P)D = -PD + D \in \mathcal{A}_{11} + \mathcal{Z}(\mathcal{A})$. It follows that $\varphi(A) = A_{11} + A_{22} = A_{11} - PD + D \in \mathcal{A}_{11} + \mathcal{Z}(\mathcal{A})$ for all $A \in \mathcal{A}_{11}$. Likewise, $\varphi(B) \in \mathcal{A}_{22} + \mathcal{Z}(\mathcal{A})$ for all $B \in \mathcal{A}_{22}$. ∎

LEMMA 3.6. *For every $A \in \mathcal{A}$, we have*

$$(1)\quad \begin{cases} \varphi(PAQ - QAP) = P\varphi(A)Q - Q\varphi(A)P & \textit{if } n \textit{ is even,} \\ \varphi(PAQ + QAP) = P\varphi(A)Q + Q\varphi(A)P & \textit{if } n \textit{ is odd,} \end{cases}$$

(2) $\begin{cases} P\varphi(A)Q = 0 & \textit{if } PAQ = 0, \\ Q\varphi(A)P = 0 & \textit{if } QAP = 0. \end{cases}$

Proof. (1) It is easy to verify that

$$q_n(A, P, \ldots, P) = [P, [P, \ldots, [P, A] \cdots]] = PAQ + (-1)^{n-1}QAP.$$

Applying φ to the above polynomial yields the desired results.

(2) Assume that $PAQ = 0$. When n is odd, then

$$\varphi(QAP) = \varphi(PAQ + QAP) = P\varphi(A)Q + Q\varphi(A)P \in \mathcal{A}_{21}$$

by Lemma 3.3. Therefore $P\varphi(A)Q = 0$. When n is even, the same result holds. Similarly, the other case can be proved. ∎

LEMMA 3.7. *For any $A \in \mathcal{A}$, $A_{12}, B_{12} \in \mathcal{A}_{12}$, we have*

(1) $[\varphi(A + A_{12}) - \varphi(A), B_{12}] = 0$,
(2) $\varphi(A+A_{12})-\varphi(A) = P(\varphi(A+A_{12})-\varphi(A))Q+C$, *where* $C \in \mathcal{Z}(\mathcal{A})$.

Proof. (1) Since $[A + A_{12}, B_{12}] = [A, B_{12}]$, we have

$$[B, [A + A_{12}, [P, \ldots, [P, B_{12}] \cdots]]] = [B, [A, [P, \ldots, [P, B_{12}] \cdots]]]$$

for all $B \in \mathcal{A}$. Note that $\varphi(B_{12}) \in \mathcal{A}_{12}$. Applying φ to the above polynomial gives

$$\begin{aligned} &[B, [\varphi(A + A_{12}), B_{12}]] + [B, [A + A_{12}, \varphi(B_{12})]] \\ &\qquad\qquad = [B, [\varphi(A), B_{12}]] + [B, [A, \varphi(B_{12})]]. \end{aligned}$$

Thus $[B, [\varphi(A + A_{12}) - \varphi(A), B_{12}]] = 0$ for all $B \in \mathcal{A}$. This implies that $[\varphi(A + A_{12}) - \varphi(A), B_{12}]$ is central quasi-nilpotent in $\mathcal{A}$ and hence is zero.

(2) This follows from (1) and [BD, Lemma 2]. ∎

LEMMA 3.8. *For $1 \leq i \neq j \leq 2$, we have*

(1) $\varphi(A_{ii} + A_{ij}) - \varphi(A_{ii}) - \varphi(A_{ij}) \in \mathcal{Z}(\mathcal{A})$,
(2) $\varphi(A_{ii} + A_{ji}) - \varphi(A_{ii}) - \varphi(A_{ji}) \in \mathcal{Z}(\mathcal{A})$.

Proof. We only prove (1) for of $i = 1$, $j = 2$. The other cases can be proved similarly. In view of Lemma 3.7 we know that

$$\varphi(A_{11} + A_{12}) - \varphi(A_{11}) = P(\varphi(A_{11} + A_{12}) - \varphi(A_{11}))Q + C$$

for some $C \in \mathcal{Z}(\mathcal{A})$. It is sufficient to show that

$$\varphi(A_{12}) = P(\varphi(A_{11} + A_{12}) - \varphi(A_{11}))Q.$$

By Lemma 3.6 we arrive at

$$\begin{aligned} \varphi(A_{12}) &= \varphi(P(A_{11} + A_{12})Q \pm Q(A_{11} + A_{12})P) \\ &= P(\varphi(A_{11} + A_{12}))Q \pm Q(\varphi(A_{11} + A_{12}))P, \end{aligned}$$

where the signs $\pm$ depend on the parity of n. Taking into account that $\varphi(A_{12}) \in \mathcal{A}_{12}$ and $\varphi(A_{11}) \in \mathcal{A}_{11} + \mathcal{Z}(\mathcal{A})$ we immediately obtain

$$\varphi(A_{12}) = P(\varphi(A_{11} + A_{12}))Q = P(\varphi(A_{11} + A_{12}) - \varphi(A_{11}))Q. \blacksquare$$

LEMMA 3.9. *φ is additive on $\mathcal{A}_{12}$ and $\mathcal{A}_{21}$.*

Proof. For all $A_{12}, B_{12} \in \mathcal{A}_{12}$, we have

$$A_{12} + B_{12} = [P + A_{12}, Q + B_{12}] = [\cdots[[[P + A_{12}, Q + B_{12}], Q], Q], \dots, Q].$$

