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L^p-L^q ESTIMATES FOR SOME CONVOLUTION OPERATORS WITH SINGULAR MEASURES ON THE HEISENBERG GROUP

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Abstract. We consider the Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$. Let ν be the Borel measure on $\mathbb{H}^n$ defined by $\nu(E) = \int_{\mathbb{C}^n} \chi_E(w, \varphi(w))\eta(w)\,dw$, where $\varphi(w) = \sum_{j=1}^n a_j|w_j|^2$, $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$, $a_j \in \mathbb{R}$, and $\eta(w) = \eta_0(|w|^2)$ with $\eta_0 \in C_c^\infty(\mathbb{R})$. We characterize the set of pairs (p, q) such that the convolution operator with ν is $L^p(\mathbb{H}^n)$-$L^q(\mathbb{H}^n)$ bounded. We also obtain L^p-improving properties of measures supported on the graph of the function $\varphi(w) = |w|^{2m}$.

1. Introduction. Let $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ be the Heisenberg group with group law $(z, t) \cdot (w, s) = (z + w, t + s + \langle z, w\rangle)$ where $\langle z, w\rangle = \frac{1}{2}\operatorname{Im}(\sum_{j=1}^n z_j \cdot \overline{w_j})$. For $x = (x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n}$, we write $x = (x', x'')$ with $x' \in \mathbb{R}^n$, $x'' \in \mathbb{R}^n$. So, $\mathbb{R}^{2n}$ can be identified with $\mathbb{C}^n$ via the map $\Psi(x', x'') = x' + ix''$. In this setting the form $\langle z, w\rangle$ agrees with the standard symplectic form on $\mathbb{R}^{2n}$. Thus $\mathbb{H}^n$ can be viewed as $\mathbb{R}^{2n} \times \mathbb{R}$ endowed with the group law

$$(x, t) \cdot (y, s) = \big(x + y, t + s + \tfrac{1}{2}W(x, y)\big)$$

where the symplectic form W is given by $W(x, y) = \sum_{j=1}^n (y_{n+j}x_j - y_j x_{n+j})$, with $x = (x_1, \ldots, x_{2n})$ and $y = (y_1, \ldots, y_{2n})$, with neutral element $(0, 0)$, and with inverse $(x, t)^{-1} = (-x, -t)$.

Let $\varphi : \mathbb{R}^{2n} \to \mathbb{R}$ be a measurable function and let ν be the Borel measure on $\mathbb{H}^n$ supported on the graph of φ, given by

$$\nu(E) = \int_{\mathbb{R}^{2n}} \chi_E(w, \varphi(w))\eta(w)\,dw, \tag{1}$$

with $\eta(w) = \prod_{j=1}^n \eta_j(|w_j|^2)$, where for $j = 1, \ldots, n$, η_j is a function in $C_c^\infty(\mathbb{R})$ such that $0 \le \eta_j \le 1$, $\eta_j(t) \equiv 1$ if $t \in [-1, 1]$ and $\operatorname{supp}(\eta_j) \subset (-2, 2)$. Let T_ν be the right convolution operator by ν, defined by

$$T_\nu f(x, t) = (f * \nu)(x, t) = \int_{\mathbb{R}^{2n}} f\big((x, t) \cdot (w, \varphi(w))^{-1}\big)\eta(w)\,dw. \tag{2}$$

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We are interested in the *type set*

$$E_\nu = \{(1/p, 1/q) \in [0,1] \times [0,1] : \|T_\nu\|_{pq} < \infty\}$$

where the L^p-spaces are taken with respect to the Lebesgue measure on $\mathbb{R}^{2n+1}$. We say that the measure ν defined in (1) is L^p-*improving* if E_ν does not reduce to the diagonal $1/p = 1/q$.

This problem is well known if in (2) we replace the Heisenberg group convolution with the ordinary convolution in $\mathbb{R}^{2n+1}$. If the graph of φ has non-zero Gaussian curvature at each point, a theorem of Littman (see [3]) implies that E_ν is the closed triangle with vertices $(0,0)$, $(1,1)$, and $\left(\frac{2n+1}{2n+2}, \frac{1}{2n+2}\right)$ (see [4]). A very interesting survey of results concerning the type set for convolution operators with singular measures can be found in [5]. Returning to our setting of $\mathbb{H}^n$, in [7] S. Secco obtains L^p-improving properties of measures supported on curves in $\mathbb{H}^1$, under the assumption that

$$\begin{vmatrix} \phi_1^{(2)} & \phi_2^{(2)} \\ \phi_1^{(3)} & \phi_2^{(3)} \end{vmatrix}(s) \neq -\frac{(\phi_1^{(2)}(s))^2}{2}, \qquad \forall s \in I,$$

$$\begin{vmatrix} \phi_1^{(2)} & \phi_2^{(2)} \\ \phi_1^{(3)} & \phi_2^{(3)} \end{vmatrix}(s) \neq \frac{(\phi_1^{(2)}(s))^2}{2}, \qquad \forall s \in I,$$

where $\Phi(s) = (s, \phi_1(s), \phi_2(s))$ is the curve on which the measure is supported. In [6] F. Ricci and E. Stein showed that the type set of the measure given by (1), for the case $\varphi(w) = 0$ and $n = 1$, is the triangle with vertices $(0,0)$, $(1,1)$, and $(3/4, 1/4)$.

