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INDUCED OPEN PROJECTIONS AND C^*-SMOOTHNESS

BY

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Abstract. We show that there exists a C^*-smooth continuum X such that for every continuum Y the induced map $C(f)$ is not open, where $f : X \times Y \to X$ is the projection. This answers a question of Charatonik et al. (2000).

1. Introduction. A *continuum* is a nondegenerate, compact, connected metric space. A *map* is a continuous function. Given a continuum X, let 2^X denote the hyperspace of all nonempty closed subsets of X, endowed with the Hausdorff metric H [6, Definition 2.1]. Let $C(X)$ denote the hyperspace of connected elements of 2^X. Given a map between continua $f : X \to Y$, we consider the induced maps $2^f : 2^X \to 2^Y$ and $C(f) : C(X) \to C(Y)$ given by $2^f(A) = f(A)$ (the image of A under f) and $C(f)(A) = f(A)$. A map between continua $f : X \to Y$ is *open* provided that the image of each open subset of X is an open subset of Y. A continuum X is said to be C^**-smooth* provided that the map $A \mapsto C(A)$ from $C(X)$ into $C(C(X))$ is continuous.

Openness of induced maps has been studied by several authors. For a surjective map $f : X \to Y$, consider the following conditions: (a) $f : X \to Y$ is open, (b) $2^f : 2^X \to 2^Y$ is open, and (c) $C(f) : C(X) \to C(Y)$ is open. It is known that (a) and (b) are equivalent and each one of them is implied by (c). In [4] an example is shown of an open map $f : X \to Y$ between locally connected continua X and Y such that the induced map $C(f) : C(X) \to C(Y)$ is not open. In [5] it was proved that if the induced map $C(C(f)) : C(C(X)) \to C(C(Y))$ is open, then f is a homeomorphism. A recent result about openness of the induced map of $C(f)$, when the domain of f is a dendroid, has been obtained in [1].

Given continua X and Y, a natural open map is given by the projection $\pi_X^Y : X \times Y \to X$ on the first coordinate. In [2] some results on the openness of $C(\pi_X^Y)$ were obtained. In [3, Theorem 4], it was proved that if there exists a continuum Y such that $C(\pi_X^Y)$ is open, then X is C^*-smooth, and it was

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asked if the converse holds [3, Problem 6]. In this paper we answer this question in the negative. We also show that if X is a compactification of the ray $[0,1)$ with an arc as remainder, then $C(\pi_X^{[0,1]})$ is open.

2. Atriodicity. A continuum X is said to have the *open projection property* provided that $C(\pi_X^Y) : C(X \times Y) \to C(X)$ is open for each continuum Y. Given $A \subset X$ and $\varepsilon > 0$, let $N(\varepsilon, A) = \bigcup\{B(\varepsilon, a) : a \in A\}$, where $B(\varepsilon, a)$ is the ε-neighborhood of a in X. An *n-od* in the continuum X is a subcontinuum B of X for which there exists an element $A \in C(B)$ such that $B - A$ has at least n components. A *triod* is a 3-od. An *atriodic* continuum is a continuum containing no triods. A *simple triod* is a continuum $X = J_1 \cup J_2 \cup J_3$, where each J_i is an arc, $J_i \cap J_j = \{p\}$ if $i \neq j$ and p is an end point of each J_i. A *weak triod* is a continuum $W = C_1 \cup C_2 \cup C_3$, where each C_i is a subcontinuum of W, $C_1 \cap C_2 \cap C_3 \neq \emptyset$ and C_i is not contained in $\bigcup\{C_j : j \in \{1,2,3\} - \{i\}\}$. By Theorem 1.8 of [11] a continuum X contains a weak triod if and only if X contains a triod.

THEOREM 2.1. *Let X and Y be continua. Suppose that the map $C(\pi_X^Y) : C(X \times Y) \to C(X)$ is open and Z is a nondegenerate subcontinuum of X. Then the map $C(\pi_Z^Y) = C(\pi_X^Y)|_{C(Z\times Y)} : C(Z \times Y) \to C(Z)$ is open.*

Proof. Let $\mathcal{U}$ be an open subset of $C(Z \times Y)$. Let $\mathcal{V}$ be an open subset of $C(X \times Y)$ such that $\mathcal{V} \cap C(Z \times Y) = \mathcal{U}$. By hypothesis $C(\pi_X^Y)(\mathcal{V})$ is open in $C(X)$. Since $C(\pi_Z^Y)(\mathcal{U}) = C(\pi_X^Y)(\mathcal{V}) \cap C(Z)$, $C(\pi_Z^Y)(\mathcal{U})$ is open in $C(Z)$. ■

THEOREM 2.2. *Let X be a continuum. Suppose that T is a triod in X and there exists a sequence of arcs $\{J_m\}_{m=1}^{\infty}$ in X such that $\lim J_m = T$. Then for each continuum Y, $C(\pi_X^Y) : C(X \times Y) \to C(X)$ is not open.*

Proof. Suppose to the contrary that there exists a continuum Y such that $C(\pi_X^Y) : C(X \times Y) \to C(X)$ is open. For each $m \in \mathbb{N}$, let x_m, y_m be the end points of J_m. We can consider the natural order in J_m satisfying $x_m < y_m$. Let $A \in C(T)$ be such that $T - A = K_1 \cup K_2 \cup K_3$, where $\mathrm{cl}_X(K_i) \cap K_j = \emptyset$ if $i \neq j$ and each K_i is nonempty. For each $i \in \{1,2,3\}$, fix a point $q_i \in K_i$ and an open subset Q_i of X such that $q_i \in Q_i$, $\mathrm{cl}_X(Q_i) \cap T \subset K_i$ and $\mathrm{cl}_X(Q_1)$, $\mathrm{cl}_X(Q_2)$ and $\mathrm{cl}_X(Q_3)$ are pairwise disjoint. Fix points $w, z \in Y$ such that $w \neq z$. Let

$$M = ((K_2 \cup A \cup K_1) \times \{w\}) \cup ((K_3 \cup A \cup K_1) \times \{z\}) \cup (\{q_1\} \times Y).$$

Then M is a subcontinuum of $X \times Y$. Fix open subsets W and Z of Y such that $\mathrm{cl}_Y(W) \cap \mathrm{cl}_Y(Z) = \emptyset$, $w \in W$ and $z \in Z$.

By [3, Theorem 4], X is C^*-smooth, so $\lim C(J_m) = C(T)$. Thus, for each $m \in \mathbb{N}$ we can choose an element $L_m \in C(J_m)$ such that $\lim L_m = K_2 \cup A \cup K_3$. Shortening L_m a little if necessary, we may assume that $x_m, y_m \notin L_m$.

Let $V = \big((X - \mathrm{cl}_X(Q_3)) \times W\big) \cup \big((X - \mathrm{cl}_X(Q_2)) \times Z\big) \cup (Q_1 \times Y)$. Then V is an open subset of $X \times Y$ containing M. Let $\mathcal{V} = \{B \in C(X \times Y) : B \subset V\}$. Since $\mathcal{V}$ is open in $C(X \times Y)$ and $C(\pi_X^Y)$ is open, $C(\pi_X^Y)(\mathcal{V})$ is an open subset of $C(X)$. Since $T = \pi_X^Y(M) \in C(\pi_X^Y)(\mathcal{V})$ and $\lim J_m = T$, there exists $m \in \mathbb{N}$ such that $J_m \in C(\pi_X^Y)(\mathcal{V})$. We may assume that $L_m \cap Q_2 \neq \emptyset \neq L_m \cap Q_3$ and $\mathrm{cl}_X(Q_1) \cap L_m = \emptyset$. Thus, there exists $B \in \mathcal{V}$ such that $\pi_X^Y(B) = J_m$. Choose points $a \in L_m \cap Q_2$ and $b \in L_m \cap Q_3$. Without loss of generality we may assume that $x_m < a < b < y_m$. Let E, F, G be the respective subarcs of J_m joining the pairs of points x_m and a; a and b; b and y_m. Notice that $F \subset L_m$.

Let

$$\begin{aligned}
B_1 &= B \cap (\pi_X^Y)^{-1}(E),\\
B_2 &= B \cap (\pi_X^Y)^{-1}(F) \cap ((X - Q_3) \times \mathrm{cl}_X(W)),\\
B_3 &= B \cap (\pi_X^Y)^{-1}(F) \cap ((X - Q_2) \times \mathrm{cl}_X(Z)),\\
B_4 &= B \cap (\pi_X^Y)^{-1}(G).
\end{aligned}$$

Clearly, $B_1 \cup B_2 \cup B_3 \cup B_4 \subset B$. Given $p \in B$ such that $\pi_X^Y(p) \notin E \cup G$, we have $\pi_X^Y(p) \in F \subset L_m \subset X - \mathrm{cl}_X(Q_1)$. Since $B \subset V$ and $\pi_X^Y(p) \notin \mathrm{cl}_X(Q_1)$, we obtain $p \in \big((X - \mathrm{cl}_X(Q_3)) \times W\big) \cup \big((X - \mathrm{cl}_X(Q_2)) \times Z\big)$. Thus, $p \in B_2 \cup B_3$. We have shown that $B = B_1 \cup B_2 \cup B_3 \cup B_4$. Notice that each B_i is closed in $X \times Y$.

Now, suppose that there exists a point $p \in (B_1 \cup B_2) \cap (B_3 \cup B_4)$. In the case that $p \in B_1$, since $B_1 \cap B_4 \subset (\pi_X^Y)^{-1}(E) \cap (\pi_X^Y)^{-1}(G) = \emptyset$, we have $p \in B_3$. This implies that $\pi_X^Y(p) = a \in Q_2$ and $p \notin B_3$, a contradiction. A similar contradiction can be obtained by supposing that $p \in B_4$. Thus, $p \in B_2 \cap B_3$, but this is impossible since $\mathrm{cl}_X(Z) \cap \mathrm{cl}_X(W) = \emptyset$. We have proved that $(B_1 \cup B_2) \cap (B_3 \cup B_4) = \emptyset$. Since $\pi_X^Y(B) = J_m$, there exists a point $p \in B$ such that $\pi_X^Y(p) = x_m$. Thus, $p \in B_1$ and $B_1 \cup B_2 \neq \emptyset$. Similarly, $B_3 \cup B_4 \neq \emptyset$. We have obtained a separation of the connected set B. This contradiction completes the proof of the theorem. ∎

Problem 2.3. *Is Theorem 2.2 true when we replace arcs by atriodic continua? That is, suppose that X is a continuum, T is a triod in X and there exists a sequence $\{J_m\}_{m=1}^{\infty}$ of atriodic subcontinua of X such that $\lim J_m = T$. Is it true that for each continuum Y, $C(\pi_X^Y) : C(X \times Y) \to C(X)$ is not open?*

Related to Problem 2.3, we have the following result.

Theorem 2.4. *Let X be a continuum. Suppose that K is a 4-od in X and there exists a sequence $\{J_m\}_{m=1}^{\infty}$ of atriodic subcontinua of X such that $\lim J_m = K$. Then for each continuum Y, $C(\pi_X^Y) : C(X \times Y) \to C(X)$ is not open.*

Proof. Suppose to the contrary that there exists a continuum Y such that $C(\pi_X^Y) : C(X \times Y) \to C(X)$ is open. Let $A \in C(K)$ be such that $K - A = K_1 \cup K_2 \cup K_3 \cup K_4$, where $\mathrm{cl}_X(K_i) \cap K_j = \emptyset$ if $i \neq j$ and each K_i is nonempty. For each $i \in \{1,2,3,4\}$, fix a point $q_i \in K_i$ and an open subset Q_i of X such that $q_i \in Q_i$, $\mathrm{cl}_X(Q_i) \cap K \subset K_i$, and $\mathrm{cl}_X(Q_1)$, $\mathrm{cl}_X(Q_2)$, $\mathrm{cl}_X(Q_3)$ and $\mathrm{cl}_X(Q_4)$ are pairwise disjoint. Fix points $w, z \in Y$ such that $w \neq z$ and fix open subsets W and Z of Y such that $\mathrm{cl}_Y(W) \cap \mathrm{cl}_Y(Z) = \emptyset$, $w \in W$ and $z \in Z$. For each $i \in \{1,2,3\}$, let

$$M_i = ((K_4 \cup A \cup K_i) \times \{w\}) \cup ((K_1 \cup K_2 \cup K_3 \cup A) \times \{z\}) \cup (\{q_i\} \times Y).$$

Then M_i is a subcontinuum of $X \times Y$. Let

$$V_i = \Big(\Big(X - \bigcup\{\mathrm{cl}_X(Q_j) : j \in \{1,2,3\} - \{i\}\}\Big) \times W\Big) \cup (Q_i \times Y) \cup \big((X - \mathrm{cl}_X(Q_4)) \times Z\big).$$

Then V_i is an open subset of $X \times Y$ such that $M_i \subset V_i$. Let $\mathcal{V}_i = \{B \in C(X \times Y) : B \subset V_i\}$. Since $\mathcal{V}_i$ is open in $C(X \times Y)$ and $C(\pi_X^Y)$ is open, $C(\pi_X^Y)(\mathcal{V}_i)$ is an open subset of $C(X)$ that contains $K = \pi_X^Y(M_i)$. Thus, there exists $m \in \mathbb{N}$ such that $J_m \in C(\pi_X^Y)(\mathcal{V}_i)$ and $J_m \cap Q_i \neq \emptyset$ for each $i \in \{1,2,3,4\}$. For $i \in \{1,2,3,4\}$, fix a point $p_i \in J_m \cap Q_i$.

