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ON A GENERALISATION OF THE HAHN–JORDAN DECOMPOSITION FOR REAL CÀDLÀG FUNCTIONS

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Abstract. For a real càdlàg function f and a positive constant c we find another càdlàg function which has the smallest total variation among all functions uniformly approximating f with accuracy $c/2$. The solution is expressed in terms of truncated variation, upward truncated variation and downward truncated variation introduced in earlier work of the author. They are always finite even if the total variation of f is infinite, and they may be viewed as a generalisation of the Hahn–Jordan decomposition for real càdlàg functions. We also present partial results for more general functions.

1. Introduction. The notion of a real-valued *signed measure* and its Hahn–Jordan decomposition plays a fundamental role in a measure theory and the theory of integration. They are also related to the *upper, lower* and *total variations* of the signed measure [H, Sect. IV.29]. A generalisation to vector-valued measures is also possible. When the measurable space is $[a;b]$, $-\infty < a < b < \infty$ (with the σ-field of all Borel measurable sets), instead of signed or vector-valued measures one may consider functions of finite total variation.

The total variation may be defined for any function $f : [a;b] \to E$ with values in a general metric space E. Namely, when ρ is the metric on E we define the total variation of f by the formula

$$\mathrm{TV}(f,[a;b]) = \sup_n \sup_{\pi_n} \sum_{i=1}^{n} \rho(f(t_i), f(t_{i-1})),$$

where the second supremum is over all partitions $\pi_n = \{a \le t_0 < t_1 < \cdots < t_n \le b\}$.

In general, the total variation of f may be (and in many important cases is) infinite. For example, almost all paths of standard Brownian motion, which is widely used in stochastic modeling and optimisation, are continuous functions with infinite total variation on any interval $[0;t]$, $t > 0$. This

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fact was arguably the main reason for the introduction of the Itô stochastic integral.

However, after imposing some mild regularity conditions on f we can easily find functions approximating f with arbitrary accuracy and having finite total variation, even if the total variation of f is infinite. Obviously, the better the approximation, the greater the total variation of the approximating function. Let us fix $c > 0$. A natural question arises about the greatest lower bound for the total variation of a function $g : [a; b] \to E$, uniformly approximating f with accuracy $c/2 > 0$, and the *first problem* we will deal with is the determination of

$$\inf_{g \in B(f, c/2)} \mathrm{TV}(g, [a; b]),$$

where $B(f, d)$ denotes the ball

$$B(f, d) := \Big\{ g : [a; b] \to E : \sup_{t \in [a;b]} \rho(f(t), g(t)) \le d \Big\}.$$

The immediate bound from below for $\inf_{g \in B(f,c/2)} \mathrm{TV}(g, [a; b])$ is

$$\inf_{g \in B(f,c/2)} \mathrm{TV}(g, [a; b]) \ge \sup_n \sup_{\pi_n} \sum_{i=1}^{n} \max\{\rho(f(t_i), f(t_{i-1})) - c, 0\}, \tag{1.1}$$

which follows directly from the triangle inequality,

$$\begin{aligned} \rho(g(t_i), g(t_{i-1})) &\ge \rho(f(t_i), f(t_{i-1})) - \rho(f(t_i), g(t_i)) - \rho(f(t_{i-1}), g(t_{i-1})) \\ &\ge \rho(f(t_i), f(t_{i-1})) - c. \end{aligned}$$

We will call the quantity on the right hand side of (1.1), i.e.

$$\sup_n \sup_{\pi_n} \sum_{i=1}^{n} \max\{\rho(f(t_i), f(t_{i-1})) - c, 0\},$$

the *truncated variation of* f *at level* c and denote it by $\mathrm{TV}^c(f, [a; b])$; it was first introduced in [L1].

The lower bound for $\inf_{g \in B(f,c/2)} \mathrm{TV}(g, [a; b])$ just obtained may well be infinite, but from inequality (1.1) it follows that it is finite for any $c > 0$ iff the function f is a uniform limit of finite variation functions. We prove this fact and identify the family of such functions in Section 2 (Fact 2.2).