In this article we consider first $\varphi(w) = \sum_{j=1}^{n} a_j |w_j|^2$, with $w_j \in \mathbb{R}^2$ and $a_j \in \mathbb{R}$. The Riesz–Thorin theorem implies that the type set E_ν is a convex subset of $[0,1] \times [0,1]$. In Lemmas 3 and 4 we obtain the following necessary conditions for the pair $(1/p, 1/q)$ to be in E_ν:

$$\frac{1}{q} \leq \frac{1}{p}, \qquad \frac{1}{q} \geq \frac{2n+1}{p} - 2n, \qquad \frac{1}{q} \geq \frac{1}{(2n+1)p}.$$

Thus E_ν is contained in the closed triangle with vertices $(0,0)$, $(1,1)$, and $\left(\frac{2n+1}{2n+2}, \frac{1}{2n+2}\right)$. In Section 3 we prove that E_ν is exactly that triangle:

Theorem 1. *If ν is the Borel measure defined by* (1), *supported on the graph of the function $\varphi(w) = \sum_{j=1}^{n} a_j |w_j|^2$, for some $n \in \mathbb{N}$, with $w_j \in \mathbb{R}^2$ and $a_j \in \mathbb{R}$, then the type set E_ν is the closed triangle with vertices*

$$A = (0,0), \qquad B = (1,1), \qquad C = \left(\frac{2n+1}{2n+2}, \frac{1}{2n+2}\right).$$

In a similar way we also obtain L^p-improving properties of the measure supported on the graph of the function $\varphi(w) = |w|^{2m}$. In fact we prove

THEOREM 2. *For $m, n \in \mathbb{N}_{\geq 2}$ let ν_m be the measure given by* (1) *with $\varphi(y) = |y|^{2m}$, $y \in \mathbb{R}^{2n}$. Then the type set E_{ν_m} contains the closed triangle with vertices*

$$(0,0), \qquad (1,1), \qquad \left(\frac{2(1+mn)-m}{2(1+mn)}, \frac{m}{2(1+mn)}\right).$$

Throughout this work, c will denote a positive constant not necessarily the same at each occurrence.

2. Necessary conditions. We denote $B(r)$ the $2n+1$-dimensional ball centered at the origin with radius r.

LEMMA 3. *Let ν be the Borel measure defined by* (1)*, where φ is a bounded measurable function. If $(1/p, 1/q) \in E_\nu$ then $p \leq q$.*

Proof. For $(y,s) \in \mathbb{H}^n$ we define the operator $\tau_{(y,s)}$ by $(\tau_{(y,s)}f)(x,t) = f((y,s)^{-1} \cdot (x,t))$. Since $\tau_{(y,s)}T_\nu = T_\nu \tau_{(y,s)}$, it is easy to see that the $\mathbb{R}^n$ argument utilized in the proof of Theorem 1.1 in [2] works on $\mathbb{H}^n$ as well. ∎

LEMMA 4. *Let ν be the Borel measure defined by* (1)*, where φ is a smooth function. Then E_ν is contained in the closed triangle with vertices*

$$(0,0), \qquad (1,1), \qquad \left(\frac{2n+1}{2n+2}, \frac{1}{2n+2}\right).$$

Proof. We will prove that if $(1/p, 1/q) \in E_\nu$ then

$$\frac{1}{q} \geq \frac{2n+1}{p} - 2n \quad \text{and} \quad \frac{1}{q} \geq \frac{1}{(2n+1)p}.$$

Then the lemma will follow by the Riesz–Thorin theorem. Let $f_\delta = \chi_{Q_\delta}$, where $Q_\delta = B(2\delta)$. Let $D = \{x \in \mathbb{R}^{2n} : \|x\| \leq 1\}$ and

$$A_\delta = \{(x,t) \in \mathbb{R}^{2n} \times \mathbb{R} : x \in D,\ |t - \varphi(x)| \leq \delta/4\}.$$

For each $(x,t) \in A_\delta$ fixed, we define $F_{\delta,x}$ by

$$F_{\delta,x} = \left\{ y \in D : \|x-y\|_{\mathbb{R}^{2n}} \leq \frac{\delta}{4n(1 + \|\nabla\varphi|_{\mathrm{supp}(\eta)}\|_\infty)} \right\}.$$