Given $i \in \{1,2,3\}$, let $B_i \subset V_i$ be such that B_i is a subcontinuum of $X \times Y$ and $\pi_X^Y(B_i) = J_m$. Fix a point $b_i \in B_i$ such that $\pi_X^Y(b_i) = p_4 \in Q_4$. Let $U_i = (X - \bigcup\{\mathrm{cl}_X(Q_j) : j \in \{1,2,3\} - \{i\}\}) \times W$. Notice that $b_i \in U_i$. Fix $j \in \{1,2,3\} - \{i\}$. Since $p_j \in \pi_X^Y(B_i) \cap Q_j$ and $B_i \subset V_i$, we see that $B_i \not\subset U_i$. Let S_i be the component of $B_i \cap U_i$ that contains b_i. By [8, Theorem 20.3], $\mathrm{cl}_{B_i}(S_i) \cap \mathrm{bd}_{B_i}(B_i \cap U_i) \neq \emptyset$. Take a point $s_i \in \mathrm{cl}_{B_i}(S_i) \cap \mathrm{bd}_{B_i}(B_i \cap U_i)$ and let $C_i = \mathrm{cl}_{B_i}(S_i)$. Then C_i is a subcontinuum of B_i and $s_i \in V_i \cap (\mathrm{cl}_{X\times Y}(U_i) - U_i)$. Since $\mathrm{cl}_Y(W) \cap \mathrm{cl}_Y(Z) = \emptyset$, we have $s_i \in Q_i \times Y$. Thus, $\pi_X^Y(C_i)$ is a subcontinuum of J_m that contains p_4 and intersects Q_i. Moreover, since

$$\pi_X^Y(C_i) \subset \mathrm{cl}_X(\pi_X^Y(S_i)) \subset \mathrm{cl}_X(\pi_X^Y(U_i)) \subset X - \bigcup\{Q_k : k \in \{1,2,3\} - \{i\}\},$$

we have $\pi_X^Y(C_i) \cap Q_k = \emptyset$ for each $k \in \{1,2,3\} - \{i\}$. Hence, $\pi_X^Y(C_1)$, $\pi_X^Y(C_2)$ and $\pi_X^Y(C_3)$ are subcontinua of J_m such that $p_4 \in \pi_X^Y(C_1) \cap \pi_X^Y(C_2) \cap \pi_X^Y(C_3)$ and no $\pi_X^Y(C_i)$ is contained in the union of the other two. Hence, $T = \pi_X^Y(C_1) \cup \pi_X^Y(C_2) \cup \pi_X^Y(C_3)$ is a weak triod. Therefore [11, Theorem 1.8], J_m is not atriodic. This contradiction completes the proof of the theorem. ■

3. The example. In Problem 6 of [3] it was asked if each C^*-smooth continuum X has the open projection property. In the following example, we give a negative answer to this question. Bruce Hughes (see [8, p. 495]) constructed a continuum X that is a compactification of the ray $[0, \infty)$ with remainder a simple triod T in such a way that each subcontinuum of T is a limit of subcontinua of $[0,\infty)$. In our example we construct a continuum X that is a compactification of the ray $[0,\infty)$ with remainder a simple triod T in such a way that X is C^*-smooth. Thus, X must have the following property: if A is a subtriod of T and $\{A_n\}_{n=1}^{\infty}$ is a sequence of arcs in the ray such that $\lim A_n = A$, then for each subtriod B of A, there exists a sequence $\{B_n\}_{n=1}^{\infty}$ of arcs in the ray such that $B_n \subset A_n$ for each $n \in \mathbb{N}$ and $\lim B_n = B$. That is, if A_n describes a path close to the triod A, then the path A_n must also contain subarcs approximating many of the subtriods of A. The construction of such X requires a very careful and technical description of the ray.

Example 3.1. There exists a C^*-smooth continuum X such that for each continuum Z the induced map $C(\pi_X^Z) : C(X \times Z) \to C(X)$ is not open.

Let $\mathbb{R}^3$ be the Euclidean 3-dimensional space. Let $\pi, \pi_0 : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $\pi(x,y,z) = (x,y)$ and $\pi_0(x,y,z) = (y,z)$. For each $i \in \{1,2,3\}$, let $\pi_i : \mathbb{R}^3 \to \mathbb{R}$ be the projection on the ith coordinate. Given $p, q \in \mathbb{R}^3$, with $p \neq q$, let pq be the convex segment in $\mathbb{R}^3$ that joins p and q. Let $\theta = (0,0,0)$, $e_1 = (1,0,0)$, $e_2 = (\cos(2\pi/3), \sin(2\pi/3), 0)$ and $e_3 = (\cos(4\pi/3), \sin(4\pi/3), 0)$. Let $T = \theta e_1 \cup \theta e_2 \cup \theta e_3$. Then T is a simple triod. We will construct a compactification X of the ray $[0,\infty)$ with remainder T such that X is C^*-smooth.

Let Y be the infinite triod defined by $Y = \{se_i : i \in \{1,2,3\}$ and $s \in [0,\infty)\}$. Given two maps $\delta_1 : [u,v] \to Y$ and $\delta_2 : [v,w] \to Y$ such that $\delta_1(v) = \delta_2(v)$, let $\delta_1 * \delta_2 : [u,w] \to Y$ be the common extension of the maps δ_1 and δ_2.

Let $\mathbb{N}^3 = \mathbb{N} \times \mathbb{N} \times \mathbb{N}$. Given $\alpha = (n,m,r) \in \mathbb{N}^3 - \{(1,1,1)\}$, let $M(\alpha) = \{(a,b,c) \in \mathbb{N}^3 : a \leq n,\ b \leq m \text{ and } c \leq r\} - \{\alpha\}$ and $K(\alpha) = M(\alpha) \cup \{\alpha\}$. Then $|M(\alpha)| = nmr - 1$. Given $t \in \mathbb{N} \cup \{0\}$, define

$$\omega(t) : \{t+1, \dots, t+2n\} \to K(\alpha) \cap ([1,n] \times \{1\} \times \{1\})$$

by

$$\omega(t)(i) = \begin{cases} (i-t,1,1) & \text{if } i \in \{t+1,\dots,t+n\}, \\ (t+2n+1-i,1,1) & \text{if } i \in \{t+n+1,\dots,t+2n\}. \end{cases}$$

Notice that $\omega(t)$ covers two times the set $K(\alpha) \cap ([1,n] \times \{1\} \times \{1\})$, first it runs in the natural order and then in the opposite. The discrete path $\omega(t)$ starts and finishes at $(1,1,1)$.

Define

$$\psi_0(t) : \{t+1, \dots, t+4nm\} \to K(\alpha) \cap ([1,n] \times [1,m] \times \{1\})$$

by

$$\psi_0(t)(i) = \omega(t+(j-1)2n)(i) + (0, j-1, 0)$$

if $i \in \{t+(j-1)2n+1, \ldots, t+j2n\}$ for some $j \in \{1, \ldots, m\}$, and

$$\psi_0(t)(i) = \omega(t+(j-1)2n)(i) + (0, 2m-j, 0)$$

if $i \in \{t+(j-1)2n+1, \ldots, t+j2n\}$ for some $j \in \{m+1, \ldots, 2m\}$.

Notice that $\psi_0(t)$ is a discrete path that starts filling twice the discrete segment $([1,n] \times \{1\} \times \{1\}) \cap K(\alpha)$, starting and finishing at $(1,1,1)$, then it fills twice the discrete segment $([1,n] \times \{2\} \times \{1\}) \cap K(\alpha)$, starting and finishing at $(1,2,1)$, next it continues filling the discrete segments of the form $([1,n] \times \{s\} \times \{1\}) \cap K(\alpha)$, starting and finishing at $(1,s,1)$, until it fills the discrete segment $([1,n] \times \{m\} \times \{1\}) \cap K(\alpha)$; then it fills this segment again and then the segment $([1,n] \times \{m-1\} \times \{1\}) \cap K(\alpha)$, and continues until it finishes filling again the segment $([1,n] \times \{1\} \times \{1\}) \cap K(\alpha)$. The discrete path $\psi_0(t)$ starts and finishes at $(1,1,1)$.

Define $\varphi(\alpha) : \{1, \ldots, 4nmr\} \to K(\alpha)$ by

$$\varphi(\alpha)(i) = \psi_0((j-1)4nm)(i) + (0,0,j-1)$$

if $i \in \{4(j-1)nm+1, \ldots, 4jnm\}$ for some $j \in \{1, \ldots, r\}$.

Notice that $\varphi(\alpha)$ is a discrete path that uses the first discrete segment $\{1, \ldots, 4nm\}$ to fill the bottom $K(\alpha) \cap ([1,n] \times [1,m] \times \{1\})$ of $K(\alpha)$ (level one) finishing at the point $(1,1,1)$, then it climbs up to the next level (level two) and then fills level two, starting and finishing at the point $(1,1,2)$. Then it climbs up to the next level (to the point $(1,1,3)$) and then it fills level three and so on. Notice also that $\varphi(\alpha)$ finishes at the point $(1,1,r)$.

Define $g(\alpha) = \min\{i \in \{1, \ldots, nmr\} : \varphi(\alpha)(i) = (n,m,r)\} - 1$.

Let $\varphi(\alpha) = (\varphi_1(\alpha), \varphi_2(\alpha), \varphi_3(\alpha))$. We will need the following properties of $\varphi(\alpha)$:

(a) $\varphi(\alpha)(1) = (1,1,1)$.
(b) $\varphi(\alpha)(g(\alpha)+1) = (n,m,r)$.
(c) $\varphi(\alpha)(\{1, \ldots, g(\alpha)\}) = M(\alpha)$.
(d) For each $1 \le i \le g(\alpha)$,

$$|\varphi_1(\alpha)(i) - \varphi_1(\alpha)(i+1)| + |\varphi_2(\alpha)(i) - \varphi_2(\alpha)(i+1)| + |\varphi_3(\alpha)(i) - \varphi_3(\alpha)(i+1)| \le 1.$$

(e) If $1 \le i \le j \le g(\alpha)$, then there exists $i \le k \le j$ such that $\{\varphi(\alpha)(l) \in \mathbb{N}^3 : i \le l \le j\} \subset K(\varphi(\alpha)(k) + (1,1,1))$.
(f) Let $\beta = (n_1, m_1, r_1) \in \mathbb{N}^3 - \{(1,1,1)\}$ be such that $|n - n_1| \le 1$, $|m - m_1| \le 1$ and $|r - r_1| \le 1$. Let $i \in \{1, \ldots, g(\alpha)\}$, $j \in \{1, \ldots, g(\beta)\}$, $A = \{\varphi(\alpha)(l) \in \mathbb{N}^3 : 1 \le l \le i\}$, $B = \{\varphi(\beta)(l) \in \mathbb{N}^3 : 1 \le l \le j\}$ and for each $k \in \{1,2,3\}$, let $u_k = \max \pi_k(A)$ and $v_k = \max \pi_k(B)$. Then $K((u_1,u_2,u_3)) \subset K((v_1,v_2,v_3) + (1,1,1))$ or $K((v_1,v_2,v_3)) \subset K((u_1,u_2,u_3) + (1,1,1))$.