The family of real càdlàg functions, i.e. right-continuous functions with left limits, will be of our special interest, since càdlàg functions with finite total variations correspond naturally to finite signed measures on $(a; b]$. Moreover, in this paper we will show that for càdlàg f, for $E = \mathbb{R}$ with the standard Euclidean metric $\rho(x, y) = |x - y|$ and for any $c > 0$ we have in fact equality, i.e.

$$\inf_{\|g - f\|_\infty \le c/2} \mathrm{TV}(g, [a; b]) = \mathrm{TV}^c(f, [a; b]), \tag{1.2}$$

where $g : [a; b] \to \mathbb{R}$ and $\|g - f\|_\infty = \sup_{t \in [a;b]} |g(t) - f(t)|$. Morever, there exists a càdlàg function $f^c : [a; b] \to \mathbb{R}$ such that

$$\|f^c - f\|_\infty \le c/2 \quad \text{and} \quad \mathrm{TV}(f^c, [a; b]) = \mathrm{TV}^c(f, [a; b]).$$

REMARK 1.1. In general, the function f^c is not unique, but imposing the stronger condition that $\|f^c - f\|_\infty \le c/2$ and for any $s \in (a; b]$,

$$\mathrm{TV}(f^c, [a; s]) = \mathrm{TV}^c(f, [a; s]), \tag{1.3}$$

we will find that the function f^c exists and is uniquely determined for any $c \le \sup_{s,u \in [a;b]} |f(s) - f(u)|$ (cf. Corollary 3.8).

REMARK 1.2. A natural question appears to be whether the truncated variation is an attainable lower bound for $\inf_{g \in B(f,c/2)} \mathrm{TV}(g, [a; b])$ for functions with values in other metric spaces, but the answer is not known to the author. In [TV, Lemma 9] it was proven that if f is continuous and E is a general, multidimensional (and complete metric) space then the value $\inf_{g \in B(f,c/2)} \mathrm{TV}(g, [a; b])$ is attained for some function g_0; however, the authors do not identify this quantity as the truncated variation. The proof of [TV, Lemma 9] works for any càdlàg function f.

Since for $E = \mathbb{R}$ with $\rho(x, y) = |x - y|$ the total variation depends only on the increments of the function, in this case a more natural problem, which we will call the *second problem*, is the following. For a càdlàg function $f : [a; b] \to \mathbb{R}$ and $c > 0$ find

$$\inf\{\mathrm{TV}(f + h, [a; b]) : \|h\|_{\mathrm{osc}} \le c\},$$

where for $h : [a; b] \to \mathbb{R}$, $\|h\|_{\mathrm{osc}} := \sup_{s,u \in [a;b]} |h(s) - h(u)|$. Note that $\|\cdot\|_{\mathrm{osc}}$ is a norm on the equivalence classes of bounded functions which differ by a constant.

Solution to the second problem is the same as the solution to the first problem, i.e.

$$\inf\{\mathrm{TV}(f + h, [a; b]) : \|h\|_{\mathrm{osc}} \le c\} = \mathrm{TV}^c(f, [a; b]), \tag{1.4}$$

and one of the optimal representatives of the class of functions for which equality (1.4) is attained is $h^c = f^c - f$. To this class also belongs some $h^{0,c}$ such that $h^{0,c}(a) = 0$. We will prove that $f^{0,c} = f + h^{0,c} - f(a)$ is a càdlàg function with possible jumps only at points where f has jumps, and that it may be represented in the form

$$f^{0,c}(s) = \mathrm{UTV}^c(f; [a; s]) - \mathrm{DTV}^c(f; [a; s]), \tag{1.5}$$

where

(1.6) $$\mathrm{UTV}^c(f,[a;s]) := \sup_n \sup_{a\le t_0<t_1<\cdots<t_n\le s} \sum_{i=1}^n \max\{f(t_i)-f(t_{i-1})-c,0\},$$

(1.7) $$\mathrm{DTV}^c(f,[a;s]) := \sup_n \sup_{a\le t_0<t_1<\cdots<t_n\le s} \sum_{i=1}^n \max\{f(t_{i-1})-f(t_i)-c,0\}.$$

The functionals $\mathrm{UTV}^c(f,[a;s])$ and $\mathrm{DTV}^c(f,[a;s])$ are non-decreasing functions of s and are called the *upward* and *downward truncated variations* of f of order c on $[a;s]$ respectively. They were first introduced in [L2] with a bit different formulas, equivalent to (1.6) and (1.7).