It follows from Lemmas 3.4 and 3.8 that

$$\begin{aligned}\varphi(A_{12} + B_{12}) &= [\cdots[[[\varphi(P + A_{12}), Q + B_{12}], Q], Q], \dots, Q] \\ &\quad + [\cdots[[[P + A_{12}, \varphi(Q + B_{12})], Q], Q], \dots, Q] \\ &= [\cdots[[[\varphi(P) + \varphi(A_{12}), Q + B_{12}], Q], Q], \dots, Q] \\ &\quad + [\cdots[[[P + A_{12}, \varphi(Q) + \varphi(B_{12})], Q], Q], \dots, Q] \\ &= \varphi(A_{12}) + \varphi(B_{12}).\end{aligned}$$

Similarly, φ is also additive on $\mathcal{A}_{21}$. $\blacksquare$

LEMMA 3.10. *For all $A_{11} \in \mathcal{A}_{11}, A_{22} \in \mathcal{A}_{22}$, we have $\varphi(A_{11} + A_{22}) - \varphi(A_{11}) - \varphi(A_{22}) \in \mathcal{Z}(\mathcal{A})$.*

Proof. Clearly $[A_{11} + A_{22}, A_{12}] = A_{11}A_{12} - A_{12}A_{22}$ for all $A_{12} \in \mathcal{A}_{12}$. By Lemma 3.9 we get

$$\begin{aligned}(5)\qquad &\varphi(A_{11}A_{12} - A_{12}A_{22}) = \varphi(A_{11}A_{12}) + \varphi(-A_{12}A_{22}) \\ &= \varphi([\cdots[[[A_{11}, A_{12}], Q], Q], \dots, Q]) + \varphi([\cdots[[[A_{22}, A_{12}], Q], Q], \dots, Q]) \\ &= [\cdots[[[\varphi(A_{11}), A_{12}], Q], Q], \dots, Q] + [\cdots[[[A_{11}, \varphi(A_{12})], Q], Q], \dots, Q] \\ &\quad + [\cdots[[[\varphi(A_{22}), A_{12}], Q], Q], \dots, Q] + [\cdots[[[A_{22}, \varphi(A_{12})], Q], Q], \dots, Q] \\ &= [\varphi(A_{11}) + \varphi(A_{22}), A_{12}] + [A_{11} + A_{22}, \varphi(A_{12})].\end{aligned}$$

On the other hand,

$$\begin{aligned}&[A, [A_{11} + A_{22}, [P, \dots, [P, A_{12}] \cdots]]] \\ &\qquad = [A, [P, [P, \dots, [P, A_{11}A_{12} - A_{12}A_{22}] \cdots]]]\end{aligned}$$

for all $A \in \mathcal{A}$. Applying φ to the above polynomial and using (5) we obtain

$$\begin{aligned}&[A, [\varphi(A_{11} + A_{22}), A_{12}]] + [A, [A_{11} + A_{22}, \varphi(A_{12})]] \\ &\qquad = [A, [P, [P, \dots, [P, \varphi(A_{11}A_{12} - A_{12}A_{22})] \cdots]]] \\ &\qquad = [A, \varphi(A_{11}A_{12} - A_{12}A_{22})] \\ &\qquad = [A, [\varphi(A_{11}) + \varphi(A_{22}), A_{12}]] + [A, [A_{11} + A_{22}, \varphi(A_{12})]].\end{aligned}$$

Hence $[A, [\varphi(A_{11} + A_{22}) - \varphi(A_{11}) - \varphi(A_{22}), A_{12}]] = 0$ for all $A \in \mathcal{A}$. This shows that $[\varphi(A_{11} + A_{22}) - \varphi(A_{11}) - \varphi(A_{22}), A_{12}]$ is central quasi-nilpotent and hence is zero. It follows from [BD, Lemma 2] that $\varphi(A_{11}+A_{22})-\varphi(A_{11})-\varphi(A_{22})\in\mathcal{A}_{12}+\mathcal{Z}(\mathcal{A})$.

Since $P(A_{11}+A_{22})Q=0$, we have $P(\varphi(A_{11}+A_{22}))Q=0$ by Lemma 3.6(2). Applying Lemma 3.5 gives

$$\varphi(A_{11}+A_{22})-\varphi(A_{11})-\varphi(A_{22})\in\mathcal{Z}(\mathcal{A}). \blacksquare$$

LEMMA 3.11. *For all $A_{ii},B_{ii}\in\mathcal{A}_{ii}$, we have $\varphi(A_{ii}+B_{ii})-\varphi(A_{ii})-\varphi(B_{ii})\in\mathcal{Z}(\mathcal{A})$ $(i=1,2)$.*

Proof. We only prove the case $i=1$, as the case $i=2$ can be proved similarly. For any $A_{12}\in\mathcal{A}_{12}$, $[A_{11}+B_{11},A_{12}]=A_{11}A_{12}+B_{11}A_{12}$. Applying Lemma 3.9 yields

$$\begin{aligned}(6)\qquad &\varphi(A_{11}A_{12}+B_{11}A_{12})=\varphi(A_{11}A_{12})+\varphi(B_{11}A_{12})\\ &=\varphi([\cdots[[[A_{11},A_{12}],Q],Q],\dots,Q])+\varphi([\cdots[[[B_{11},A_{12}],Q],Q],\dots,Q])\\ &=[\varphi(A_{11}),A_{12}]+[A_{11},\varphi(A_{12})]+[\varphi(B_{11}),A_{12}]+[B_{11},\varphi(A_{12})]\\ &=[\varphi(A_{11})+\varphi(B_{11}),A_{12}]+[A_{11}+B_{11},\varphi(A_{12})].\end{aligned}$$