Properties (a)–(d) are immediate. We prove property (e). Take $1 \leq i < j \leq g(\alpha)$. Let $A = \{\varphi(\alpha)(l) \in \mathbb{N}^3 : i \leq l \leq j\}$. For each $s \in \{1, 2, 3\}$, let $u_s = \min \pi_s(A)$ and $v_s = \max \pi_s(A)$. Then $[u_1, v_1] \times [u_2, v_2] \times [u_3, v_3]$ is the minimal box in $\mathbb{N}^3$ containing A. We analyze three cases.

Case 1: $2 \leq v_3 - u_3$. Since $\varphi(\alpha)$ fills each level of the form $K(\alpha) \cap ([1, n] \times [1, m] \times \{s\})$ before going to the next one, $\{1, \ldots, n\} \times \{1, \ldots, m\} \times \{v_3 - 1\} \subset A$. So, there exists $i \leq k \leq j$ such that $\varphi(\alpha)(k) = (n, m, v_3 - 1) \in A$. Clearly, $A \subset K(\varphi(\alpha)(k) + (1, 1, 1))$.

Case 2: $v_3 = u_3$. If $2 \leq v_2 - u_2$, since $\varphi(\alpha)$ fills each row of the form $[1, n] \times \{s\} \times \{v_3\}$ before going to the next one, $\{1, \ldots, n\} \times \{v_2 - 1\} \times \{v_3\} \subset A$. So, there exists $i \leq k \leq j$ such that $\varphi(\alpha)(k) = (n, v_2 - 1, v_3) \subset A$. Clearly, $A \subset K(\varphi(\alpha)(k) + (1, 1, 1))$. Thus, we may assume that $v_2 - u_2 \leq 1$. In the case that $v_2 = u_2$, $A = ([u_1, v_1] \cap \mathbb{N}) \times \{u_2\} \times \{u_3\}$. Thus, taking $i \leq k \leq j$ such that $\varphi(\alpha)(k) = (v_1, u_2, u_3)$, we are done. In the case that $v_2 = u_2 + 1$, A is of the form

$$A = (([1, x] \cap \mathbb{N}) \times \{u_2\} \times \{v_3\}) \cup (([1, y] \cap \mathbb{N}) \times \{v_2\} \times \{v_3\}).$$

Then taking $i \leq k \leq j$ such that $\varphi(\alpha)(k) = (x, u_2, v_3)$ if $y \leq x$, and $\varphi(\alpha)(k) = (y, v_2, v_3)$ if $x \leq y$, we are done.

Case 3: $v_3 = u_3 + 1$. The case that $v_2 = 1$ is similar to the last subcase of Case 2. Thus, we may assume that $v_2 > 1$ and $\{u_2, v_2\} \neq \{n\}$. Then either

(i) $\pi_0(A) = (([1, y] \cap \mathbb{N}) \times \{u_3\}) \cup (([1, v_2] \cap \mathbb{N}) \times \{v_3\})$, or
(ii) $\pi_0(A) = (([1, y] \cap \mathbb{N}) \times \{v_3\}) \cup (([1, v_2] \cap \mathbb{N}) \times \{u_3\})$.

In case (i), we have $\{1, \ldots, n\} \times \{v_2 - 1\} \times \{v_3\} \subset A$; then it is enough to take $i \leq k \leq j$ such that $\varphi(\alpha)(k) = (n, v_2 - 1, v_3)$. In case (ii), we have $\{1, \ldots, n\} \times \{v_2 - 1\} \times \{u_3\} \subset A$; then it is enough to take $i \leq k \leq j$ such that $\varphi(\alpha)(k) = (n, v_2 - 1, u_3)$.

This completes the proof of (e).

Finally, we prove (f). We consider three cases.

Case 1: $\max\{u_3, v_3\} > 1$. Suppose, for example, that $u_3 \geq v_3$. In this case, $u_1 = n$, $u_2 = m$ and $([1, n] \times [1, m] \times [1, u_3 - 1]) \cap \mathbb{N}^3 \subset A$, so $K((v_1, v_2, v_3)) \subset ([1, v_1] \times [1, m+1] \times [1, v_3]) \cap \mathbb{N}^3 \subset ([1, n+1] \times [1, m+1] \times [1, u_3]) \cap \mathbb{N}^3 \subset K((u_1, u_2, u_3) + (1, 1, 1))$.

Case 2: $u_3 = v_3 = 1$ and $\max\{u_2, v_2\} > 1$. Suppose, for example, that $u_2 \geq v_2$. In this case, $u_1 = n$ and $([1, n] \times [1, u_2 - 1] \times \{1\}) \cap \mathbb{N}^3 \subset A$, so $K((v_1, v_2, v_3)) \subset ([1, n+1] \times [1, v_2] \times \{1\}) \cap \mathbb{N}^3 \subset ([1, n+1] \times [1, u_2] \times \{1\}) \cap \mathbb{N}^3 \subset K((u_1, u_2, u_3) + (1, 1, 1))$.

Case 3: $u_3 = v_3 = 1$ and $u_2 = v_2 = 1$. Suppose, for example, that $u_1 \geq v_1$. In this case $K((v_1, v_2, v_3)) \subset K((u_1, u_2, u_3)) \subset K((u_1, u_2, u_3) + (1, 1, 1))$.

This completes the proof of (f).

Given a subcontinuum A of Y such that $\theta \in A$, for each $i \in \{1,2,3\}$, let

$$\lambda_i(A) = \text{length of } A \cap \{te_i : t \in [0,\infty)\}.$$

Given subcontinua A, B of Y such that $\theta \in A \cap B$, set

$$D(A,B) = |\lambda_1(A) - \lambda_1(B)| + |\lambda_2(A) - \lambda_2(B)| + |\lambda_3(A) - \lambda_3(B)|.$$

Given $i \in \{1,2,3\}$, $n \in \mathbb{N}$ and $u < v$, let $\eta(i,n,u,v)$ be the map

$$\eta(i,n,u,v) : [u,v] \to Y$$

given by the conditions: $\eta(i,n,u,v)$ is linear on each one of the intervals $[u,(u+v)/2]$ and $[(u+v)/2,v]$, $\eta(i,n,u,v)(u) = \theta$, $\eta(i,n,u,v)((u+v)/2) = ne_i$ and $\eta(i,n,u,v)(v) = \theta$. Notice that

$$\max\{|\eta(i,n,u,v)(t)| : t \in [u,v]\} = n. \tag{3.1}$$

Given $\alpha = (n,m,r) \in \mathbb{N}^3$ and $u < v$, we will define a map

$$\sigma_\alpha : [u,v] \to Y$$

(we write $\sigma_\alpha(u,v)$ when it is necessary to mention the interval $[u,v]$) by induction on the number of elements of $M(\alpha)$. In order to define $\sigma_{(1,1,1)}$, divide the interval $[u,v]$ by a partition $u = s_0 < s_1 < s_2 < s_3 = v$, where $s_{i+1} - s_i = (v-u)/3$ for each i and define $\sigma_{(1,1,1)}$ by the following conditions: $\sigma_{(1,1,1)}|[s_0,s_1] = \eta(1,1,s_0,s_1)$, $\sigma_{(1,1,1)}|[s_1,s_2] = \eta(2,1,s_1,s_2)$ and $\sigma_{(1,1,1)}|[s_2,s_3] = \eta(3,1,s_2,s_3)$. This defines σ_α for the case that $|M(\alpha)| = 0$. Notice that $\sigma_{(1,1,1)}(u) = \sigma_{(1,1,1)}(v) = \theta$.

In the case that $\alpha = (2,1,1)$, divide the interval $[u,v]$ by a partition $u = s_0 < s_1 < s_2 < s_3 = v$, where $s_{i+1} - s_i = (v-u)/3$ for each i, and define σ_α as the map $\sigma_{(1,1,1)}(s_0,s_1) * \eta(1,2,s_1,s_2) * \sigma_{(1,1,1)}(s_2,s_3)$. Inductively, in the case $\alpha = (k,1,1)$ for some $k \geq 3$, divide the interval $[u,v]$ by a partition $u = s_0 < s_1 < \cdots < s_{2k-1} = v$, where $s_{i+1} - s_i = (v-u)/(2k-1)$ for each i, and define σ_α as the map $\sigma_{(1,1,1)}(s_0,s_1) * \cdots * \sigma_{(k-1,1,1)}(s_{k-1},s_k) * \eta(1,k,s_k,s_{k+1}) * \sigma_{(k-1,1,1)}(s_{k+1},s_{k+2}) * \cdots * \sigma_{(1,1,1)}(s_{2k-2},s_{2k-1})$.

In a similar way, define $\sigma_{(1,k,1)}$ and $\sigma_{(1,1,k)}$ for each $k \geq 2$.

Now, suppose that $0 < |M(\alpha)|$ and σ_β has been defined for every $\beta \in \mathbb{N}^3$ and $u < v$; when $|M(\beta)| < |M(\alpha)|$, suppose also that α is not of any of the forms $(k,1,1), (1,k,1), (1,1,k)$ $(k \in \mathbb{N})$ and suppose that each σ_β satisfies

$$\sigma_\beta(u) = \theta = \sigma_\beta(v). \tag{3.2}$$

Here, we use the map $\varphi(\alpha)$ defined before. Divide the interval $[u,v]$ by a partition $u = s_0 < s_1 < \cdots < s_{2g(\alpha)-1} = v$, where $s_{i+1} - s_i = (v-u)/(2g(\alpha)-1)$ for each i, and define

$$\sigma_\alpha = \sigma_{\varphi(\alpha)(1)} * \cdots * \sigma_{\varphi(\alpha)(g(\alpha)-1)} * \sigma_{\varphi(\alpha)(g(\alpha))} * \sigma_{\varphi(\alpha)(g(\alpha)-1)} * \cdots * \sigma_{\varphi(\alpha)(1)}.$$

Using (c), it can be shown that

$$ne_1 \in \operatorname{Im}\sigma_\alpha, \qquad me_2 \in \operatorname{Im}\sigma_\alpha, \qquad re_3 \in \operatorname{Im}\sigma_\alpha. \tag{3.3}$$

Using (3.1) and (3.2), it can be proved by induction that for each $\alpha = (n, m, r) \in \mathbb{N}^3$,

$$\sigma_\alpha(u) = \theta = \sigma_\alpha(v) \tag{3.4}$$

and

$$\begin{aligned} &\lambda_1(\sigma_\alpha([u,v])) = n, \quad \lambda_2(\sigma_\alpha([u,v])) = m, \quad \lambda_3(\sigma_\alpha([u,v])) = r, \\ &\max\{|\sigma_\alpha(t)| : t \in [u,v]\} = \max\{n, m, r\}. \end{aligned} \tag{3.5}$$

Inductively, the following properties can be shown:

(3.6.1) $\sigma_\alpha^{-1}(\theta)$ can be ordered as a partition $u = u_0 < u_1 < \cdots < u_k = v$;
(3.6.2) each interval $[u_{j-1}, u_j]$ can be divided in two subintervals $[u_{j-1}, v_j]$ and $[v_j, u_j]$ such that $\sigma_\alpha|[u_{j-1}, v_j]$ and $\sigma_\alpha|[v_j, u_j]$ are linear.

In the definition of $\sigma_\alpha(u,v)$, for each $i \in \{2, \ldots, g(\alpha)-1\}$, the map $\sigma_{\varphi(\alpha)(i)}$ is defined on two possible subintervals of $[u,v]$, and the map $\sigma_{\alpha_{\varphi(\alpha)(g(\alpha))}}$ is defined on one subinterval of $[u,v]$. The total number of these specific functions is $2g(\alpha) - 1$.

We use the notation $\sigma_\gamma(x,y) \lhd \sigma_\alpha(u,v)$ to indicate that $\gamma = \sigma_{\varphi(\alpha)(i)}$ for some $i \in \{2, \ldots, g(\alpha)\}$ and $[x,y]$ is one of the intervals mentioned in the previous paragraph, so $[x,y] \subset [u,v]$ and $[x,y]$ is the domain of $\sigma_\gamma(x,y)$.

For each $m \in \mathbb{N}$, let $\beta_{m-1} = (m,m,m)$ and consider the map $\sigma_{\beta_m}(m-1,m)$ (defined on the interval $[m-1,m]$), then define $\xi_m : [m-1,m] \to Y$ and $\psi_m : [m-1,m] \to T$ by

$$\xi_m(t) = \sigma_{\beta_m}(m-1,m)(t) \quad \text{and} \quad \psi_m(t) = \frac{1}{m+1}\xi_m(t).$$

By (3.5), the image of ψ_m is contained in the set T.