Finally, for $s\in(a;b]$ we will show that

(1.8) $$\mathrm{TV}(f^{0,c},[a;s]) = \mathrm{TV}^c(f,[a;s]) = \mathrm{UTV}^c(f,[a;s]) + \mathrm{DTV}^c(f,[a;s]).$$

The equalities (1.5) and (1.8) give the Hahn–Jordan decomposition of the finite signed measure induced by the function $f^{0,c}$ (or f^c). This measure assigns to any interval $(a_1,b_1]\subset(a;b]$ the number

$$\mu(a_1,b_1] = f^{0,c}(b_1) - f^{0,c}(a_1)$$

and we have

$$\mu(a_1,b_1] = \mu_+(a_1,b_1] - \mu_-(a_1,b_1],$$

where

$$\mu_+(a_1,b_1] = \mathrm{UTV}^c(f,[a;b_1]) - \mathrm{UTV}^c(f,[a;a_1]),$$
$$\mu_-(a_1,b_1] = \mathrm{DTV}^c(f,[a;b_1]) - \mathrm{DTV}^c(f,[a;a_1]).$$

However, since $c>0$ is arbitrary, the equalities (1.5) and (1.8) may also be viewed as a generalisation of the Hahn–Jordan decomposition for any real càdlàg function f.

REMARK 1.3. The truncated variation and its decomposition into the sum of the upward and downward truncated variations appears naturally when the uniform approximation of the càdlàg function f by finite variation functions is considered. The truncated variation is obtained by the composition of increments of f with a convex function $\varphi(\cdot) = (|\cdot|-c)_+$. Naturally, for any Young function (convex, non-decreasing, non-constant and vanishing at 0) $\varphi:[0;\infty)\to\mathbb{R}$ the notion of *φ-variation* defined as

$$\mathrm{TV}^\varphi(f,[a;b]) := \sup_n \sup_{a\le t_0<t_1<\cdots<t_n\le b} \sum_{i=1}^n \varphi(|f(t_i)-f(t_{i-1})|)$$

is of importance. More on φ-variation may be found in [DN, Chapt. 3]. The authors of [DN] consider only the case when φ is strictly increasing, since for such φ, the φ-variation leads to interesting estimates for integrals (generalisations of the Love–Young inequality).

However, for any Young function $\varphi : [0;\infty) \to \mathbb{R}$ the functional

$$\|f\|_{(\varphi)} := \inf\{C > 0 : \mathrm{TV}^{\varphi}(f/C, [a;b]) \le 1\}$$

is a seminorm on the space of functions $f : [a;b] \to \mathbb{R}$ such that $\mathrm{TV}^{\varphi}(f/C, [a;b]) < \infty$ for some $C > 0$ (cf. [DN, Chapt. 3, proof of Theorem 3.7]). $\|\cdot\|_{(\varphi)}$ is also a norm on the space of equivalence classes of such functions, differing by a constant. Defining

$$\mathrm{UTV}^{\varphi}(f, [a;b]) := \sup_n \sup_{a \le t_0 < t_1 < \cdots < t_n \le b} \sum_{i=1}^{n} \varphi\big((f(t_i) - f(t_{i-1}))_+\big),$$

$$\mathrm{DTV}^{\varphi}(f, [a;b]) := \sup_n \sup_{a \le t_0 < t_1 < \cdots < t_n \le b} \sum_{i=1}^{n} \varphi\big((f(t_i) - f(t_{i-1}))_-\big),$$