On the other hand,

$$\begin{aligned}&[A,[A_{11}+B_{11},[P,\dots,[P,A_{12}]\cdots]]]\\ &\qquad=[A,[P,[P,\dots,[P,A_{11}A_{12}+B_{11}A_{12}]\cdots]]]\end{aligned}$$

for all $A\in\mathcal{A}$. Applying φ to the above polynomial and using (6) we arrive at

$$\begin{aligned}&[A,[\varphi(A_{11}+B_{11}),A_{12}]]+[A,[A_{11}+B_{11},\varphi(A_{12})]]\\ &\qquad=[A,[P,[P,\dots,[P,\varphi(A_{11}A_{12}+B_{11}A_{12})]\cdots]]]\\ &\qquad=[A,\varphi(A_{11}A_{12}+B_{11}A_{12})]\\ &\qquad=[A,[\varphi(A_{11})+\varphi(B_{11}),A_{12}]]+[A,[A_{11}+B_{11},\varphi(A_{12})]].\end{aligned}$$

Therefore $[A,[\varphi(A_{11}+B_{11})-\varphi(A_{11})-\varphi(B_{11}),A_{12}]]=0$ for all $A\in\mathcal{A}$. It follows that $[\varphi(A_{11}+B_{11})-\varphi(A_{11})-\varphi(B_{11}),A_{12}]$ is central quasi-nilpotent and so is zero. By [BD, Lemma 2] we again get $\varphi(A_{11}+B_{11})-\varphi(A_{11})-\varphi(B_{11})\in\mathcal{A}_{12}+\mathcal{Z}(\mathcal{A})$.

Note that $P(A_{11}+B_{11})Q=0$. Thus $P(\varphi(A_{11}+B_{11}))Q=0$ by Lemma 3.6(2). Taking Lemma 3.5 into account we conclude that

$$\varphi(A_{11}+B_{11})-\varphi(A_{11})-\varphi(B_{11})\in\mathcal{Z}(\mathcal{A}). \blacksquare$$

LEMMA 3.12. *For all $A_{ii}\in\mathcal{A}_{ii}$, $A_{ij}\in\mathcal{A}_{ij}$, $A_{jj}\in\mathcal{A}_{jj}$ $(1\le i\ne j\le 2)$, we have $\varphi(A_{ii}+A_{jj}+A_{ij})-\varphi(A_{ii})-\varphi(A_{jj})-\varphi(A_{ij})\in\mathcal{Z}(\mathcal{A})$.*

Proof. It suffices to consider the case $i=1$, $j=2$. The other case can be proved similarly. Combining Lemma 3.10 with Lemma 3.7 leads to

$$\begin{aligned}\varphi(A_{11}+A_{22}+A_{12})-\varphi(A_{11})-\varphi(A_{22})&\\ &=\varphi(A_{11}+A_{22}+A_{12})-\varphi(A_{11}+A_{22})+C_0\\ &=P(\varphi(A_{11}+A_{22}+A_{12})-\varphi(A_{11}+A_{22}))Q+C\\ &=P\varphi(A_{11}+A_{22}+A_{12})Q+C\in\mathcal{A}_{12}+\mathcal{Z}(\mathcal{A}).\end{aligned}$$

for some $C_0, C\in\mathcal{Z}(\mathcal{A})$. By Lemma 3.6 it follows that

$$\begin{aligned}\varphi(A_{12})&=\varphi(P(A_{11}+A_{22}+A_{12})Q\pm Q(A_{11}+A_{22}+A_{12})P)\\ &=P\varphi(A_{11}+A_{22}+A_{12})Q\pm Q\varphi(A_{11}+A_{22}+A_{12})P\\ &=P\varphi(A_{11}+A_{22}+A_{12})Q,\end{aligned}$$

where the signs $\pm$ are related to the parity of n. Therefore

$$\varphi(A_{11}+A_{22}+A_{12})-\varphi(A_{11})-\varphi(A_{22})-\varphi(A_{12})=C\in\mathcal{Z}(\mathcal{A}).\ \blacksquare$$

LEMMA 3.13. *φ is almost additive on $\mathcal{A}$. That is, $\varphi(A+B)-\varphi(A)-\varphi(B)\in\mathcal{Z}(\mathcal{A})$ for all $A, B\in\mathcal{A}$.*

Proof. By Lemmas 3.9 and 3.11, we only need to prove

$$\varphi(A_{11}+A_{12}+A_{21}+A_{22})-\varphi(A_{11})-\varphi(A_{12})-\varphi(A_{21})-\varphi(A_{22})\in\mathcal{Z}(\mathcal{A})$$

for all $A_{ij}\in\mathcal{A}_{ij}$ $(1\le i,j\le 2)$.

Applying Lemmas 3.12 and 3.7 we have

$$\begin{aligned}&[\varphi(A_{11}+A_{12}+A_{21}+A_{22})-\varphi(A_{11})-\varphi(A_{12})-\varphi(A_{21})-\varphi(A_{22}),B_{12}]\\ &\quad=[\varphi(A_{11}+A_{12}+A_{21}+A_{22})-\varphi(A_{11})-\varphi(A_{21})-\varphi(A_{22}),B_{12}]\\ &\quad=[\varphi(A_{11}+A_{12}+A_{21}+A_{22})-\varphi(A_{11}+A_{21}+A_{22}),B_{12}]=0\end{aligned}$$

for all $B_{12}\in\mathcal{A}_{12}$. Similarly,

$$[\varphi(A_{11}+A_{12}+A_{21}+A_{22})-\varphi(A_{11})-\varphi(A_{12})-\varphi(A_{21})-\varphi(A_{22}),B_{21}]=0$$

for all $B_{21}\in\mathcal{A}_{21}$. In view of Lemma 2.1(2) we deduce that

$$\varphi(A_{11}+A_{12}+A_{21}+A_{22})-\varphi(A_{11})-\varphi(A_{12})-\varphi(A_{21})-\varphi(A_{22})\in\mathcal{Z}(\mathcal{A})$$

for all $A_{ij}\in\mathcal{A}_{ij}$ $(1\le i,j\le 2)$. $\blacksquare$

Now we are in a position to prove our main result.