Finally, define $\xi : [0,\infty) \to Y$ and $\psi : [0,\infty) \to \mathbb{R}^3$ by

$$\xi(t) = \xi_m(t) \quad \text{and} \quad \psi(t) = \left(\psi_m(t), \frac{1}{t+1}\right)$$

if $t \in [m-1,m]$ for some $m \in \mathbb{N}$. By (3.4), $\xi(m) = \theta = \xi(m+1)$ and ξ and ψ are well defined.

Now, we can define

$$R = \{\psi(t) : t \in [0,\infty)\} \quad \text{and} \quad X = T \cup R.$$

Notice that R is a ray in $\mathbb{R}^3$ and $(\pi(\psi(t)), 0) \in T$ for each $t \in [0,\infty)$. For each $m \in \mathbb{N}$, $\{(m+1)e_1, (m+1)e_2, (m+1)e_3\} \subset \operatorname{Im}\sigma_{\beta_m} = \xi_m([m-1,m])$. Hence, $\{e_1, e_2, e_3\} \subset \operatorname{Im}\psi_m \subset \pi(\operatorname{Im}\psi) \times \{0\}$. This implies that $\{e_1, e_2, e_3\} \subset \operatorname{cl}(\operatorname{Im}\psi)$. Thus, $T \subset \operatorname{cl}(\operatorname{Im}\psi)$. Therefore, X is a compactification of the ray $[0,\infty)$ with remainder T.

A nondegenerate subinterval $[u,v]$ of $[0,\infty)$ is called *basic* provided that there exists $w \in (u,v)$ such that $\xi|[u,w]$ and $\xi|[w,v]$ are linear and $\xi(u) = \theta = \xi(v)$. With an easy induction it can be shown that there exists a unique infinite partition $0 = t_0 < t_1 < \cdots$ of $[0,\infty)$ such that each interval $[t_{j-1}, t_j]$ is basic.

For each $m \in \mathbb{N}$, the interval $[m-1, m]$ is called *canonical of order* 1. An interval $[u, v]$ is called *canonical of order* 2 provided that there exist $m \in \mathbb{N}$ and $\gamma \in M(\beta_m) - \{(1,1,1)\}$ such that $\sigma_\gamma(u, v) \lhd \sigma_{\beta_m}(m-1, m)$. By definition, $\xi|[u, v] = \sigma_\gamma(u, v)$. Inductively, an interval $[u, v]$ is called *canonical of order* $k+1$ provided that there exist a canonical interval $[x, y]$ of order k and $\alpha, \gamma \in \mathbb{N}^3 - \{(1,1,1)\}$ such that $\sigma_\gamma(u, v) \lhd \sigma_\alpha(x, y)$. An interval is called *canonical* if it is canonical of some order.

CLAIM 0. *Suppose that $x \in [0,\infty)$ is such that $2 \le |\xi(x)|$. Then there exist $j, k \in \mathbb{N}$, $i \in \{1,2,3\}$ and $0 \le u < v$ such that $\xi(x) = \sigma_\alpha(x) = \eta(i, k, t_{j-1}, t_j)(x)$, where $\alpha \in \{(k,1,1),(1,k,1),(1,1,k)\} - \{(1,1,1)\}$, $[u, v]$ is the canonical interval that is a domain of σ_α and $\xi|[u, v] = \sigma_\alpha(u, v)$.*

Proof. Let $m \in \mathbb{N}$ be such that $x \in [m-1, m]$. By the inductive definition of β_m, there exist finite sequences $[u_1, v_1], \ldots, [u_r, v_r]$ and $\alpha_1, \ldots, \alpha_r$ such that $\sigma_{\alpha_r}(u_r, v_r) \lhd \sigma_{\alpha_{r-1}}(u_{r-1}, v_{r-1}) \lhd \cdots \lhd \sigma_{\alpha_1}(u_1, v_1) = \sigma_{\beta_m}(m-1, m)$, $x \in [u_r, v_r]$ and r is the maximum possible integer. Since $2 \le |\xi(x)| = |\sigma_{\alpha_r}(x)|$, we have $\alpha_r \ne (1,1,1)$. Let $\alpha = \alpha_r$. By the maximality of r, the interval $[u_r, v_r]$ cannot be partitioned into canonical subintervals, so $\alpha \in \{(k,1,1),(1,k,1),(1,1,k)\}$ for some $k \ge 2$ and $\xi(x) = \sigma_\alpha(x) = \eta(i, k, u_r, v_r)(x)$ for some $i \in \{1,2,3\}$. Let $j \in \mathbb{N}$ be such that $[u_r, v_r] = [t_{j-1}, t_j]$ and let $[u, v] = [u_r, v_r]$. This finishes the proof of Claim 0. ∎

The following claim is the key to proving that X is C^*-smooth.

CLAIM 1. *Let P, Q be subtriods of Y such that $Q \subset P$ and $\lambda_i(Q) > 5$ for each $i \in \{1,2,3\}$. Suppose that $0 \le t < u$ are such that $\xi([t, u]) = P$. Then there exist $v, w \in [t, u]$ such that $v \le w$ and for each $i \in \{1,2,3\}$, $|\lambda_i(\xi([v, w])) - \lambda_i(Q)| \le 3$.*

Proof. We will need two preliminary results.

CLAIM 1.1. *If there exist $\alpha = (a, b, c) \in \mathbb{N}^3 - \{(1,1,1)\}$ and a canonical interval $[x, y]$ such that $\xi|[x, y] = \sigma_\alpha(x, y)$, $[x, y] \subset [t, u]$, and for each $i \in \{1,2,3\}$, $|\lambda_i(\operatorname{Im}\sigma_\alpha) - \lambda_i(P)| \le 3$, then there exist $v, w \in [t, u]$ with the required properties.*

Proof. Since $\lambda_i(P) > 5$ for each $i \in \{1,2,3\}$, by (3.5), $5 < \min\{a, b, c\}$. Let $(a_1, b_1, c_1) \in \mathbb{N}^3$ be such that $Q \subset \theta(a_1 e_1) \cup \theta(b_1 e_2) \cup \theta(c_1 e_3)$ and a_1, b_1, c_1 are minimal (that is, $\theta(a_1 e_1) \cup \theta(b_1 e_2) \cup \theta(c_1 e_3)$ is the minimal subtriod of Y containing Q, with integer length of its legs). By the hypothesis of Claim 1, we find that $5 \le \min\{a_1, b_1, c_1\}$. By the hypothesis of Claim 1.1,

$$\begin{aligned} \operatorname{Im}\sigma_\alpha(x, y) &= \theta(a e_1) \cup \theta(b e_2) \cup \theta(c e_3) \subset P \\ &\subset \theta((a+3)e_1) \cup \theta((b+3)e_2) \cup \theta((c+3)e_3). \end{aligned}$$

Since $Q \subset P$, we get $2 \leq \min\{a_1 - 3, b_1 - 3, c_1 - 3\}$ and $a_1 - 3 \leq a$, $b_1 - 3 \leq b$ and $c_1 - 3 \leq c$. Let $\beta = (a_1 - 3, b_1 - 3, c_1 - 3)$. Notice that either $\beta = \alpha$ or $\beta \in M(\alpha) - \{(1,1,1)\}$. In both cases, by (c), there exists a canonical interval $[v, w]$ such that $[v, w] \subset [x, y] \subset [t, u]$ and $\xi|[v, w] = \sigma_\beta(v, w)$. By (3.5), $\xi([v, w]) = \theta((a_1 - 3)e_1) \cup \theta((b_1 - 3)e_2) \cup \theta((c_1 - 3)e_3)$. Clearly, for each $i \in \{1, 2, 3\}$, $|\lambda_i(\xi([v, w])) - \lambda_i(Q)| \leq 3$. This proves Claim 1.1. ∎

Claim 1.2. *There exist $\alpha \in \mathbb{N}^3 - \{(1,1,1), (2,1,1), (1,2,1), (1,1,2)\}$ and a canonical interval $[x, y]$ such that $t \leq x < y \leq u$ and $\xi|[x, y] = \sigma_\alpha(x, y)$.*

Proof. Let $x_1, x_2 \in [t, u]$ be such that $\xi(x_1)$ and $\xi(x_2)$ are in different legs of Y and $5 \leq \min\{|\xi(x_1)|, |\xi(x_2)|\}$. Let $j_1, j_2, k_1, k_2 \in \mathbb{N}$, $i_1, i_2 \in \{1, 2, 3\}$, $0 \leq u_1 < v_1$, $0 \leq u_2 < v_2$, α_1 and α_2 be as in Claim 0 applied to the points x_1 and x_2, respectively. We may assume that $j_1 = 1$. Since $x_1, x_2 \in [t, u]$, we have $[u_1, x_1] \subset [t, u]$ or $[x_1, v_1] \subset [t, u]$. Notice that $5 \leq k_1$ and the intervals $[u_1, x_1]$ and $[x_1, v_1]$ contain a canonical interval $[x, y]$ which is the domain of the map $\sigma_{(3,1,1)}$. Hence, $t \leq x < y \leq u$ and $\xi|[x, y] = \sigma_{(3,1,1)}(x, y)$. Let $\alpha = (3, 1, 1)$. This proves Claim 1.2. ∎

Let x, y and α be as in Claim 1.2. By (3.2), $\xi(x) = \theta = \xi(y)$. Since $\xi^{-1}(\theta) \cap [t, u]$ is finite, the number of possible intervals $[x, y]$ is finite. From all the possible choices of intervals $[x, y]$, we choose one having minimal order. We analyze two cases.

Case 1: $k = 1$. In this case there exists $m \in \mathbb{N}$ such that $\alpha = \beta_m$ and we can take the maximum such m. Then $[x, y] = [m - 1, m]$. We will prove that for each $i \in \{1, 2, 3\}$, $|\lambda_i(\operatorname{Im} \alpha) - \lambda_i(P)| \leq 1$.

Given $n < m$, by (3.5), we have $\operatorname{Im} \sigma_{\beta_n} \subset \operatorname{Im} \sigma_{\beta_m}$. Then $\operatorname{Im} \xi([1, m]) = \operatorname{Im} \xi([m - 1, m]) = \xi([x, y]) \subset P$. By the maximality of m, $[m, m + 1]$ cannot be contained in $[x, y]$. Hence, $m + 1 \notin [x, y]$. Thus, $[x, y] \subset [1, m + 1)$ and

$$\begin{aligned}\theta((m+1)e_1) \cup \theta((m+1)e_2) \cup \theta((m+1)e_3) &= \operatorname{Im} \sigma_{\beta_m} \subset P \subset \operatorname{Im} \xi([1, m+1)) \\ &\subset \operatorname{Im} \xi([1, m + 1]) = \operatorname{Im} \xi([m, m + 1]) = \operatorname{Im} \sigma_{\beta_{m+1}}(m, m + 1) \\ &= \theta((m + 2)e_1) \cup \theta((m + 2)e_2) \cup \theta((m + 2)e_3).\end{aligned}$$

This implies that for each $i \in \{1, 2, 3\}$, $|\lambda_i(\operatorname{Im} \alpha) - \lambda_i(P)| \leq 1$. By the hypothesis of Claim 1, $\lambda_i(P) > 5$ for each $i \in \{1, 2, 3\}$. Therefore $5 < m + 2$. Hence, we can apply Claim 1.1 to conclude that there exist $v, w \in [t, u]$ with the required properties.

Case 2: $k > 1$. By the definition of k, there exists a canonical interval $[v_0, w_0]$ of order $k - 1$ and there exists $\gamma = (a_1, b_1, c_1) \in \mathbb{N}^3 - \{(1, 1, 1)\}$ such that $\sigma_\gamma(v_0, w_0) = \xi|[v_0, w_0]$ and $\sigma_\alpha(x, y) \lhd \sigma_\gamma(v_0, w_0)$. Then $\alpha \in M(\gamma)$. This implies that $a \leq a_1$, $b \leq b_1$ and $c \leq c_1$ and one of these inequalities is proper. By the minimality of k, $[v_0, w_0]$ is not contained in $[t, u]$. Since $[x, y] \subset [v_0, w_0] \cap [t, u]$, we have $[v_0, w_0] \cap [t, u] \neq \emptyset$.

Let $[v_1, w_1]$ be a canonical interval of order $k-1$ such that $[v_1, w_1]$ is adjacent to $[v_0, w_0]$ (that is, $v_0 = w_1$ or $w_0 = v_1$). By the minimality of k, $[v_1, w_1]$ is not contained in $[t, u]$.