$$\|f\|_{\mathrm{U},(\varphi)} := \inf\{C > 0 : \mathrm{UTV}^{\varphi}(f/C, [a;b]) \le 1\},$$

$$\|f\|_{\mathrm{D},(\varphi)} := \inf\{C > 0 : \mathrm{DTV}^{\varphi}(f/C, [a;b]) \le 1\}$$

we also have

$$\|f\|_{(\varphi)} \le \|f\|_{\mathrm{U},(\varphi)} + \|f\|_{\mathrm{D},(\varphi)}.$$

For two Young functions φ and ψ, $\|\cdot\|_{(\varphi)}$ and $\|\cdot\|_{(\psi)}$ are equivalent when the ratio of the right-continuous inverse functions, φ^{-1}/ψ^{-1}, is separated from 0 and from ∞. Let us notice, however, that not for every Young function φ can the corresponding φ-variation be decomposed into the sum of the upward and downward φ-variation. To see this, consider the following example. Let φ be such that $\varphi(0) = \varphi(1) = 0$, $\varphi(2) = 1$, $\varphi(3) = 2$ and $\varphi(4) = 6$; let f be increasing on $[0;1]$, decreasing on $[1;2]$ and increasing on $[2;3]$ with $f([0;1]) = [0;3]$, $f([1;2]) = [1;3]$ and $f([2;3]) = [1;4]$. We have $\mathrm{TV}^{\varphi}(f, [0;3]) = 6$, $\mathrm{UTV}^{\varphi}(f, [0;3]) = 6$ and $\mathrm{DTV}^{\varphi}(f, [0;3]) = 1$, thus

$$\mathrm{TV}^{\varphi}(f, [0;3]) < \mathrm{UTV}^{\varphi}(f, [0;3]) + \mathrm{DTV}^{\varphi}(f, [0;3]).$$

These and other properties of TV^{φ}, UTV^{φ} and DTV^{φ} for a general Young function φ will be the subject of further investigation.

Remark 1.4. Since we deal with càdlàg functions, a more natural setting of the first problem would be the investigation of

$$\inf\{\mathrm{TV}(g, [a;b]) : g \text{ càdlàg}, d_{\mathrm{D}}(g, f) \le c/2\},$$

where d_{D} denotes the Skorokhod metric (cf. [B, Chapt. 3]). However, the total variation does not change under (continuous and strictly increasing) transformations of the argument and for $E = \mathbb{R}$ with $\rho(x, y) = |x - y|$ the function f^c minimizing $\mathrm{TV}(g, [a;b])$ appears to be a càdlàg one, hence solutions of both problems coincide in this case.

Let us comment on the organisation of the paper. In the next section we deal with functions with values in general metric spaces and prove Fact 2.2. In the third section we deal with real càdlàg functions: we introduce some necessary definitions and notation, and present the construction of the functions f^c and $f^{0,c}$ for the first and the second problem. In the fourth section we establish the connection between $f^{0,c}$ and truncated variations. In the last section we summarise some other general properties of truncated variations, e.g. we show that for any real càdlàg function f, $\mathrm{TV}^c(f,[a;b])$ is a continuous, convex and decreasing function of the parameter $c>0$.

2. Truncated variation of functions with values in metric spaces. In this section we consider families of functions $f:[a;b]\to E$, with finite truncated variation for any $c>0$, even if their total variation appears to be infinite.

DEFINITION 2.1. Let $-\infty<a<b<\infty$ and $f:[a;b]\to E$. The function f is called *regulated* if for any $s\in(a;b)$ it has left and right limits, $f(s-)$ and $f(s+)$, and the limits $f(a+)$ and $f(b-)$ exist.

Each regulated function has an at most countable number of discontinuities (this follows easily from [DN, Chapt. 2, Corollary 2.2]), but this property is not sufficient for a function to be regulated.