Proof of Theorem 2.3. By Lemmas 3.3 and 3.5, if $A_{ij}\in\mathcal{A}_{ij}$ with $i\neq j$, then $\varphi(A_{ij})=B_{ij}\in\mathcal{A}_{ij}$; if $A_{ii}\in\mathcal{A}_{ii}$, then $\varphi(A_{ii})=B_{ii}+C$, where $B_{ii}\in\mathcal{A}_{ii}$ and $C\in\mathcal{Z}(\mathcal{A})$ are uniquely determined by Lemma 2.1(3). Therefore it is reasonable to define $d\colon\mathcal{A}\to\mathcal{A}$ by $d(A_{11}+A_{12}+A_{21}+A_{22})=B_{11}+B_{12}+B_{21}+B_{22}$. It is clear that $\varphi(A)-d(A)\in\mathcal{Z}(\mathcal{A})$ for all $A\in\mathcal{A}$. So we can define $f\colon\mathcal{A}\to\mathcal{Z}(\mathcal{A})$ by $f(A)=\varphi(A)-d(A)$ for $A\in\mathcal{A}$).

STEP 1. *d is additive.*

Lemma 3.9 shows that d is additive on $\mathcal{A}_{12}$ and $\mathcal{A}_{21}$. We claim that it is also additive on $\mathcal{A}_{ii}$ $(i = 1, 2)$. For any $A_{ii}, B_{ii} \in \mathcal{A}_{ii}$, we have

$$\begin{aligned} d(A_{ii} + B_{ii}) - d(A_{ii}) - d(B_{ii}) = \varphi(A_{ii} + B_{ii}) - f(A_{ii} + B_{ii}) - \varphi(A_{ii}) \\ + f(A_{ii}) - \varphi(B_{ii}) + f(B_{ii}) \in \mathcal{A}_{ii} \cap \mathcal{Z}(\mathcal{A}). \end{aligned}$$

Lemma 2.1(3), yields $d(A_{ii} + B_{ii}) - d(A_{ii}) - d(B_{ii}) = 0$ for all $A_{ii}, B_{ii} \in \mathcal{A}_{ii}$.

Assume that $A = \sum_{1 \le i,j \le 2} A_{ij}$, $B = \sum_{1 \le i,j \le 2} B_{ij}$, where $A_{ij}, A_{ij} \in \mathcal{A}_{ij}$. According to the definition of d, we get

$$\begin{aligned} d(A + B) &= d\Big(\sum_{1 \le i,j \le 2} (A_{ij} + B_{ij})\Big) = \sum_{1 \le i,j \le 2} d(A_{ij} + B_{ij}) \\ &= \sum_{1 \le i,j \le 2} (d(A_{ij}) + d(B_{ij})) = d(A) + d(B). \end{aligned}$$

Step 2. *d is a derivation of $\mathcal{A}$.*

Choose $A_{ij}, B_{ij}, C_{ij} \in \mathcal{A}_{ij}$ with $1 \le i, j \le 2$. If $i \ne j$, we have

$$\begin{aligned} d(A_{ii}B_{ij}) &= \varphi(A_{ii}B_{ij}) = \varphi([\cdots[[A_{ii}, B_{ij}], P_j], \dots, P_j]) \\ &= [\cdots[[\varphi(A_{ii}), B_{ij}] + [A_{ii}, \varphi(B_{ij})], P_j], \dots, P_j] \\ &= [\cdots[d(A_{ii})B_{ij} + A_{ii}d(B_{ij}), P_j], \dots, P_j] \\ &= d(A_{ii})B_{ij} + A_{ii}d(B_{ij}), \end{aligned} \tag{7}$$

where $P_j = P$ if $j = 1$, $P_j = Q$ if $j = 2$. Similarly,

$$d(A_{ij}B_{jj}) = d(A_{ij})B_{jj} + A_{ij}d(B_{jj}). \tag{8}$$

Using (7) it is easy to check that

$$d(A_{ii}B_{ii}C_{ij}) = d(A_{ii}B_{ii})C_{ij} + A_{ii}B_{ii}d(C_{ij}).$$

On the other hand,

$$\begin{aligned} d(A_{ii}B_{ii}C_{ij}) &= d(A_{ii})B_{ii}C_{ij} + A_{ii}d(B_{ii}C_{ij}) \\ &= d(A_{ii})B_{ii}C_{ij} + A_{ii}d(B_{ii})C_{ij} + A_{ii}B_{ii}d(C_{ij}). \end{aligned}$$