By the choice of α, $\alpha \notin \{(1,1,1), (2,1,1), (1,2,1), (1,1,2)\}$. This implies that there are two different canonical intervals $[v_1, w_1]$ and $[v_2, w_2]$, of order $k-1$, adjacent to the interval $[v_0, w_0]$ and there exist $\zeta, \vartheta \in \mathbb{N}^3 - \{(1,1,1)\}$ such that $\xi|[v_1, w_1] = \sigma_\zeta(v_1, w_1)$ and $\xi|[v_2, w_2] = \sigma_\vartheta(v_2, w_2)$. By the previous paragraph, $[v_0, w_0]$ is not contained in $[t, u]$ and $[v_j, w_j]$ is not contained in $[t, u]$ for each $j \in \{1, 2\}$. This implies that $[t, u] \subset [v_0, w_0] \cup [v_j, w_j]$ for some $j \in \{1, 2\}$. We may assume that $[t, u] \subset [v_1, w_1] \cup [v_0, w_0]$. Let $\zeta = (a_2, b_2, c_2)$.

We will prove that $\lambda_1(P) \le \min\{a_1+1, a_2+1\}$, $\lambda_2(P) \le \min\{b_1+1, b_2+1\}$ and $\lambda_3(P) \le \min\{c_1+1, c_2+1\}$. We only prove that $\lambda_1(P) \le a_1+1$, the rest of the proof is similar. We consider two cases.

If $k = 2$, then there exists $m \in \mathbb{N}$ such that $\gamma = \beta_m$, so $w_0 = m$ and $(a_1, b_1, c_1) = (m+1, m+1, m+1)$. Notice that

$$\begin{aligned} P \subset \xi([0, m] \cup [m, m+1]) &= \beta_{m+1}([m, m+1]) \\ &= \theta((m+2)e_1) \cup \theta((m+2)e_2) \cup \theta((m+2)e_3). \end{aligned}$$

Hence, $\lambda_1(P) \le m+2 = a_1+1$.

If $k > 2$, there exist a canonical interval $[v_3, w_3]$ of order $k-2$, $\kappa \in \mathbb{N}^3 - \{(1,1,1)\}$ and $i \in \{1, \dots, g(\kappa)\}$ such that $\sigma_\gamma(v_0, w_0) \lhd \sigma_\kappa(v_3, w_3)$, $\sigma_\zeta(v_1, w_1) \lhd \sigma_\kappa(v_3, w_3)$, $\gamma = \sigma_{\varphi(\kappa)(i)}$ and $\zeta \in \{\sigma_{\varphi(\kappa)(i-1)}, \sigma_{\varphi(\kappa)(i+1)}\}$. By (d), $a_2 \le a_1 + 1$. Since

$$\begin{aligned} P = \xi([t, u]) &\subset \xi([v_1, w_1] \cup [v_0, w_0]) \\ &= \theta(a_2e_1) \cup \theta(b_2e_2) \cup \theta(c_2e_3) \cup \theta(a_1e_1) \cup \theta(b_1e_2) \cup \theta(c_1e_3), \end{aligned}$$

we obtain $\lambda_1(P) \le a_1 + 1$.

Therefore, $\lambda_1(P) \le \min\{a_1+1, a_2+1\}$, $\lambda_2(P) \le \min\{b_1+1, b_2+1\}$ and $\lambda_3(P) \le \min\{c_1+1, c_2+1\}$.

If the canonical interval $[x_1, y_1]$ contained in $[v_0, w_0]$, where $[x_1, y_1]$ is the domain for $\gamma_{\varphi(\gamma)(g(\gamma))}$, satisfies $[x_1, y_1] \subset [t, u]$, let $\gamma_{\varphi(\gamma)(g(\gamma))} = (a_4, b_4, c_4)$. Then $a_4 \le \lambda_1(P)$ and, by (b) and (d) applied to γ, $|a_1 - a_4| + |b_1 - b_4| + |c_1 - c_4| \le 1$. Thus, $a_1 \le a_4 + 1 \le \lambda_1(P) + 1 \le a_1 + 2$ and $|a_1 - \lambda_1(P)| \le 1$. Proceeding similarly, $\max\{|a_1-\lambda_1(P)|, |b_1-\lambda_2(P)|, |c_1-\lambda_3(P)|\} \le 1$. Hence, $\max\{|a_4-\lambda_1(P)|, |b_4-\lambda_2(P)|, |c_4-\lambda_3(P)|\} \le 2$. Therefore, Claim 1.1 implies that there exist $v, w \in [t, u]$ with the required properties. Hence, we may assume that the canonical interval $[x_1, y_1]$, contained in $[v_0, w_0]$, that is the domain for $\gamma_{\varphi(\gamma)(g(\gamma))}$ is not contained in $[t, u]$.

Similarly, we can assume that the canonical interval $[x_2, y_2]$, contained in $[v_1, w_1]$, that is the domain for $\zeta_{\varphi(\zeta)(g(\zeta))}$ is not contained in $[t, u]$.

We consider two subcases.

SUBCASE 2.1: $[t,u]$ is not contained in $[v_0,w_0]$. We may assume that $w_1=v_0$. Then $v_0\in[t,u]$. By definition, $[v_0,w_0]$ is the domain of the map

$$\sigma_\gamma=\sigma_{\varphi(\gamma)(1)}*\cdots*\sigma_{\varphi(\gamma)(g(\gamma)-1)}*\sigma_{\varphi(\gamma)(g(\gamma))}*\sigma_{\varphi(\gamma)(g(\gamma)-1)}*\cdots*\sigma_{\varphi(\gamma)(1)}$$

and $[v_1,w_1]$ is the domain of

$$\sigma_\zeta=\sigma_{\varphi(\zeta)(1)}*\cdots*\sigma_{\varphi(\zeta)(g(\zeta)-1)}*\sigma_{\varphi(\zeta)(g(\zeta))}*\sigma_{\varphi(\zeta)(g(\zeta)-1)}*\cdots*\sigma_{\varphi(\zeta)(1)}.$$

For each $i\in\{1,\ldots,g(\gamma)\}$, let J_i be the domain on the left of the map $\sigma_{\varphi(\gamma)(i)}$ in the interval $[v_0,w_0]$ and, for each $j\in\{1,\ldots,g(\zeta)\}$, let L_j be the domain on the right of the map $\sigma_{\varphi(\zeta)(j)}$ in the interval $[v_1,w_1]$. Since $v_0\in[t,u]$ and $J_{g(\gamma)}$ and (by the fact we mention three paragraphs above) $L_{g(\zeta)}$ are not contained in $[t,u]$, we see that $[v_0,w_0]\cap[t,u]$ is contained in $J_1\cup\cdots\cup J_{g(\gamma)}$ and $[v_1,w_1]\cap[t,u]$ is contained in $L_1\cup\cdots\cup L_{g(\zeta)}$. Let $i\in\{1,\ldots,g(\gamma)\}$ and $j\in\{1,\ldots,g(\zeta)\}$ be such that $J_{i-1}\subset[t,u]$ and $L_{j-1}\subset[t,u]$ and j and l are maximal (we define $J_0=\{v_0\}=L_0$ in order that i and j be well defined). Then

$$J_1\cup\cdots\cup J_{i-1}\cup L_1\cup\cdots\cup L_{j-1}\subset[t,u]\subset J_1\cup\cdots\cup J_i\cup L_1\cup\cdots\cup L_j.$$

So the only possible intervals of order k in $[t,u]$ are the intervals $J_1,\ldots,J_{i-1}$, $L_1,\ldots,L_{j-1}$. Thus, $1<i$ or $1<j$.

Since $[v_0,w_0]$ and $[v_1,w_1]$ are consecutive intervals of order $k-1$, either they are two intervals of the form $[m-1,m]$ or $[m,m+1]$ (in some order), or there exist $\gamma_0\in\mathbb{N}^3-\{(1,1,1)\}$ and $i_0\in\{2,\ldots,\varphi(\gamma_0)(g(\gamma_0))\}$ such that $\{\gamma,\zeta\}=\{\sigma_{\varphi(\gamma_0)(i_0)},\sigma_{\varphi(\gamma_0)(i_0+1)}\}$. In both cases (see (d)), γ and ζ satisfy the hypothesis of (f).

If $1<i$ and $1<j$, let $A=\{\varphi(\gamma)(l)\in\mathbb{N}^3:1\le l<i\}$, $B=\{\varphi(\zeta)(l)\in\mathbb{N}^3:1\le l<j\}$ and for each $e\in\{1,2,3\}$, let $r_e=\max\pi_e(A)$ and $s_e=\max\pi_e(B)$. By (f), we may assume that

$$K((s_1,s_2,s_3))\subset K((r_1,r_2,r_3)+(1,1,1)).$$

By (d),

$$\begin{aligned}K(\varphi(\gamma)(i))&\subset K(\varphi(\gamma)(i-1)+(1,1,1))\subset K((r_1,r_2,r_3)+(1,1,1)),\\ K(\varphi(\zeta)(j))&\subset K(\varphi(\zeta)(j-1)+(1,1,1))\subset K((s_1,s_2,s_3)+(1,1,1))\\ &\subset K((r_1,r_2,r_3)+(2,2,2)).\end{aligned}$$

Thus, $\varphi(\gamma)(i)\in K((r_1+1,r_2+1,r_3+1))$, so for each $e\in\{1,2,3\}$, $\pi_e(\varphi(\gamma)(i))\le r_e+1$. Similarly, for each $e\in\{1,2,3\}$, $\pi_e(\varphi(\gamma)(i))\le r_e+2$.

Applying (3.5), we obtain

$$\begin{aligned}P=\xi([t,u])&\subset\xi(J_1\cup\cdots\cup J_i\cup L_1\cup\cdots\cup L_j)\\ &=\sigma_{\varphi(\gamma)(1)}(J_1)\cup\cdots\cup\sigma_{\varphi(\gamma)(i)}(J_i)\cup\sigma_{\varphi(\zeta)(1)}(L_1)\cup\cdots\cup\sigma_{\varphi(\zeta)(j)}(L_j)\\ &\subset\theta(r_1+2)e_1\cup\theta(r_2+2)e_2\cup\theta(r_3+2)e_3.\end{aligned}$$

In the case $j = 1$, we have $i > 1$ (the case $i = 1$ is similar). In this case we can also define A, r_1, r_2 and r_3 and we also obtain

$$\begin{aligned} P &\subset \sigma_{\varphi(\gamma)(1)}(J_1) \cup \theta(r_1+2)e_1 \cup \theta(r_2+2)e_2 \cup \theta(r_3+2)e_3 \\ &= \theta(r_1+2)e_1 \cup \theta(r_2+2)e_2 \cup \theta(r_3+2)e_3. \end{aligned}$$

Therefore, we can assume that $i > 1$, A, r_1, r_2 and r_3 are defined, and $P \subset \theta(r_1+2)e_1 \cup \theta(r_2+2)e_2 \cup \theta(r_3+2)e_3$.

On the other hand, by (e), there exists $1 \le k_0 < i$ with

$$\{\varphi(\gamma)(l) \in \mathbb{N}^3 : 1 \le l < i\} \subset K(\varphi(\gamma)(k_0) + (1,1,1)).$$

This implies that

$$r_1 \le \pi_1(\varphi(\gamma)(k_0)) + 1, \quad r_2 \le \pi_2(\varphi(\gamma)(k_0)) + 1, \quad r_3 \le \pi_3(\varphi(\gamma)(k_0)) + 1.$$

Thus,

$$\begin{aligned} &\theta(r_1+2)e_1 \cup \theta(r_2+2)e_2 \cup \theta(r_3+2)e_3 \\ &\quad \subset \theta\big(\pi_1(\varphi(\gamma)(k_0)) + 3\big)e_1 \cup \theta\big(\pi_2(\varphi(\gamma)(k_0)) + 3\big)e_2 \cup \theta\big(\pi_3(\varphi(\gamma)(k_0)) + 3\big)e_3. \end{aligned}$$

By (3.5),

$$\begin{aligned} &\sigma_{\varphi(\gamma)(k_0)}(J_{k_0}) \\ &\quad = \theta\big(\pi_1(\varphi(\gamma)(k_0))\big)e_1 \cup \theta\big(\pi_2(\varphi(\gamma)(k_0))\big)e_2 \cup \theta\big(\pi_3(\varphi(\gamma)(k_0))\big)e_3 \subset P \\ &\quad \subset \theta\big(\pi_1(\varphi(\gamma)(k_0)) + 3\big)e_1 \cup \theta\big(\pi_2(\varphi(\gamma)(k_0)) + 3\big)e_2 \cup \theta\big(\pi_3(\varphi(\gamma)(k_0)) + 3\big)e_3. \end{aligned}$$

We can apply Claim 1.1 to deduce that there exist v and w with the required properties.