Now, for each $(x,t) \in A_\delta$ fixed, we have

$$(x,t) \cdot (y, \varphi(y))^{-1} \in Q_\delta, \qquad \forall y \in F_{\delta,x}; \tag{3}$$

indeed,

$$\begin{aligned} \|(x,t) \cdot (y,\varphi(y))^{-1}\|_{\mathbb{R}^{2n+1}} \leq \|x-y\|_{\mathbb{R}^n \times \mathbb{R}^n} + |t - \varphi(x)| \\ + |\varphi(x) - \varphi(y)| + \tfrac{1}{2}|W(x,y)|, \end{aligned}$$

and since

$$\tfrac{1}{2}|W(x,y)| \leq n\|x\|_{\mathbb{R}^{2n}} \|x-y\|_{\mathbb{R}^{2n}},$$

(3) follows. Then for $(x,t) \in A_\delta$ we obtain

$$T_\nu f_\delta(x,t) \geq \int_{F_{\delta,x}} \eta(y)\,dy \geq c\delta^{2n},$$

where c is independent of δ, x and t. If $(1/p,1/q) \in E_\nu$ then

$$c\delta^{1/q+2n} = c\delta^{2n}|A_\delta|^{1/q} \leq \Big(\int_{A_\delta} |T_\nu f_\delta(x,t)|^q\,dx\,dt\Big)^{1/q}$$
$$\leq \|T_\nu f_\delta\|_q \leq c_{p,q}\|f_\delta\|_p = c\delta^{(2n+1)/p},$$

thus $\delta^{2n+1/q} \leq C\delta^{(2n+1)/p}$ for all $0 < \delta < 1$ small enough. This implies that

$$\frac{1}{q} \geq \frac{2n+1}{p} - 2n.$$

Now, the adjoint operator of T_ν is given by

$$T_\nu^* g(x,t) = \int_{\mathbb{R}^{2n}} g((x,t)\cdot(y,\varphi(y)))\eta(y)\,dy$$

and let E_ν^* be the corresponding type set. Since $T_\nu = (T_\nu^*)^*$, by duality it follows that $(1/p,1/p') \in E_\nu$ if and only if $(1/p,1/p') \in E_\nu^*$, thus if $(1/p,1/q) \in E_\nu^*$ then $\frac{1}{q} \geq \frac{2n+1}{p} - 2n$. Finally, by duality it is also necessary that

$$\frac{1}{q} \geq \frac{1}{(2n+1)p}.$$

Therefore E_ν is contained in the region determined by these two conditions and by the condition $p \leq q$, i.e. the closed triangle with vertices $(0,0)$, $(1,1)$, $\left(\frac{2n+1}{2n+2}, \frac{1}{2n+2}\right)$. ∎

Remark. Lemma 4 holds if we replace the smoothness condition with a Lipschitz condition.

3. The main results. For each $N \in \mathbb{N}$ fixed, we consider an auxiliary operator T_N which will be embedded in an analytic family $\{T_{N,z}\}$ of operators on the strip $-n \leq \mathrm{Re}(z) \leq 1$ such that

$$\begin{cases} \|T_{N,z}(f)\|_{L^\infty(\mathbb{H}^n)} \leq c_z\|f\|_{L^1(\mathbb{H}^n)}, & \mathrm{Re}(z) = 1, \\ \|T_{N,z}(f)\|_{L^2(\mathbb{H}^n)} \leq c_z\|f\|_{L^2(\mathbb{H}^n)}, & \mathrm{Re}(z) = -n, \end{cases} \tag{4}$$

where c_z will depend admissibly on the variable z and it will not depend on N. We denote $T_N = T_{N,0}$. By Stein's theorem on complex interpolation, it will follow that the operator T_N will be bounded from $L^{\frac{2n+2}{2n+1}}(\mathbb{H}^n)$ in $L^{2n+2}(\mathbb{H}^n)$ uniformly in N, if we see that $T_N f(x,t) \to T_\nu f(x,t)$ as $N \to \infty$, a.e. $(x,t) \in \mathbb{R}^{2n+1}$. Theorem 1 will then follow from Fatou's lemma and Lemmas 3 and 4.

To prove the second inequality in (4) we will see that such a family will admit the expression

$$T_{N,z}(f)(x,t) = (f * K_{N,z})(x,t),$$

where $K_{N,z} \in L^1(\mathbb{H}^n)$, moreover it is a *polyradial* function (i.e. the values of $K_{N,z}$ depend on $|w_1|, \dots, |w_n|$ and t). Now our operator $T_{N,z}$ can be realized as a pointwise product of operators via the group Fourier transform, i.e.