Comparing with the above two expressions, we obtain

$$(d(A_{ii}B_{ii}) - d(A_{ii})B_{ii} - A_{ii}d(B_{ii}))C_{ij} = 0.$$

By Lemma 2.1(1) it follows that

$$d(A_{ii}B_{ii}) = d(A_{ii})B_{ii} + A_{ii}d(B_{ii}). \tag{9}$$

Note that d is additive. Thus there exists $C \in \mathcal{Z}(\mathcal{A})$ such that

$$\begin{aligned} &d(A_{12}B_{21}) - d(B_{21}A_{12}) = d(A_{12}B_{21} - B_{21}A_{12}) = d([A_{12}, B_{21}]) \\ &\quad = \varphi([A_{12}, B_{21}]) + C = \varphi([[\cdots[A_{12}, Q], \dots, Q], B_{21}]) + C \\ &\quad = [[\cdots[\varphi(A_{12}), Q], \dots, Q], B_{21}] + [[\cdots[A_{12}, Q], \dots, Q], \varphi(B_{21})] + C \\ &\quad = [d(A_{12}), B_{21}] + [A_{12}, d(B_{21})] + C \\ &\quad = d(A_{12})B_{21} + A_{12}d(B_{21}) - B_{21}d(A_{12}) - d(B_{21})A_{12} + C. \end{aligned}$$

Hence $d(A_{12}B_{21}) - d(A_{12})B_{21} - A_{12}d(B_{21}) \in P\mathcal{Z}(\mathcal{A})$ and $d(B_{21}A_{12}) - B_{21}d(A_{12}) - d(B_{21})A_{12} \in Q\mathcal{Z}(\mathcal{A})$.

Let $A = \sum_{1\le i,j\le 2} A_{ij}$, $B = \sum_{1\le i,j\le 2} B_{ij}$, where $A_{ij}, B_{ij} \in \mathcal{A}_{ij}$. A direct computation shows that

$$\begin{aligned} d(AB) &= d(A_{11}B_{11}) + d(A_{11}B_{12}) + d(A_{12}B_{21}) + d(A_{12}B_{22}) \\ &\quad + d(A_{21}B_{11}) + d(A_{21}B_{12}) + d(A_{22}B_{21}) + d(A_{22}B_{22}) \\ &= d(A)B + Ad(B) + PC_1 + QC_2 \end{aligned}$$

for some $C_1, C_2 \in \mathcal{Z}(\mathcal{A})$. Define $\theta\colon \mathcal{A}\times\mathcal{A} \to P\mathcal{Z}(\mathcal{A}) \oplus Q\mathcal{Z}(\mathcal{A})$ by

$$\theta(A,B) = d(AB) - d(A)B - Ad(B).$$

It is not difficult to see that θ is bi-additive. Hence, for any $A \in \mathcal{A}_{11} \oplus \mathcal{A}_{22}$ and $B \in \mathcal{A}$, we obtain $\theta(A,B) = \theta(B,A) = 0$ by (7)–(9).

It suffices to show that $\theta(A,B) \equiv 0$ for all $A,B,C \in \mathcal{A}$. For all $A,B,C \in \mathcal{A}$, we have

$$\begin{aligned} d(ABC) &= d((AB)C) = d(AB)C + ABd(C) + \theta(AB,C) \\ &= d(A)BC + Ad(B)C + ABd(C) + \theta(A,B)C + \theta(AB,C). \end{aligned}$$

On the other hand,

$$\begin{aligned} d(ABC) &= d(A(BC)) = d(A)BC + Ad(BC) + \theta(A,BC) \\ &= d(A)BC + Ad(B)C + ABd(C) + A\theta(B,C) + \theta(A,BC). \end{aligned}$$

Therefore

$$\theta(A,B)C + \theta(AB,C) = A\theta(B,C) + \theta(A,BC).$$

(Hence θ is a Hochschild 2-cocycle.) For all $A_{ii} \in \mathcal{A}_{ii}$ $(i = 1,2)$, we have $\theta(A,B)A_{ii} = \theta(A,BA_{ii})$. Assume that $\theta(A,B) = PC_1 + QC_2$ for some $C_1, C_2 \in \mathcal{Z}_{\mathcal{A}}$. Then $\theta(A,B)A_{11} = PC_1A_{11} = \theta(A,BA_{11}) \in P\mathcal{Z}(\mathcal{A}) \oplus Q\mathcal{Z}(\mathcal{A})$. Thus $PC_1A_{11} \in P\mathcal{Z}(\mathcal{A}) = \mathcal{Z}(\mathcal{A}_{11})$. Since $\mathcal{A}$ has no abelian central summands, neither has $\mathcal{A}_{11}$. Applying Lemma 2.1(4) gives $PC_1 = 0$. Similarly, we can prove that $QC_2 = 0$. So $\theta(A,B) = 0$.

Step 3. *$f([\cdots[[A_1,A_2],A_3],\ldots,A_n]) = 0$ for all $A_i \in \mathcal{A}$.*

In fact,

$$\begin{aligned} &f([\cdots[[A_1,A_2],A_3],\ldots,A_n]) \\ &\quad = \varphi([\cdots[[A_1,A_2],A_3],\ldots,A_n]) - d([\cdots[[A_1,A_2],A_3],\ldots,A_n]) \\ &\quad = [\cdots[[\varphi(A_1),A_2],A_3],\ldots,A_n] + [\cdots[[A_1,\varphi(A_2)],A_3],\ldots,A_n] + \cdots \\ &\qquad + [\cdots[[A_1,A_2],A_3],\ldots,\varphi(A_n)] - d([\cdots[[A_1,A_2],A_3],\ldots,A_n]) \\ &\quad = [\cdots[[d(A_1),A_2],A_3],\ldots,A_n] + [\cdots[[A_1,d(A_2)],A_3],\ldots,A_n] + \cdots \\ &\qquad + [\cdots[[A_1,A_2],A_3],\ldots,d(A_n)] - d([\cdots[[A_1,A_2],A_3],\ldots,A_n]) \\ &\quad = 0. \ \blacksquare \end{aligned}$$

Sakai's well-known theorem [Sa] states that every linear derivation of a von Neumann algebra $\mathcal{A}$ is inner. The anonymous referee advised us to consider whether the additive derivation d of $\mathcal{A}$ in the decomposition $\varphi = d + f$ of Theorem 2.3 is inner. We do not know how to answer this question in our nonlinear context. However, the referee helped us to prove the following result.