Case 2.2: $[t, u]$ is contained in $[v_0, w_0]$. Recall that $[v_0, w_0]$ is the domain of the map

$$\sigma_\gamma = \sigma_{\varphi(\gamma)(1)} * \cdots * \sigma_{\varphi(\gamma)(g(\gamma)-1)} * \sigma_{\varphi(\gamma)(g(\gamma))} * \sigma_{\varphi(\gamma)(g(\gamma)-1)} * \cdots * \sigma_{\varphi(\gamma)(1)}.$$

For each $i \in \{1, \dots, g(\gamma)\}$, let J_i be the domain on the left of the map $\sigma_{\varphi(\gamma)(i)}$ in the interval $[v_0, w_0]$ and let J_i' be the domain on the right of the map $\sigma_{\varphi(\gamma)(i)}$ in the interval $[v_0, w_0]$. Since $J_{g(\gamma)}$ is not contained in $[t, u]$, we see that either $[t, u]$ is contained in $J_1 \cup \cdots \cup J_{g(\gamma)}$ or it is contained in $J_1' \cup \cdots \cup J_{g(\gamma)}'$. We analyze the case $[t, u] \subset J_1 \cup \cdots \cup J_{g(\gamma)}$, the other one is similar. Let $i, j \in \{1, \dots, g(\gamma)\}$ be such that $[t, u] \subset J_i \cup \cdots \cup J_j$ and i is the maximum and j is the minimum. By the choice of $[x, y]$, we note that one of the intervals $J_2, \dots, J_{g(\gamma)-1}$ coincides with $[x, y]$, so i and j are well defined. Then $1 \le i \le j \le g(\gamma)$ and $J_{i+1} \cup \cdots \cup J_{j-1} \subset [t, u] \subset J_i \cup \cdots \cup J_j$.

By (e), there exists $i < k_0 < j$ such that $\{\varphi(\gamma)(l) \in \mathbb{N}^3 : i < l < j\} \subset K(\varphi(\gamma)(k_0) + (1,1,1))$. By (d),

$$K(\varphi(\gamma)(j)) \subset K(\varphi(j-1) + (1,1,1)) \subset K(\varphi(\gamma)(k_0) + (2,2,2)).$$

Similarly, $K(\varphi(\gamma)(i)) \subset K(\varphi(\gamma)(k_0) + (2,2,2))$. This implies that

$$P = \xi([t,u]) \subset \xi(J_i \cup \cdots \cup J_j) = \sigma_{\varphi(\gamma)(i)}(J_i) \cup \cdots \cup \sigma_{\varphi(\gamma)(j)}(J_j)$$
$$\subset \theta\big(\pi_1(\varphi(\gamma)(k_0))+2\big)e_1 \cup \theta\big(\pi_2(\varphi(\gamma)(k_0))+2\big)e_2 \cup \theta\big(\pi_3(\varphi(\gamma)(k_0))+2\big)e_3.$$

By (3.5),

$$\sigma_{\varphi(\gamma)(k_0)}(J_{k_0})$$
$$= \theta\big(\pi_1(\varphi(\gamma)(k_0))\big)e_1 \cup \theta\big(\pi_2(\varphi(\gamma)(k_0))\big)e_2 \cup \theta\big(\pi_3(\varphi(\gamma)(k_0))\big)e_3 \subset P$$
$$\subset \theta\big(\pi_1(\varphi(\gamma)(k_0)) + 2\big)e_1 \cup \theta\big(\pi_2(\varphi(\gamma)(k_0)) + 2\big)e_2 \cup \theta\big(\pi_3(\varphi(\gamma)(k_0)) + 2\big)e_3.$$

We can apply Claim 1.1 to conclude that there exist v and w with the required properties.

This completes the proof of Claim 1. ∎

Claim 2. *X is C^*-smooth.*

Proof. First, consider a triod $A \subset T$ and a sequence of arcs $\{A_n\}_{n=1}^{\infty}$ in R such that $\lim A_n = A$. Let $B \in C(A)$. We need to show that $B \in \lim C(A_n)$, that is, there exists a sequence $\{B_n\}_{n=1}^{\infty}$ such that $B_n \in C(A_n)$ for each $n \in \mathbb{N}$ and $\lim B_n = B$. If B is a one-point set, it is easy to see that $B \in \lim C(A_n)$. So, suppose that B is nondegenerate.

First, we consider the case that B is a triod. Then $\theta \in B$ and $\lambda_0 = \min\{\lambda_1(B), \lambda_2(B), \lambda_3(B)\}$ is positive. For each $n \in \mathbb{N}$, choose $B_n \in C(A_n)$ such that $H(B, B_n) = \min\{H(B, D) : D \in C(A_n)\}$; we need to show that $\lim B_n = B$.

Take $\varepsilon > 0$. We are going to find $N \in \mathbb{N}$ such that for each $n \geq N$, $H(B, B_n) < \varepsilon$. Let $B_0 \in C(B)$ be such that B_0 is a triod, $H(B, B_0) < \varepsilon/4$ and $\lambda_0/2 < \lambda_i(B_0) < \lambda_i(B)$ for each $i \in \{1,2,3\}$. Since $B \subset A = \lim(\pi(A_n) \times \{0\})$, there exists $N_1 \in \mathbb{N}$ such that $10/N_1 < \min\{\lambda_0, \varepsilon\}$ and $B_0 \subset \pi(A_n) \times \{0\}$ for all $n \geq N_1$.

For each $n \in \mathbb{N}$, let $A_n = \psi([t_n, u_n])$, with $0 \leq t_n \leq u_n$. Since $\lim A_n = A$, $\lim t_n = \infty = \lim u_n$. Thus, there exists $N_2 \in \mathbb{N}$ such that $N_1 \leq t_n$ for each $n \geq N_2$. Let $n = \max\{N_1, N_2\}$.

Let $n \geq N$. We consider two cases.

Case 1: There exists $m \in \mathbb{N}$ such that $[m-1, m] \subset [t_n, u_n]$. In this case

$$B_0 \subset T = \psi_m([m-1,m]) = \frac{1}{m+1}\xi([m-1,m]) \subset \frac{1}{m+1}\xi([t_n,u_n]).$$

Thus, $(m+1)B_0$ is a triod contained in $\xi([m-1,m])$. Since $m \geq t_n \geq N_1$, $10/(m+1) < 10/N_1 < \lambda_0 \leq \min\{2\lambda_1(B_0), 2\lambda_2(B_0), 2\lambda_3(B_0)\}$. This implies that $5 < \min\{(m+1)\lambda_1(B_0), (m+1)\lambda_2(B_0), (m+1)\lambda_3(B_0)\}$. By Claim 1, there exist $v, w \in [m-1, m]$ such that $v \leq w$ and for each $i \in \{1,2,3\}$, we have $|\lambda_i(\xi([v,w])) - \lambda_i((m+1)B_0)| \le 3$. Therefore,

$$\begin{aligned}\big|\lambda_i\big(\pi(\psi([v,w]))\times\{0\}\big)-\lambda_i(B_0)\big| &= |\lambda_i(\psi_m([v,w]))-\lambda_i(B_0)|\\ &= \left|\lambda_i\left(\frac{1}{m+1}(\xi([v,w]))\right)-\lambda_i(B_0)\right| \le \frac{3}{m+1} < \frac{\varepsilon}{3}.\end{aligned}$$

Given $t \in [u,v]$, there exists $i \in \{1,2,3\}$ such that $\big(\pi(\psi(t)),0\big) \in \theta e_i$. Since $\big|\lambda_i\big(\pi(\psi([v,w]))\times\{0\}\big)-\lambda_i(B_0)\big| < \varepsilon/3$, there exists $q \in B_0 \cap \theta e_i$ such that $|\psi_m(t)-q| = \big|\big(\pi(\psi(t)),0\big)-q\big| < \varepsilon/3$. Since $m-1 \le v \le t$, we get $1/(t+1) \le 1/m < \varepsilon/9$. Thus, $|\psi(t)-q| < \varepsilon/2$. We have shown that $\psi([u,v]) \subset N(\varepsilon/2, B_0)$. Similarly, $B_0 \subset N(\varepsilon/2, \psi([u,v]))$. Hence, $\psi([v,w])$ is a subcontinuum of A_n such that $H(\psi([v,w]),B_0) < \varepsilon/2$. Therefore, $H(\psi([v,w]),B) < \varepsilon$. This implies that $H(B_n,B) < \varepsilon$.

Case 2: For each $m \in \mathbb{N}$, $[m-1,m]$ is not contained in $[t_n,u_n]$. In this case, there exists $m \in \mathbb{N}$ such that $[t_n,u_n] \subset [m-1,m+1]$. We suppose that $m \in [t_n,u_n]$; the reasoning for $[t_n,u_n] \subset [m-1,m]$ is similar but easier. Note that $A_n = \psi([t_n,u_n]) = \psi([t_n,m]) \cup \psi([m,u_n])$. Let

$$D_1 = \big(\pi(\psi([t_n,m])\times\{0\})\big)\cap B_0, \qquad D_2 = \big(\pi(\psi([m,u_n]))\times\{0\}\big)\cap B_0.$$

Since $B_0 \subset \pi(A_n)\times\{0\}$, we have $B_0 = D_1 \cup D_2$. Since

$$D_1 \subset \pi(\psi([m-1,m])) = \psi_m([m-1,m]) = \frac{1}{m+1}\xi_m([m-1,m]),$$

we have $(m+1)D_1 \subset \xi_m([t_n,m])$. Similarly, $(m+2)D_1 \subset \xi_{m+1}([m,u_n])$. Thus,

$$(m+1)B_0 = (m+1)(D_1\cup D_2) \subset \xi_m([t_n,m]) \cup \xi_{m+1}([m,u_n]) = \xi([t_n,u_n]).$$

As in Case 1, we deduce

$$5 < \min\{(m+1)\lambda_1(B_0),(m+1)\lambda_2(B_0),(m+1)\lambda_3(B_0)\}.$$

By Claim 1, there exist $v,w \in [t_n,u_n]$ such that $v \le w$ and for each $i \in \{1,2,3\}$, $|\lambda_i(\xi([v,w]))-\lambda_i((m+1)B_0)| \le 3$. Given $t \in [v,w]\cap[m-1,m]$, there exists $i \in \{1,2,3\}$ such that $\big(\pi(\psi(t)),0\big) \in \theta e_i$. Since

$$\big(\pi(\psi(t)),0\big) = \psi_m(t) = \frac{1}{m+1}\xi_m(t) = \frac{1}{m+1}\xi(t), \qquad \xi(t) \in \xi([v,w])\cap\theta e_i,$$

there exists $q \in B_0$ such that $|\xi(t)-(m+1)q| < 3$. Thus, $\big|\big(\pi(\psi(t)),0\big)-q\big| < 3/(m+1) < \varepsilon/3$. Hence, $\big(\pi(\psi(t)),0\big) \in N(\varepsilon/3,B_0)$. Similarly, for each $t \in [v,w]\cap[m,m+1]$, we have $\big(\pi(\psi(t)),0\big) \in N(\varepsilon/3,B_0)$. This implies that $\psi(t) \in N(\varepsilon/2,B_0)$. We have proved that $\psi([v,w]) \subset N(\varepsilon/2,B_0)$. Similarly, $B_0 \subset N(\varepsilon/2,\psi([v,w]))$. Hence, $\psi([v,w])$ is a subcontinuum of A_n such that $H(\psi([v,w]),B_0) < \varepsilon/2$. Therefore, $H(\psi([v,w]),B) < \varepsilon$. This implies that $H(B_n,B) < \varepsilon$.

This completes the proof that $\lim B_n = B$.

Now, we consider the case that B is an arc and $\theta \in B$. Since A is a triod, there exists a sequence $\{B'_m\}_{m=1}^{\infty}$ of triods in A such that $\lim B'_m = B$. By the first case we considered, for each $m \in \mathbb{N}$, there exists a sequence $\{B_n^{(m)}\}_{n=1}^{\infty}$ such that for each $n \in \mathbb{N}$, $B_n^{(m)} \in C(A_n)$ and $\lim B_n^{(m)} = B'_m$. Now, it is easy to see $B \in \lim C(A_n)$.