$$\widehat{T_{N,z}(f)}(\lambda) = \widehat{f}(\lambda)\widehat{K_{N,z}}(\lambda)$$

where, for each $\lambda \neq 0$, $\widehat{K_{N,z}}(\lambda)$ is an operator on the Hilbert space $L^2(\mathbb{R}^n)$ given by

$$\widehat{K_{N,z}}(\lambda)g(\xi) = \int_{\mathbb{H}^n} K_{N,z}(\varsigma,t)\pi_\lambda(\varsigma,t)g(\xi)\,d\varsigma\,dt.$$

It then follows from Plancherel's theorem for the group Fourier transform that

$$\|T_{N,z}f\|_{L^2(\mathbb{H}^n)} \leq A_z\|f\|_{L^2(\mathbb{H}^n)}$$

if and only if

$$\|\widehat{K_{N,z}}(\lambda)\|_{\mathrm{op}} \leq A_z \tag{5}$$

uniformly over N and $\lambda \neq 0$. Since $K_{N,z}$ is a polyradial integrable function, by a well known result of Geller (see [1, Lemma 1.3, p. 213]) the operators $\widehat{K_{N,z}}(\lambda) : L^2(\mathbb{H}^n) \to L^2(\mathbb{H}^n)$ are, for each $\lambda \neq 0$, diagonal with respect to a Hermite basis for $L^2(\mathbb{R}^n)$, namely

$$\widehat{K_{N,z}}(\lambda) = C_n(\delta_{\gamma,\alpha}\mu_{N,z}(\alpha,\lambda))_{\gamma,\alpha\in\mathbb{N}_0^n}$$

where $C_n = (2\pi)^n$, $\alpha = (\alpha_1,\dots,\alpha_n)$, $\delta_{\gamma,\alpha} = 1$ if $\gamma = \alpha$ and $\delta_{\gamma,\alpha} = 0$ if $\gamma \neq \alpha$, and the diagonal entries $\mu_{N,z}(\alpha_1,\dots,\alpha_n,\lambda)$ can be expressed explicitly in terms of the Laguerre transform. We have in fact

$$\begin{aligned}\mu_{N,z}&(\alpha_1,\dots,\alpha_n,\lambda)\\ &= \int_0^\infty \dots \int_0^\infty K^\lambda_{N,z}(r_1,\dots,r_n)\prod_{j=1}^n \left(r_j L^0_{\alpha_j}\left(\tfrac{1}{2}|\lambda|r_j^2\right)e^{-\frac{1}{4}|\lambda|r_j^2}\right)dr_1\dots dr_n\end{aligned}$$

where $L^0_k(s)$ are the Laguerre polynomials, i.e. $L^0_k(s) = \sum_{i=0}^k \frac{k!}{(k-i)!i!}\frac{(-s)^i}{i!}$ and $K^\lambda_{N,z}(\varsigma) = \int_{\mathbb{R}} K_{N,z}(\varsigma,t)e^{i\lambda t}\,dt$. Now (5) is equivalent to

$$\|T_{N,z}f\|_{L^2(\mathbb{H}^n)} \leq A_z\|f\|_{L^2(\mathbb{H}^n)}$$

if and only if

$$|\mu_{N,z}(\alpha_1,\dots,\alpha_n,\lambda)| \leq A_z \tag{6}$$

uniformly over N, α_j and $\lambda \neq 0$. If $\mathrm{Re}(z) = -n$, in the proof of Theorem 1 we find that (6) holds with A_z independent of N, $\lambda \neq 0$ and α_j, and then we obtain the boundedness on $L^2(\mathbb{H}^n)$ that is stated in (4).

We consider the family $\{I_z\}_{z\in\mathbb{C}}$ of distributions on $\mathbb{R}$ that arises by analytic continuation of the family $\{I_z\}$ of functions, initially given when $\mathrm{Re}(z) > 0$ and $s \in \mathbb{R}\setminus\{0\}$ by

$$I_z(s) = \frac{2^{-z/2}}{\Gamma(z/2)}|s|^{z-1}. \tag{7}$$

In particular, we have $\widehat{I_z} = I_{1-z}$, also $I_0 = c\delta$ where $\widehat{\cdot}$ denotes the Fourier transform on $\mathbb{R}$ and δ is the Dirac distribution at the origin on $\mathbb{R}$.

Let $H \in \mathcal{S}(\mathbb{R})$ be such that $\mathrm{supp}(\widehat{H}) \subseteq (-1,1)$ and $\int \widehat{H}(t)\,dt = 1$. Now we put $\phi_N(t) = H(t/N)$, thus $\widehat{\phi_N}(\xi) = N\widehat{H}(N\xi)$ and $\widehat{\phi_N} \to \delta$ in the sense of distributions as $N \to \infty$.