Corollary 3.14. *The additive derivation d of $\mathcal{A}$ in the decomposition $\varphi = d + f$ of Theorem 2.3 is inner whenever φ is continuous in the weak operator topology on $\mathcal{A}$.*

Proof. Assume that φ is continuous in the weak operator topology on $\mathcal{A}$. Note that $A \in \mathcal{A} \mapsto A_{11} = PAP, A_{12} = PAQ, A_{21} = QAP, A_{22} = QAQ$ are continuous in the weak operator topology. Hence $A \in \mathcal{A} \mapsto B_{12} = \varphi(A_{12}), B_{21} = \varphi(A_{21})$ are continuous. Write $\varphi(A_{11}) = B_{11}(A) + C(A)$, where $B_{11}(A) \in \mathcal{A}_{11}$ and $C(A) \in \mathcal{Z}(A)$. Since $C(A)Q = \varphi(A_{11})Q$ and $A \mapsto \varphi(A_{11})Q$ is continuous in the weak operator topology, it follows that $\{C(A)Q : A \in \mathcal{A}, \|A\| \leq R\}$ is compact in the weak operator topology and hence norm-bounded for each $R > 0$. Since $C \in \mathcal{Z}(A) \mapsto CQ \in \mathcal{Z}(A)Q$ is an isomorphism due to $\overline{Q} = I$, $\{C(A) : A \in \mathcal{A}, \|A\| \leq R\}$ is norm-bounded.

Now, let $\{A_\alpha\}$ be a net in $\mathcal{A}$ with $\|A_\alpha\| \leq R$ and assume that $A_\alpha \to A \in \mathcal{A}$ in the weak operator topology. By the boundedness of $\{C(A_\alpha)\}$ one can choose a subnet $\{A_{\alpha(\beta)}\}$ such that $C(A_{\alpha(\beta)}) \to C$ for some $C \in \mathcal{Z}(A)$. Since $\varphi((A_{\alpha(\beta)})_{11}) = B_{11}(A_{\alpha(\beta)}) + C(A_{\alpha(\beta)})$ and $\varphi((A_{\alpha(\beta)})_{11}) \to \varphi(A_{11})$, one has $B_{11}(A_{\alpha(\beta)}) \to B_{11}$ for some $B_{11} \in \mathcal{A}_{11}$ so that $\varphi(A_{11}) = B_{11} + C$. By the uniqueness property, it follows that $B_{11} = B_{11}(A)$ and $C = C(A)$. This implies that $C(A)$ is a unique limit point of $\{C(A_\alpha)\}$ so that $C(A_\alpha) \to C(A)$ and hence $B_{11}(A_\alpha) \to B_{11}(A)$ in the weak operator topology. Therefore, $A \in \mathcal{A} \mapsto B_{11} = B_{11}(A)$ is continuous in the weak operator topology on each bounded subset of $\mathcal{A}$. Since the same holds for $A \mapsto B_{22}$, we see that $A \in \mathcal{A} \mapsto d(A) = B_{11} + B_{12} + B_{21} + B_{22}$ is continuous in the weak operator topology for each bounded subset of $\mathcal{A}$.

Note that the derivation d obtained in Theorem 2.3 is additive. Thus $d(A) + d(-A) = d(0) = 0$ for all $A \in \mathcal{A}$ and hence $d(-A) = -d(A)$ for all $A \in \mathcal{A}$. Whenever d is continuous on $\mathcal{A}$ in some linear topology, it is automatically linear on $\mathcal{A}$. It follows from Sakai's theorem that d is inner. ∎

4. Topics for further research. Although the main aim of this paper is to study nonlinear Lie-type derivations of a class of von Neumann algebras, the structure of nonlinear Lie-type derivations on noncommutative algebras and operator algebras is also of great interest. In the light of the motivation and contents of this article, we will propose several topics for future research in this field.

Recall that all von Neumann algebras are semiprime and each von Neumann algebra without central summands of type I_1 is a generalized matrix algebra (i.e. a unital algebra $\mathcal{A}$ having a nontrivial idempotent e such that $e\mathcal{A}(1-e)$ or $(1-e)\mathcal{A}e$ is nonzero). Nonlinear Lie n-derivations of triangular algebras and those of full matrix algebras were already studied in [BeEr, JLZ, XW4]. Some similar problems were recently considered in the context of generalized matrix algebras [LW, XW1, XW2, XW3]. One would expect that the next step is to investigate nonlinear Lie-type mappings of generalized matrix algebras. It is worth pointing out that the notion of generalized matrix algebras efficiently combines triangular algebras and full matrix algebras. The eventual goal of our systematic work is to deal with all questions related to additive (or multiplicative) mappings of triangular algebras and full matrix algebras in a unified framework, which is the desirable generalized matrix algebras framework. Let us recall the definition of generalized matrix algebras given by a Morita context.