Finally, we consider the case that B is an arc and $\theta \notin B$. We assume that B is nondegenerate. Since B is the limit of its proper subarcs, as in the paragraph above, it is enough to show that, if B_0 is a proper subarc of B and B_0 does not contain the end points of B, then $B_0 \in \lim C(A_n)$. Notice that there exists $N \in \mathbb{N}$ such that $B_0 \subset \pi(A_n) \times \{0\}$ for each $n \geq N$. Since maps onto arcs are weakly confluent, for each $n \geq N$ there exists a subarc $\psi([v_n, w_n])$ of A_n such that $B_0 = \pi(\psi([v_n, w_n])) \times \{0\}$. Clearly, $\lim \psi([v_n, w_n]) = B_0$.

This completes the proof that if A is a subtriod of T and $\{A_n\}_{n=1}^{\infty}$ is a sequence of arcs in R such that $\lim A_n = A$, then $C(A) \subset \lim C(A_n)$.

From this, it is easy to conclude that X is C^*-smooth, as asserted in Claim 2. ∎

By Theorem 2.2, for each continuum Z the induced map $C(\pi_X^Z) : C(X \times Z) \to C(X)$ is not open.

This completes the proof of the properties of the example X.

4. Chainable continua. For every $n \in \mathbb{N}$ and $i \in \{1, \dots, n\}$, let $\rho_i : \mathbb{R}^n \to \mathbb{R}$ be the projection on the ith coordinate. Given a map $g : [a, b] \to [-1, 1]$, let $\mathrm{Gr}(g) = \{(t, g(t)) \in \mathbb{R}^2 : t \in [a, b]\}$. Given a continuum X, $B \in C(X)$ and $\varepsilon > 0$, let $B^H(\varepsilon, B)$ be the ε-ball around B in $C(X)$.

The classical Mountain Climbing Theorem [10, Theorem 1] claims that if $f, g : [0, 1] \to [0, 1]$ are piecewise monotone maps such that $f(0) = 0 = g(0)$ and $f(1) = 1 = g(1)$, then there exist piecewise monotone maps α, β such that $\alpha(0) = 0 = \beta(0)$ and $\alpha(1) = 1 = \beta(1)$. In this theorem the word "monotone" can be changed to "linear" [7, Theorem 2]. The linear version can be used to prove the following lemma.

LEMMA 4.1. *Let $f, g : [a, b] \to [r, s]$ be piecewise linear maps such that $f(a) = r$ and $f(b) = s$. Then there exist piecewise linear maps $\alpha, \beta : [0, 1] \to [a, b]$ such that $f \circ \alpha = g \circ \beta$, $\beta(0) = a$ and $\beta(1) = b$.*

LEMMA 4.2. *Let $g : [a, b] \to [r, s] \subset [-1, 1]$ be a piecewise linear map, where $0 < a < b \leq 1$. Let $[r, s] = \mathrm{Im}\, g$ and let $c, e \in [a, b]$ be such that $g(c) = r$ and $g(e) = s$. Let B be a subcontinuum of $\{0\} \times [-1, 1] \times [0, 1]$ such that $\rho_2(B) = [r, s]$. Let $t_0, t_1 \in [0, 1]$ be such that $(0, r, t_0), (0, s, t_1) \in B$. Then there exists a subcontinuum E of $\mathrm{Gr}(g) \times [0, 1]$ such that $\mathrm{Gr}(g) = \{(\rho_1(w), \rho_2(w)) : w \in E\}$, $(c, r, t_0), (e, s, t_1) \in E$ and $H(B, E) < 2b$.*

Proof. Let $\sigma : [a,b] \to \{0\} \times [r,s] \times [0,1]$ be a piecewice linear map such that $\sigma(a) = (0,r,t_0)$, $\sigma(b) = (0,s,t_1)$, a is the unique value for which $\rho_2(\sigma(a)) = r$, and b is the unique value for which $\rho_2(\sigma(b)) = s$ and $H(\operatorname{Im}\sigma, B) < b$. Since $\rho_2(\sigma(a)) = r$ and $\rho_2(\sigma(b)) = s$, we can apply Lemma 4.1, so there exist piecewise linear maps $\alpha, \beta : [0,1] \to [a,b]$ such that $\rho_2 \circ \sigma \circ \alpha = g \circ \beta$, $\beta(0) = a$ and $\beta(1) = b$.

Let $\varphi : [0,1] \to \mathbb{R}^3$ be given by

$$\varphi(t) = (\beta(t), g(\beta(t)), (\rho_3 \circ \sigma \circ \alpha)(t)).$$

Let $E = \operatorname{Im}\varphi$. Then E is a subcontinuum of $\operatorname{Gr}(g) \times [0,1]$. Since $\varphi(0) = (a, g(a), (\rho_3 \circ \sigma \circ \alpha)(a))$ and $\varphi(1) = (b, g(b), (\rho_3 \circ \sigma \circ \alpha)(b))$, we find that $\operatorname{Im}(\rho_1 \circ \varphi, \rho_2 \circ \varphi) = \operatorname{Gr}(g)$. Thus, $\operatorname{Gr}(g) = \{(\rho_1(e), \rho_2(e)) : e \in E\}$. Given $t \in [0,1]$, we deduce $\big|\big(\beta(t), g(\beta(t)), (\rho_3 \circ \sigma \circ \alpha)(t)\big) - \sigma \circ \alpha(t)\big| = \beta(t) \le b$. Hence, $H(E, \operatorname{Im}(\sigma \circ \alpha)) \le b$.

Since β is onto, there exist $t_2, t_3 \in [0,1]$ such that $\beta(t_2) = c$ and $\beta(t_3) = e$. Then $\rho_2\big(\sigma(\alpha(t_2))\big) = g(\beta(t_2)) = g(c) = r$ and $\rho_2\big(\sigma(\alpha(t_3))\big) = g(\beta(t_3)) = g(e) = s$. By the choice of σ, $\alpha(t_2) = a$ and $\alpha(t_3) = b$. Thus, $\operatorname{Im}\alpha = [a,b]$ and $\operatorname{Im}\sigma = \operatorname{Im}(\sigma \circ \alpha)$. Therefore, $H(B,E) < 2b$. Finally, $\varphi(t_2) = \big(\beta(t_2), g(\beta(t_2)), (\rho_3 \circ \sigma \circ \alpha)(t_2)\big) = (c, r, t_0)$ and $\varphi(t_3) = \big(\beta(t_3), g(\beta(t_3)), (\rho_3 \circ \sigma \circ \alpha)(t_3)\big) = (e, s, t_1)$. Therefore, $(c,r,t_0), (e,s,t_1) \in E$. ■

In Example 37 of [2], it was shown that the $\sin(1/x)$-continuum has the open projection property. We do not know if this result can be extended to every compactification of the ray $[0,1)$ with an arc as remainder (see Problem 4.4 below). For this family of continua we have the following partial result.

THEOREM 4.3. *Let X be a compactification of the ray $[0,\infty)$ such that the remainder of X is an arc. Then $C(\pi_X^{[0,1]}) : C(X \times [0,1]) \to C(X)$ is open.*

Proof. By [9, Lemma 5.1, p. 20], we may assume that there exists a map $g : (0,1] \to [-1,1]$ such that

$$X = (\{0\} \times [-1,1]) \cup \{(t, g(t)) \in \mathbb{R}^2 : t \in (0,1]\}.$$

Let $R = \{0\} \times [-1,1]$ and $S = \{(t,g(t)) \in \mathbb{R}^2 : t \in (0,1]\}$. It is easy to show that we may also assume that for each $n \in \mathbb{N} - \{1\}$, $g(1/n) = (-1)^n$. Let $J_n = [1/n, 1/(n-1)]$. Notice that there exists a piecewise linear map $g_n^0 : J_n \to [-1,1]$ such that for each $t \in J_n$, $|g(t) - g_n^0(t)| < 1/n$, $g(1/n) = g_n^0(1/n)$ and $g(1/(n-1)) = g_n^0(1/(n-1))$. Clearly, X is homeomorphic to $(\{0\} \times [-1,1]) \cup \{(t, g_n^0(t)) \in \mathbb{R}^2 : n \in \mathbb{N}$ and $t \in J_n\}$. Therefore, we may assume that, for every $0 < a < b \le 1$, $g|_{[a,b]}$ is piecewise linear.

In order to see that $C(\pi_X^{[0,1]})$ is open, let $B \in C(X \times [0,1])$ and let $\mathcal{U}$ be an open subset of $C(X \times [0,1])$ such that $B \in \mathcal{U}$. Let $\varepsilon > 0$ be such that

$B^H(\varepsilon, B) \subset \mathcal{U}$. We need to show that $A = \pi_X^{[0,1]}(B) \in \operatorname{int}_{C(X)}(\pi_X^{[0,1]}(\mathcal{U}))$. In the case that A is degenerate, this claim follows from Proposition 13 of [2]. Thus, we assume that A is nondegenerate. We consider three cases.

CASE 1: $A \subset S$. This case follows from Proposition 14 of [2].

CASE 2: $A \subset R$. Let $\delta > 0$ be such that $4\delta < \varepsilon$ and, in the case that $A \neq R$, we also ask that, for each $E \in B^H(\delta, A)$, $R \subsetneq E$. Let $E \in C(X)$ be such that $H(A, E) < \delta$. If E is degenerate, let $E = \{p\}$. Let $q \in A$ be such that $|p - q| < \delta$. Let $F = E \times \rho_3(B)$. Then F is a subcontinuum of $X \times [0,1]$ such that $\pi_X^{[0,1]}(F) = E$, $H(F, B) < \varepsilon$ and $F \in \mathcal{U}$. Hence, we suppose that E is nondegenerate.

In the case that $E \subset R$, there exists an onto map $h : A \to E$ such that $|h(z) - z| < \delta$ for each $z \in A$. Let $F = (h \times \mathrm{Id}_{[0,1]})(B)$. Clearly, F is a subcontinuum of $X \times [0,1]$ such that $\pi_X^{[0,1]}(F) = E$, $H(F, B) < \varepsilon$ and $F \in \mathcal{U}$. Therefore, $E \in \pi_X^{[0,1]}(\mathcal{U})$.

Now, we suppose that $E \cap R = \emptyset$. Then there exist $a, b \in [0,1]$ such that $0 < a < b \leq 1$ and $E = \mathrm{Gr}(g|_{[a,b]})$. Since $H(A, E) < \delta$, $b < \delta$. Let $E_1 = \{0\} \times \rho_2(E)$. Then $H(E_1, E) < \delta$. Since $H(A, E_1) < 2\delta$, there exists an onto map $h : A \to E_1$ such that $|h(z) - z| < 2\delta$ for each $z \in A$. Let $F_1 = (h \times \mathrm{Id}_{[0,1]})(B)$. Clearly, F_1 is a subcontinuum of X such that $F_1 \subset \{0\} \times [-1,1] \times [0,1]$, $\pi_X^{[0,1]}(F_1) = E_1$ and $H(F_1, B) < 2\delta$. Let $[r, s] = g([a,b]) = \rho_2(E) = \rho_2(E_1) = \rho_2(F_1)$. By Lemma 4.2, applied to the map $g|_{[a,b]}$ and the subcontinuum F_1 of $\{0\} \times [-1,1] \times [0,1]$, we deduce that there exists a subcontinuum F of $\mathrm{Gr}(g|_{[a,b]}) \times [0,1] = E \times [0,1]$ such that $E = \mathrm{Gr}(g|_{[a,b]}) = \{(\rho_1(w), \rho_2(w)) : w \in F\}$ and $H(F_1, F) < 2b$. Thus, $\pi_X^{[0,1]}(F) = E$, $H(F, B) < 4\delta < \varepsilon$ and $F \in \mathcal{U}$. Therefore, $E \in \pi_X^{[0,1]}(\mathcal{U})$.