For $z\in \mathbb{C}$ and $N \in \mathbb{N}$, we also define $J_{N,z}$ as the distribution on $\mathbb{H}^n$ given by the tensor product

$$J_{N,z} = \delta \otimes \cdots \otimes \delta \otimes (I_z *_{\mathbb{R}} \widehat{\phi_N}) \tag{8}$$

where $*_{\mathbb{R}}$ denotes the usual convolution on $\mathbb{R}$ and I_z is the fractional integration kernel given by (7). Finally, for $z \in \mathbb{C}$ and $N \in \mathbb{N}$ fixed, we define the operator $T_{N,z}$ by

$$T_{N,z}f(x,t) = (f * \nu * J_{N,z})(x,t). \tag{9}$$

We observe that $T_{N,0}f(x,t) \to cT_\nu f(x,t)$ as $N \to \infty$ a.e. $(x,t) \in \mathbb{R}^{2n+1}$, since $J_{N,0} = \delta\otimes\cdots\otimes\delta\otimes c\widehat{\phi_N} \to \delta\otimes\cdots\otimes\delta\otimes c\delta$ in the sense of distributions as $N \to \infty$.

Before proving Theorem 1 we need the following lemmas:

Lemma 5. *If* $\mathrm{Re}(z) \le -1$ *then* $\nu * J_{N,z} \in L^p(\mathbb{H}^n)$ *for all* $p \ge 1$.

Proof. For $\mathrm{Re}(z) \le -1$ and $N \in \mathbb{N}$ fixed, a simple calculation gives

$$(\nu * J_{N,z})(x,\sigma) = \eta(x)(I_z *_{\mathbb{R}} \widehat{\phi_N})(\sigma - \varphi(x)).$$

We see that it is enough to prove that $(I_z * \widehat{\phi_N})(\cdot) \in L^p(\mathbb{R})$ whenever $\mathrm{Re}(z) \le -1$. We observe that if $g \in \mathcal{S}(\mathbb{R})$ with $\mathrm{supp}(g)\cap[-\epsilon,\epsilon] = \emptyset$ for some $\epsilon > 0$, then for $\mathrm{Re}(z) \le -1$,

$$I_z(g) = \begin{cases} \dfrac{2^{-z/2}}{\Gamma(z/2)} \displaystyle\int_{|t|\ge\epsilon} |t|^{z-1}g(t)\,dt & \text{if } z \notin -2\mathbb{N}, \\ 0 & \text{if } z \in -2\mathbb{N}. \end{cases}$$

From this observation and the fact that

$$\mathrm{supp}(\tau_s(\widehat{\phi_N}^{\vee})) \subset [s-1/N, s+1/N] \subset [-\infty,-1]\cup[1,\infty] \qquad \text{for } |s| \ge \frac{N+1}{N}$$

(where $\phi^\vee(x) = \phi(-x)$ and $(\tau_s\phi)(x) = \phi(x-s)$), we obtain

$$|(I_z * \widehat{\phi_N})(s)| = |I_z(\tau_s(\widehat{\phi_N}^\vee))| \le c\left|s - \frac{\operatorname{sgn}(s)}{N}\right|^{-2} \quad \text{if } |s| \ge \frac{N+1}{N}.$$

Finally, since $|(I_z * \widehat{\phi_N})(s)| \le c$ for all $s \in [-2,2]$, the lemma follows. ∎

Lemma 6. *For $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$ set*

$$F_{n,k}(\sigma) := \chi_{(0,\infty)}(\sigma) L_k^{n-1}(\sigma) e^{-\sigma/2}\sigma^{n-1}.$$

Then

$$\widehat{F_{n,k}}(\xi) = \frac{(k+n-1)!}{k!}\,\frac{(-1/2+i\xi)^k}{(1/2+i\xi)^{k+n}}.$$

Proof. On $\mathbb{R}$ we define the Fourier transform by $\widehat{g}(\xi) = \int_{\mathbb{R}} g(\sigma)e^{-i\sigma\xi}\,d\sigma$, so

$$\widehat{F_{n,k}}(\xi) = \int_0^\infty L_k^{n-1}(\sigma)\sigma^{n-1}e^{-\sigma(1/2+i\xi)}d\sigma$$

and since

$$L_k^{n-1}(\sigma)\sigma^{n-1} = \frac{e^\sigma}{k!}\left(\frac{d}{d\sigma}\right)^k (e^{-\sigma}\sigma^{k+n-1})$$

for each $n \in \mathbb{N}$ and each $k \in \mathbb{N}_0$, we obtain

$$\begin{aligned}
\widehat{F_{n,k}}(\xi) &= \frac{1}{k!}\int_0^\infty \left(\frac{d}{d\sigma}\right)^k (e^{-\sigma}\sigma^{k+n-1})e^{-\sigma(-1/2+i\xi)}\,d\sigma \\
&= \frac{(-1/2+i\xi)^k}{k!}\int_0^\infty \sigma^{k+n-1}e^{-\sigma(1/2+i\xi)}\,d\sigma \\
&= \frac{(-1/2+i\xi)^k}{k!}\int_0^\infty \frac{s^{k+n-1}}{(1/2+i\xi)^{k+n-1}}e^{-s}\,\frac{ds}{1/2+i\xi} \\
&= \frac{(k+n-1)!}{k!}\,\frac{(-1/2+i\xi)^k}{(1/2+i\xi)^{k+n}}
\end{aligned}$$

where the third equality follows from the rapid decay of the function e^{-z} in $\{z : \operatorname{Re}(z) > 0\}$. Then we apply the Cauchy's theorem. ∎