Let $\mathcal{R}$ be a commutative ring with identity. A *Morita context* consists of two $\mathcal{R}$-algebras A and B, two bimodules ${}_AM_B$ and ${}_BN_A$, and two bimodule homomorphisms called pairings $\Phi_{MN}: M\otimes_B N \to A$ and $\Psi_{NM}: N\otimes_A M \to B$ making the following diagrams commutative:

$$\begin{tikzcd} M\otimes_B N\otimes_A M \arrow[r, "\Phi_{MN}\otimes I_M"] \arrow[d, "I_M\otimes\Psi_{NM}"] & A\otimes_A M \arrow[d, "\cong"] \\ M\otimes_B B \arrow[r, "\cong"] & M \end{tikzcd} \qquad \begin{tikzcd} N\otimes_A M\otimes_B N \arrow[r, "\Psi_{NM}\otimes I_N"] \arrow[d, "I_N\otimes\Phi_{MN}"] & B\otimes_B N \arrow[d, "\cong"] \\ N\otimes_A A \arrow[r, "\cong"] & N \end{tikzcd}$$

Let us write this Morita context as $(A, B, M, N, \Phi_{MN}, \Psi_{NM})$. We refer the reader to [Mor] for the basic properties of Morita contexts. If $(A, B, M, N, \Phi_{MN}, \Psi_{NM})$ is a Morita context, then the set

$$\begin{bmatrix} A & M \\ N & B \end{bmatrix} = \left\{ \begin{bmatrix} a & m \\ n & b \end{bmatrix} \;\middle|\; a\in A,\, m\in M,\, n\in N,\, b\in B \right\}$$

form an $\mathcal{R}$-algebra under matrix-like addition and matrix-like multiplication, where at least one of the two bimodules M and N is not zero. Such an $\mathcal{R}$-algebra is usually called a *generalized matrix algebra* of order 2 and is denoted by

$$\mathcal{G} = \begin{bmatrix} A & M \\ N & B \end{bmatrix}.$$

Similarly, one can define a generalized matrix algebra of order $n > 2$. It was shown in [LW, Example 2.2] that up to isomorphism, an arbitrary generalized matrix algebra of order n $(n \geq 2)$ is a generalized matrix algebra of order 2. If one of the modules M and N is zero, then $\mathcal{G}$ degenerates to an ordinary *triangular algebra*.

Let $\mathcal{R}$ be a commutative ring with identity 1 and $\mathcal{A}$ be a unital algebra over $\mathcal{R}$. In [XW1], we have shown that $\mathcal{A}$ is a natural generalized matrix algebra if and only if there exists a nontrivial idempotent $e \in \mathcal{A}$ such that $e\mathcal{A}(1-e) \neq 0$ or $(1-e)\mathcal{A}e \neq 0$. In view of the purely algebraic nature of the proof of Theorem 2.3, we believe that a much more general result can be obtained. In particular, we ask:

QUESTION 4.1. *Let $\mathcal{A}$ be a unital semiprime algebra having a nontrivial idempotent e such that*

(1) *$e\mathcal{A}(1-e)$ and $(1-e)\mathcal{A}e$ are faithful bimodules,*
(2) *$\mathcal{Z}(e\mathcal{A}e) = \mathcal{Z}(\mathcal{A})e$ and $\mathcal{Z}((1-e)\mathcal{A}(1-e)) = \mathcal{Z}(\mathcal{A})(1-e)$,*
(3) *$e\mathcal{A}e$ has no nonzero central ideals.*

Does any nonlinear Lie n-derivation of $\mathcal{A}$ have the standard form $(\clubsuit)$?

It is not difficult to see that each von Neumann algebra without central summands of type I_1 has all the above-mentioned properties.

Let us return to the case of generalized matrix algebras. Let $\mathcal{R}$ be a commutative ring with identity and A, B be unital associative algebras over $\mathcal{R}$. Suppose that M is a faithful (A, B)-bimodule and N is a faithful (B, A)-bimodule. Let $\mathcal{G} = \left[\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right]$ be a generalized matrix algebra defined by the Morita context $(A, B, M, N, \Phi_{MN}, \Psi_{NM})$. The bilinear form $\Phi_{MN} : M \otimes_B N \to A$ (resp. $\Psi_{NM} : N \otimes_A M \to B$) is said to be *nondegenerate* if for any $0 \neq m \in M$ and $0 \neq n \in N$, $\Phi_{MN}(m, N) \neq 0$ and $\Phi_{MN}(M, n) \neq 0$ (resp. $\Psi_{NM}(n, M) \neq 0$ and $\Psi_{NM}(N, m) \neq 0$). We call the generalized matrix algebra *strict* if Φ_{MN}, Ψ_{NM} are both nondegenerate.

It follows from [XW1, Lemma 3.1] that

$$\mathcal{Z}(\mathcal{G}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \,\middle|\, am = mb,\ na = bn,\ \forall m \in M,\ n \in N \right\}.$$

Let us define two natural $\mathcal{R}$-linear projections $\pi_A : \mathcal{G} \to A$ and $\pi_B : \mathcal{G} \to B$ by

$$\pi_A \colon \begin{bmatrix} a & m \\ n & b \end{bmatrix} \mapsto a \quad \text{and} \quad \pi_B \colon \begin{bmatrix} a & m \\ n & b \end{bmatrix} \mapsto b.$$

It is easy to see that $\pi_A(\mathcal{Z}(\mathcal{G}))$ is a subalgebra of $\mathcal{Z}(A)$ and that $\pi_B(\mathcal{Z}(\mathcal{G}))$ is a subalgebra of $\mathcal{Z}(B)$. Furthermore, there exists a unique algebraic isomorphism $\varphi \colon \pi_A(\mathcal{Z}(\mathcal{G})) \to \pi_B(\mathcal{Z}(\mathcal{G}))$ such that $am = m\varphi(a)$ and $na = \varphi(a)n$ for all $a \in \pi_A(\mathcal{Z}(\mathcal{G}))$, $m \in M$, $n \in N$ (see [XW1, Lemma 3.2]).