Finally, suppose that $R \subsetneq E$. In this case, E is of the form $E = R \cup \{(t, g(t)) \in \mathbb{R}^2 : t \in (0, b_0]\}$ for some $b_0 > 0$. Since $H(E, A) < \delta$, $b_0 < \delta$. By the choice of δ, $A = R$. Let $N = \min\{n \in \mathbb{N} : 1/2n \in [0, b_0]\}$. Let $g_N = g|_{[1/(2N+1), b_0]}$, $E_N = \mathrm{Gr}(g_N)$ and for each $n > N$, let $g_n = g|_{[1/(N+n+1), 1/(N+n)]}$ and $E_n = \mathrm{Gr}(g_n)$. Notice that $E = R \cup \bigcup\{E_n : n \geq N\}$ and $\operatorname{Im} g_n = \rho_2(E_n) = [-1, 1]$ for each $n \geq N$. Since $A = R$, there exist $t_{-1}, t_1 \in [0,1]$ such that $(0, -1, t_{-1}), (0, 1, t_1) \in B$. For each $n \geq N$, let u_n (resp., v_n) be the even (resp., odd) number of the set $\{N + n, N + n + 1\}$), $c_n = 1/v_n$ and $e_n = 1/u_n$. Then $g_n(c_n) = -1$ and $g_n(e_n) = 1$. Thus, we can apply Lemma 4.2 to B and g_n and infer that there exists a subcontinuum F_n of $\mathrm{Gr}(g_n) \times [0,1]$ such that $\mathrm{Gr}(g_n) = \{(\rho_1(w), \rho_2(w)) : w \in F_n\}$, $(c_n, -1, t_{-1}), (e_n, 1, t_1) \in F_n$, and if $n > N$, then $H(B, F_n) < 2/(N + n) < 1/N \leq 2b_0$ and $H(B, F_N) < 2b_0$. Let $F = B \cup \bigcup\{F_n : n \geq N\}$. Since $\lim F_n = B$, F is compact. Given $n \geq N$, since $\left(\frac{1}{N+n+1}, (-1)^{N+n+1}, t_{(-1)^{N+n+1}}\right) \in F_n \cap F_{n+1}$, we see that F is connected. Hence, F is a subcontinuum of $X \times [0,1]$. Notice that $H(B,F) < 2b_0 < 2\delta$. Hence, $F \in \mathcal{U}$. Finally, $\pi_X^{[0,1]}(F) = \pi_X^{[0,1]}(B) \cup \bigcup\{\pi_X^{[0,1]}(F_n) : n \geq N\} = A \cup \bigcup\{E_n : n \geq N\} = E$. Hence, $E \in \pi_X^{[0,1]}(\mathcal{U})$.

We have shown that, in this case, $A \in \mathrm{int}_{C(X)}(\pi_X^{[0,1]}(\mathcal{U}))$.

Case 3: $R \subsetneq A$. In this case, A is of the form $A = R \cup \{(t, g(t)) \in \mathbb{R}^2 : t \in (0, a_0]\}$ for some $a_0 > 0$. Let $\delta > 0$ be such that $12\delta < \min\{\varepsilon, a_0\}$ and, if $E \in C(X)$, $R \subset E$ and $H(A,E) < \delta$, then there exists a homeomorphism $h : A \to E$ such that $|a - h(a)| < \varepsilon/2$ for each $a \in A$. We can also ask that if $E \in C(X)$ and $H(A,E) < \delta$, then $\rho_1(E)$ is nondegenerate and $E \cap A \neq \emptyset$. Let $N \in \mathbb{N}$ be such that N is even and $1/N < \delta < a_0 - \delta$. Take $E \in C(X)$ such that $H(A,E) < 1/(N+1) < a_0 - \delta$. Since $R \subset A$, there exists $x_0 \in (0,1]$ such that $(x_0, g(x_0)) \in E$ and $x_0 < 1/(N+1)$. Moreover, there exists $x_1 \in (0,1]$ such that $(x_1, g(x_1)) \in E$ and $|a_0 - x_1| < a_0 - \delta$. Then $1/N < \delta < x_1$. Thus, $\{(x, g(x)) \in X : 1/(N+1) \leq x \leq 1/N\} \subset E$.

In the case that $R \subset E$, by the choice of δ, there exists a homeomorphism $h : A \to E$ such that $|a - h(a)| < \varepsilon/2$ for each $a \in A$. Let $F = (h \times \mathrm{Id}_{[0,1]})(B)$. Then F is a subcontinuum of $X \times [0,1]$ such that $\pi_X^{[0,1]}(F) = E$, $H(F,B) < \varepsilon/2$ and $F \in \mathcal{U}$. Therefore, $E \in \pi_X^{[0,1]}(\mathcal{U})$.

Now, suppose that $E \cap R = \emptyset$. In this case, there exist $u, v \in [0,1]$ such that $0 < u < v \leq 1$ and $E = \mathrm{Gr}(g|_{[u,v]})$. Then $u \leq 1/(N+1) < 1/N < v$ (since $1/N < x_1 \leq v$). Let $A_0 = R \cup \{(x, g(x)) \in X : x \in (0, v]\}$. Since $H(A,E) < \delta$, we have $H(A, A_0) < \delta$. By the paragraph above, there exists a subcontinuum F_0 of $X \times [0,1]$ such that $\pi_X^{[0,1]}(F_0) = A_0$ and $H(F_0, B) < \varepsilon/2$. Since $\mathrm{Gr}(g|_{[1/(N+1), 1/N]}) \subset E$, the set $M_0 = (\{1/N\} \times \{1\} \times [0,1]) \cap F_0$ is nonempty.

Let

$$M = \bigcup\{\{1/N\} \times \{1\} \times ([r - \delta/2, r + \delta/2] \cap [0,1] : (1/N, 1, r) \in F_0\}$$

and $F_1 = F_0 \cup M$. Notice that F_1 is a continuum, $H(F_1, F_0) < \delta < \varepsilon/4$, $H(F_1, B) < 3\varepsilon/4$, $\pi_X^{[0,1]}(F_1) = A_0$, $M = (\{1/N\} \times \{1\} \times [0,1]) \cap F_1$ and M has a finite number of components $D_1, \ldots, D_k$. Let

$$M^- = ([0, 1/N] \times [-1,1] \times [0,1]) \cap F_1, \quad M^+ = ([1/N, 1] \times [-1,1] \times [0,1]) \cap F_1.$$

Notice that M^-, M^+ are closed subsets of F_1 such that $F_1 = M^- \cup M^+$, $M = M^- \cap M^+$, $\mathrm{Fr}_{F_1}(M^-) \subset M$ and, since $1/N < v$, $M^- \neq F_1$.

Given a component C of M^-, by [8, Theorem 20.3], $C \cap M \neq \emptyset$. Since $M \subset M^-$ and M has a finite number of components, we deduce that M^- has a finite number of components. Similarly, M^+ has a finite number of components. Since $R \subset \pi_X^{[0,1]}(F_1)$, we can take the components $C_1, \ldots, C_m$ of M^- such that $\rho_1(C_i) \cap [0, 1/(N+1)] \neq \emptyset$. For each $i \in \{1, \ldots, m\}$, let $J_i =$

$\{j \in \{1,\dots,k\} : C_i \cap D_j \neq \emptyset\}$. Since $\emptyset \neq C_i \cap M$, we have $J_i \neq \emptyset$. Given $j \in J_i$, choose a point $(1/N, 1, t_i^j) \in C_i \cap D_j \subset M$. Let

$$B_i = \{(0, \rho_2(p), \rho_3(p)) \in \{0\} \times [-1,1] \times [0,1] : p \in C_i\}.$$

Then B_i is a continuum. Since $\rho_1(C_i) \cap [0, 1/(N+1)] \neq \emptyset$, we can choose a point $(1/(N+1), -1, s_i) \in C_i$. Then $\rho_2(B_i) = \rho_2(C_i) = [-1,1]$. For each $j \in J_i$, we apply Lemma 4.2 to $g|_{[u,1/N]}$, $c_i = 1/(N+1)$, $e_i^j = 1/N$, $r = -1$, $s = 1$, s_i and t_i^j and B_i to obtain a subcontinuum G_i^j of $\mathrm{Gr}(g|_{[u,1/N]}) \times [0,1]$ such that $\mathrm{Gr}(g|_{[u,1/N]}) = \{(\rho_1(w), \rho_2(w)) : w \in G_i^j\}$, $(1/(N+1), -1, s_i), (1/N, 1, t_i^j) \in G_i^j$ and $H(B_i, G_i^j) < 2/N$. Define $G_i = \bigcup\{G_i^j : j \in J_i\}$. Since each G_i^j contains the point $(1/(N+1), -1, s_i)$, G_i is a subcontinuum of X. Notice that $\mathrm{Gr}(g|_{[u,1/N]}) = \{(\rho_1(w), \rho_2(w)) : w \in G_i\}$, $G_i \cap D_j \neq \emptyset$ for each $j \in J_i$ and $H(B_i, G_i) < 2/N$.

Let $F = M^+ \cup (M^- - (C_1 \cup \dots \cup C_m)) \cup (G_1 \cup \dots \cup G_m)$. Clearly, F is a compact subset of $X \times [0,1]$.

Let $i \in \{1,\dots,m\}$ and let D be a component of M^+ such that $C_i \cap D \neq \emptyset$. Let $z \in C_i \cap D \subset M$. Then there exists $j \in \{1,\dots,k\}$ such that $z \in D_j$. Note that $D_j \subset D$ and $j \in J_i$, so $G_i \cap D \neq \emptyset$.

We are ready to show that F is connected. Let $\mathcal{A} = \{K : K$ is a component of M^+ or K is a component of $M^-\}$ and $\mathcal{B} = \{K : K$ is a component of M^+ or K is a component of $M^- - (C_1 \cup \dots \cup C_m)\} \cup \{G_1, \dots, G_m\}$. Notice that $\mathcal{A}$ (resp., $\mathcal{B}$) is finite, its elements are compact and the union of the elements of $\mathcal{A}$ (resp., $\mathcal{B}$) is F_1 (resp., F). Given two elements $R, S \in \mathcal{B}$, let $R_1 = R$ if $R_1 \notin \{G_1, \dots, G_m\}$ and $R_1 = C_i$ if $R = G_i$ for some $i \in \{1,\dots,m\}$. Define S_1 similarly. Then $R_1, S_1 \in \mathcal{A}$. Since F_1 is connected there exists a finite sequence $R_1 = T_1, T_2, \dots, T_{l-1}, T_l = S_1$ such that $T_h \cap T_{h+1} \neq \emptyset$ for each $h < l$. Define a sequence $Q_1, \dots, Q_l$ by making $Q_h = T_h$ if $Q_h \notin \{C_1, \dots, C_m\}$ and $Q_h = G_i$ if $T_h = C_i$ for some $i \in \{1,\dots,m\}$. By the paragraph above, $Q_h \cap Q_{h+1} \neq \emptyset$ for each $h < l$. It follows that F is connected.

Since $\pi_X^{[0,1]}(F_1) = A_0$ and $M^+ = ([1/N, 1] \times [-1,1] \times [0,1]) \cap F_1$, we have $\pi_X^{[0,1]}(M^+) = \mathrm{Gr}(g|_{[1/N,v]})$. Notice that $\pi_X^{[0,1]}(M^- - (C_1 \cup \dots \cup C_m)) \subset \mathrm{Gr}(g|_{[1/(N+1),1/N]})$ and $\pi_X^{[0,1]}(G_1 \cup \dots \cup G_m) = \mathrm{Gr}(g|_{[u,1/N]})$. Thus, $\pi_X^{[0,1]}(F) = \mathrm{Gr}(g|_{[u,v]}) = E$.

Given $p \in G_1 \cup \dots \cup G_m$, there exists $q \in B_1 \cup \dots \cup B_m$ such that $|p - q| < 2/N$. Then there exists $r \in C_1 \cup \dots \cup C_m \subset ([0, 1/N] \times [-1,1] \times [0,1]) \cap F_1$ such that $|q - r| \leq 1/N$. Thus, $|p - r| < 3/N < 3\delta < \varepsilon/4$. Similarly, for each $r \in C_1 \cup \dots \cup C_m$, there exists $p \in G_1 \cup \dots \cup G_m$ such that $|p - r| < \varepsilon/4$.

This implies that $H(F_1, F) < \varepsilon/4$. Therefore, $H(B, F) < \varepsilon$ and $F \in \mathcal{U}$. This proves $E \in \pi_X^{[0,1]}(\mathcal{U})$.

We have shown that also in this case $A \in \mathrm{int}_{C(X)}(\pi_X^{[0,1]}(\mathcal{U}))$. ∎

Problem 4.4. *Let X be a compactification of the ray $[0, \infty)$ such that the remainder of X is an arc. Does X have the open projection property?*

Problem 4.5. *Let X be a chainable continuum. Does X have the open projection property? Is the map $C(\pi_X^{[0,1]}) : C(X \times [0, 1]) \to C(X)$ open?*

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