Proof of Theorem 1. For $\operatorname{Re}(z) = 1$ we have

$$\|T_{N,z}f\|_\infty = \|f * \nu * J_{N,z}\|_\infty \le \|f\|_1\|\nu * J_{N,z}\|_\infty.$$

Since

$$(\nu * J_{N,z})(x,\sigma) = \eta(x)(I_z *_{\mathbb{R}} \widehat{\phi_N})(\sigma - \varphi(x))$$

it follows that $\|\nu * J_{N,z}\|_\infty \le c|\Gamma(z/2)|^{-1}$. Then, for $\operatorname{Re}(z) = 1$, we obtain

$$\|T_{N,z}\|_{1,\infty} \le c|\Gamma(z/2)|^{-1}.$$

From Lemma 5, in particular, we have $\nu * J_{N,z} \in L^1(\mathbb{H}^n) \cap L^2(\mathbb{H}^n)$. In addition $\nu * J_{N,z}$ is a polyradial function. Thus the operator $(\nu * J_{N,z})\widehat{\ }(\lambda)$ is diagonal with respect to a Hermite base for $L^2(\mathbb{R}^n)$, and its diagonal entries $\mu_{N,z}(\alpha, \lambda)$, with $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, are given by

$$\begin{aligned}\mu_{N,z}(\alpha,\lambda) = \int_0^\infty \dots \int_0^\infty & (\nu * J_{N,z})(r_1, \dots, r_n, \widehat{-\lambda}) \\ & \times \prod_{j=1}^n \left(r_j L_{k_j}^0\left(\tfrac{1}{2}|\lambda| r_j^2\right) e^{-|\lambda| r_j^2/4}\right) dr_1 \dots dr_n \\ = \int_0^\infty \dots \int_0^\infty & (I_z *_{\mathbb{R}} \widehat{\phi_N})\widehat{\ }(-\lambda) \\ & \times \prod_{j=1}^n \left(\eta_j(r_j^2) e^{i\lambda a_j r_j^2} r_j L_{k_j}^0\left(\tfrac{1}{2}|\lambda| r_j^2\right) e^{-|\lambda| r_j^2/4}\right) dr_1 \dots dr_n \\ = I_{1-z}(-\lambda)\phi_N(\lambda) & \prod_{j=1}^n \int_0^\infty \left(\eta_j(r_j^2) e^{i\lambda a_j r_j^2} r_j L_{\alpha_j}^0\left(\tfrac{1}{2}|\lambda| r_j^2\right) e^{-|\lambda| r_j^2/4}\right) dr_j.\end{aligned}$$

Thus, it is enough to study the integral

$$\int_0^\infty \eta_1(r^2) L_{\alpha_1}^0(|\lambda| r^2/2) e^{-|\lambda| r^2/4} e^{i\lambda a_1 r^2} r\, dr,$$

where $a_1 \in \mathbb{R}$ and $\eta_1 \in C_c^\infty(\mathbb{R})$. We make the change of variable $\sigma = |\lambda| r^2/2$ in such an integral to obtain

$$\begin{aligned}\int_0^\infty & \eta_1(r^2) L_{\alpha_1}^0(|\lambda| r^2/2) e^{-|\lambda| r^2/4} e^{i\lambda a_1 r^2} r\, dr \\ & = |\lambda|^{-1} \int_0^\infty \eta_1\left(\frac{2\sigma}{|\lambda|}\right) L_{\alpha_1}^0(\sigma) e^{-\sigma/2} e^{i2\,\mathrm{sgn}(\lambda) a_1 \sigma}\, d\sigma \\ & = |\lambda|^{-1} (F_{\alpha_1} G_\lambda)\widehat{\ }(-2\,\mathrm{sgn}(\lambda) a_1) = |\lambda|^{-1} (\widehat{F_{\alpha_1}} * \widehat{G_\lambda})(-2\,\mathrm{sgn}(\lambda) a_1)\end{aligned}$$

where

$$F_{\alpha_1}(\sigma) := \chi_{(0,\infty)}(\sigma) L_{\alpha_1}^0(\sigma) e^{-\sigma/2}, \tag{10}$$

$$G_\lambda(\sigma) := \eta_1(2\sigma/|\lambda|). \tag{11}$$

Now

$$|(\widehat{F_{\alpha_1}} * \widehat{G_\lambda})(-2\,\mathrm{sgn}(\lambda) a_1)| \le \|\widehat{F_{\alpha_1}} * \widehat{G_\lambda}\|_\infty \le \|\widehat{F_{\alpha_1}}\|_\infty \|\widehat{G_\lambda}\|_1 = \|\widehat{F_{\alpha_1}}\|_\infty \|\widehat{\eta_1}\|_1.$$