Let 1_A and 1_B be the identities of the algebras A and B, respectively, and let I be the identity of $\mathcal{G}$. We will adopt the traditional notations:

$$P = \begin{bmatrix} 1_A & 0 \\ 0 & 0 \end{bmatrix}, \qquad Q = I - P = \begin{bmatrix} 0 & 0 \\ 0 & 1_B \end{bmatrix}$$

and

$$\mathcal{G}_{11} = P\mathcal{G}P, \quad \mathcal{G}_{12} = P\mathcal{G}Q, \quad \mathcal{G}_{21} = Q\mathcal{G}P, \quad \mathcal{G}_{22} = Q\mathcal{G}Q.$$

Thus the generalized matrix algebra $\mathcal{G}$ can be expressed as

$$\mathcal{G} = \mathcal{G}_{11} + \mathcal{G}_{12} + \mathcal{G}_{21} + \mathcal{G}_{22}.$$

Note that $\mathcal{G}_{11}$ and $\mathcal{G}_{22}$ are subalgebras of $\mathcal{G}$ which are isomorphic to A and B, respectively. Moreover, $\pi_A(\mathcal{Z}(\mathcal{G}))$ and $\pi_B(\mathcal{Z}(\mathcal{G}))$ are isomorphic to $P\mathcal{Z}(\mathcal{G})P$ and $Q\mathcal{Z}(\mathcal{G})Q$, respectively.

The following theorem was proved in our preprint [XW5].

THEOREM 4.2. *Let $\mathcal{G} = \left[\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right]$ be a 2-torsion free strict generalized matrix algebra and $\varphi\colon \mathcal{G} \to \mathcal{G}$ be a nonlinear Lie 2-derivation of $\mathcal{G}$. If $\pi_A(\mathcal{Z}(\mathcal{G})) = \mathcal{Z}(A)$ and $\pi_B(\mathcal{Z}(\mathcal{G})) = \mathcal{Z}(B)$, then φ is of the standard form ($\clubsuit$).*

It is natural to consider the following question.

QUESTION 4.3. *Let $\mathcal{G} = \left[\begin{smallmatrix} A & M \\ N & B \end{smallmatrix}\right]$ be a 2-torsion free strict generalized matrix algebra. Suppose that $\pi_A(\mathcal{Z}(\mathcal{G})) = \mathcal{Z}(A)$ and $\pi_B(\mathcal{Z}(\mathcal{G})) = \mathcal{Z}(B)$. Does any nonlinear Lie n-derivation ($n \geq 2$) of $\mathcal{G}$ have the standard form ($\clubsuit$)?*

Various Lie-type derivations of nest algebras have been intensively studied [BeEr, LW, Lu2, QH, YZ, ZWC]. The most extensive results known are obtained in the case of Hilbert space. It is natural and interesting to extend these results to the Banach space case. A *nest algebra* is an operator algebra whose invariant subspace lattice is a nest. In what follows, let $\mathcal{X}$ be a Banach space over the complex field $\mathbb{C}$. By $\mathcal{B}(\mathcal{X})$ and I we denote the algebra of all bounded linear operators on $\mathcal{X}$ and the identity operator on $\mathcal{X}$, respectively. The terms *operator* on $\mathcal{X}$ and *subspace* of $\mathcal{X}$ will mean bounded linear mapping of $\mathcal{X}$ into itself and norm-closed linear manifold of $\mathcal{X}$, respectively. A *nest* $\mathcal{N}$ is a family of closed subspaces of $\mathcal{X}$ that is totally ordered by inclusion and contains $\{0\}$ and $\mathcal{X}$, and is closed under closed linear spans and intersections. The nest algebra associated to a nest $\mathcal{N}$, denoted by $\operatorname{Alg}\mathcal{N}$, is the set of all bounded linear operators on $\mathcal{X}$ which leave invariant each subspace in $\mathcal{N}$, that is,

$$\operatorname{Alg}\mathcal{N} = \{T \in \mathcal{B}(\mathcal{X}) \mid T(N) \subseteq N \text{ for all } N \in \mathcal{N}\}.$$

More recently, Sun and Ma [SM] obtained a new characterization theorem in the Banach space setting. Let $\mathcal{N}$ be a nontrivial nest on $\mathcal{X}$, $\operatorname{Alg}\mathcal{N}$ be the associated nest algebra and $\varphi\colon \operatorname{Alg}\mathcal{N} \to \mathcal{B}(\mathcal{X})$ be a linear mapping. Then φ is a Lie triple derivation if and only if there exist a derivation $d\colon \operatorname{Alg}\mathcal{N} \to$

$\mathcal{B}(\mathcal{X})$ and a linear mapping $f\colon \operatorname{Alg}\mathcal{N} \to \mathbb{C}I$ with $f([[X,Y],Z]) = 0$ for all $X,Y,Z \in \operatorname{Alg}\mathcal{N}$ such that $\varphi = d + f$ on $\operatorname{Alg}\mathcal{N}$. This implies that every Lie triple derivation from $\operatorname{Alg}\mathcal{N}$ into $\mathcal{B}(\mathcal{X})$ has the standard form ($\clubsuit$). It seems reasonable to make the following conjecture:

CONJECTURE 4.4. *Let $\mathcal{N}$ be a nontrivial nest on $\mathcal{X}$, $\operatorname{Alg}\mathcal{N}$ be the associated nest algebra and $\varphi\colon \operatorname{Alg}\mathcal{N} \to \mathcal{B}(\mathcal{X})$ be a nonlinear Lie n-derivation $(n \geq 2)$. Then φ has the standard form ($\clubsuit$).*

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