So it is enough to estimate $\|\widehat{F_{\alpha_1}}\|_\infty$. From Lemma 6, with $n = 1$ and $k = \alpha_1$, we obtain

$$|\widehat{F_{\alpha_1}}(\xi)| = \frac{1}{|1/2 + i\xi|}. \tag{12}$$

Finally, for $\mathrm{Re}(z) = -n$, we obtain

$$\begin{aligned}|\mu_{N,z}(\alpha_1,\ldots,\alpha_n,\lambda)| &\le 2^n|I_{1-z}(-\lambda)\phi_N(\lambda)|\,|\lambda|^{-n}\prod_{j=1}^n\|\widehat{\eta_j}\|_1\\ &\le 2^n\left|\Gamma\left(\frac{1-z}{2}\right)\right|^{-1}|H(\lambda/N)|\prod_{j=1}^n\|\widehat{\eta_j}\|_1\\ &\le 2^n\left|\Gamma\left(\frac{1-z}{2}\right)\right|^{-1}\|H\|_\infty\prod_{j=1}^n\|\widehat{\eta_j}\|_1\end{aligned}$$

and by (6) it follows, for $\mathrm{Re}(z) = -n$, that

$$\|T_{N,z}f\|_{L^2(\mathbb{H}^n)} \le c\frac{(2\pi)^n 2^n}{\left|\Gamma\left(\frac{1-z}{2}\right)\right|}\|f\|_{L^2(\mathbb{H}^n)}.$$

It is easy to see, with the aid of the Stirling formula (see [9, p. 326]), that the family $\{T_{N,z}\}$ satisfies, on the strip $-n \le \mathrm{Re}(z) \le 1$, the hypothesis of the complex interpolation theorem (see [10, p. 205]) and so $T_{N,0}$ is bounded from $L^{\frac{2n+2}{2n+1}}(\mathbb{H}^n)$ into $L^{2n+2}(\mathbb{H}^n)$ uniformly in N; then letting N tend to infinity, we conclude that the operator T_ν is bounded from $L^{\frac{2n+2}{2n+1}}(\mathbb{H}^n)$ into $L^{2n+2}(\mathbb{H}^n)$ for all $n \in \mathbb{N}$. ∎

Proof of Theorem 2. We consider, for each $N \in \mathbb{N}$ fixed, the analytic family $\{U_{N,z}\}$ of operators on the strip $-\left(n+\frac{1-m}{m}\right) \le \mathrm{Re}(z) \le 1$, defined by $U_{N,z}f = f*\nu_m*J_{N,z}$, where $J_{N,z}$ is given by (8) and $U_{N,0}f \to U_{\nu_m}f = f*\nu_m$ as $N \to \infty$. Proceeding as in the proof of Theorem 1 it follows, for $\mathrm{Re}(z) = 1$, that $\|U_{N,z}\|_{1,\infty} \le c|\Gamma(z/2)|^{-1}$. Also it is clear that, for $\mathrm{Re}(z) = -\left(n+\frac{1-m}{m}\right)$, the kernel $\nu_m*J_{N,z}$ is in $L^1(\mathbb{H}^n)\cap L^2(\mathbb{H}^n)$ and it is also a radial function. Now, our operator $(\nu_m * J_{N,z})\widehat{\ }(\lambda)$ is diagonal, with diagonal entries $\upsilon_{N,z}(k,\lambda)$ given by

$$\begin{aligned}\upsilon_{N,z}(k,\lambda) &= \frac{k!}{(k+n-1)!}\int_0^\infty (\nu_m * J_{N,z})(s,\widehat{-\lambda})L_k^{n-1}(|\lambda|s^2/2)e^{-|\lambda|s^2/4}s^{2n-1}\,ds\\ &= \frac{k!}{(k+n-1)!}I_{1-z}(-\lambda)\phi_N(\lambda)\int_0^\infty \eta_0(s^2)L_k^{n-1}(|\lambda|s^2/2)e^{-|\lambda|s^2/4}e^{i\lambda s^{2m}}s^{2n-1}\,ds.\end{aligned}$$

Now we study the integral

$$\int_0^\infty \eta_0(s^2)L_k^{n-1}(|\lambda|s^2/2)e^{-|\lambda|s^2/4}e^{i\lambda s^{2m}}s^{2n-1}ds.$$

We make the change of variable $\sigma = |\lambda|s^2/2$ to obtain

$$\begin{aligned}
&\int_0^\infty \eta_0(s^2) L_k^{n-1}(|\lambda|s^2/2) e^{-|\lambda|s^2/4} e^{i\lambda s^{2m}} s^{2n-1}\, ds \\
&\qquad = 2^{n-1}|\lambda|^{-n} \int_0^\infty \eta_0(2\sigma/|\lambda|) L_k^{n-1}(\sigma) e^{-\sigma/2} e^{i2^m \operatorname{sgn}(\lambda)|\lambda|^{1-m}\sigma^m} \sigma^{n-1}\, d\sigma \\
&\qquad = 2^{n-1}|\lambda|^{-n}(F_{n,k} G_\lambda R_\lambda)\widehat{\ }(0) = 2^{n-1}|\lambda|^{-n}(\widehat{F_{n,k}} * \widehat{G_\lambda R_\lambda})(0) \\
&\qquad = 2^{n-1}|\lambda|^{-n}(\widehat{F_{n,k}} * (\widehat{G_\lambda} * \widehat{R_\lambda}))(0)
\end{aligned}$$

where $F_{n,k}$ is the function defined in Lemma 6, $G_\lambda(\sigma) = \eta_0(2\sigma/|\lambda|)$ and $R_\lambda(\sigma) = \chi_{(0,|\lambda|)}(\sigma) e^{i2^m \operatorname{sgn}(\lambda)|\lambda|^{1-m}\sigma^m}$. If $n \geq 2$, from Lemma 6 we get

$$\begin{aligned}
\|\widehat{F_{n,k}} * (\widehat{G_\lambda} * \widehat{R_\lambda})\|_\infty &\leq \|\widehat{F_{n,k}}\|_1 \|\widehat{G_\lambda}\|_1 \|\widehat{R_\lambda}\|_\infty \\
&= \frac{(k+n-1)!}{k!} \left(\int_{\mathbb{R}} \frac{d\xi}{(1/4+\xi^2)^{n/2}} \right) \|\widehat{\eta_0}\|_1 \|\widehat{R_\lambda}\|_\infty .
\end{aligned}$$

Now, we estimate $\|\widehat{R_\lambda}\|_\infty$. Taking account of [8, Proposition 2, p. 332], we note that

$$|\widehat{R_\lambda}(\xi)| = \left| \int_0^{|\lambda|} e^{i(2^m \operatorname{sgn}(\lambda)|\lambda|^{1-m}\sigma^m - \xi\sigma)}\, d\sigma \right| \leq \frac{C_m}{|\lambda|^{(1-m)/m}}$$

where the constant C_m does not depend on λ. Then for $\operatorname{Re}(z) = -\big(n + \frac{1-m}{m}\big)$, we have

$$\begin{aligned}
|\upsilon_{N,z}(k,\lambda)| &\leq \frac{k!}{(k+n-1)!} |I_{1-z}(-\lambda)\phi_N(\lambda)| 2^{n-1}|\lambda|^{-n} \|\widehat{F_{n,k}} * (\widehat{G_\lambda} * \widehat{R_\lambda})\|_\infty \\
&\leq |I_{1-z}(-\lambda)||\phi_N(\lambda)| 2^{n-1}|\lambda|^{-n} \left(\int_{\mathbb{R}} \frac{d\xi}{(1/4+\xi^2)^{n/2}} \right) \|\widehat{\eta_0}\|_1 \frac{C_m}{|\lambda|^{(1-m)/m}} \\
&\leq C_m 2^{n-1} \left| \Gamma\left(\frac{1-z}{2}\right) \right|^{-1} \|H\|_\infty \left(\int_{\mathbb{R}} \frac{d\xi}{(1/4+\xi^2)^{n/2}} \right) \|\widehat{\eta_0}\|_1 .
\end{aligned}$$

Finally, by (6) it follows that, for $\operatorname{Re}(z) = -\big(n + \frac{1-m}{m}\big)$,

$$\|U_{N,z} f\|_{L^2(\mathbb{H}^n)} \leq \frac{C_{n,m}}{\left|\Gamma\big(\frac{1-z}{2}\big)\right|} \|f\|_{L^2(\mathbb{H}^n)} .$$

It is clear that the family $\{U_{N,z}\}$ satisfies, on the strip $-\big(n + \frac{1-m}{m}\big) \leq \operatorname{Re}(z) \leq 1$, the hypothesis of the complex interpolation theorem. Thus $U_{N,0}$ is bounded from $L^{\frac{2(1+nm)}{2(1+mn)-m}}(\mathbb{H}^n)$ into $L^{\frac{2(1+nm)}{m}}(\mathbb{H}^n)$ uniformly in N, and letting N tend to infinity we conclude that the operator U_{ν_m} is bounded from $L^{\frac{2(1+nm)}{2(1+mn)-m}}(\mathbb{H}^n)$ into $L^{\frac{2(1+nm)}{m}}(\mathbb{H}^n)$ for $m, n \in \mathbb{N}_{\geq 2}$. ∎

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