FACT 2.2. *Let E be a complete metric space, $-\infty<a<b<\infty$ and $f:[a;b]\to E$. The following properties are equivalent:*

(a) *f is regulated;*
(b) *f is a uniform limit of finite variation functions;*
(c) *for any $c>0$, $\mathrm{TV}^c(f,[a;b])<\infty$.*

Proof. To prove (a)⇒(b) it is enough to notice that by [DN, Chapt. 2, Theorem 2.1]), f is a uniform limit of step functions of finite total variation (the assumption of [DN, Chapt. 2, Theorem 2.1] that E is a Banach space may be relaxed and the proof works for E a complete metric space). To prove (b)⇒(a) it is enough to notice that condition (b) of [DN, Chapt. 2, Theorem 2.1] holds for any function which is a uniform limit of finite variation functions.

The implication (b)⇒(c) follows immediately from the inequality (1.1), and to prove (c)⇒(b) it is enough to notice that every function satisfying (c) also satisfies condition (b) of [DN, Chapt. 2, Theorem 2.1]. ■

REMARK 2.3. When E is not a complete metric space, the families of functions satisfying conditions (b) and (c) of Fact 2.2 are still equal and contain the family of regulated functions (the implications (a)⇒(b) and (b)⇒(c) in the proof of [DN, Chapt. 2, Theorem 2.1] hold), but the latter family may be strictly smaller. To see this it is enough to note that the function $f : [0; 2] \to [0; 1)$ such that $f(x) = x1_{x<1}$ is not regulated for $E = [0; 1)$ with standard Euclidean metric, but it has finite total variation.

REMARK 2.4. From (1.2) we may derive an upper bound for

$$\inf_{g \in B(f,c/2)} \mathrm{TV}(g, [a; b])$$

when f is càdlàg and $E = \mathbb{R}^N$ with ρ induced by the L^1 norm. Namely, for $f(t) = (f_1(t), \ldots, f_N(t)) \in \mathbb{R}^N$, $\|f(t)\|_1 := |f_1(t)| + \cdots + |f_N(t)|$ and $\rho(f(t), g(t)) := \|f(t) - g(t)\|_1$, we have

$$\begin{aligned} \inf_{g \in B(f,c/2)} \mathrm{TV}(g, [a; b]) &\le \inf_{c_1,\ldots,c_N>0,\, c_1+\cdots+c_N=c} \sum_{i=1}^{N} \inf_{g_i \in B(f_i,c_i/2)} \mathrm{TV}(g_i, [a; b]) \\ &= \inf_{c_1,\ldots,c_N>0,\, c_1+\cdots+c_N=c} \sum_{i=1}^{N} \mathrm{TV}^{c_i}(f_i, [a; b]). \end{aligned}$$

Another upper bound for $\inf_{g \in B(f,c/2)} \mathrm{TV}(g, [a; b])$ was given in [TV, Theorems 10 and 11].

3. Solution of the first and second problems for real càdlàg functions

3.1. Definitions and notation. In this subsection we introduce definitions and notation which will be used throughout the paper.

Let $f : [a; b] \to \mathbb{R}$ be a càdlàg function. For $c > 0$ we define two stopping times:

$$T_{\mathrm{D}}^{c} f = \inf\Big\{ s \ge a : \sup_{t \in [a;s]} f(t) - f(s) \ge c \Big\},$$

$$T_{\mathrm{U}}^{c} f = \inf\Big\{ s \ge a : f(s) - \inf_{t \in [a;s]} f(t) \ge c \Big\}.$$

Assume that $T_{\mathrm{D}}^{c} f \ge T_{\mathrm{U}}^{c} f$, i.e. either the first upward jump of f of size c appears before the first downward jump of the same size, or both times are infinite (there is no upward or downward jump of size c). Note that in the case $T_{\mathrm{D}}^{c} f < T_{\mathrm{U}}^{c} f$ we may simply consider the function $-f$. Now we define sequences $(T_{\mathrm{U},k}^{c})_{k=0}^{\infty}$, $(T_{\mathrm{D},k}^{c})_{k=-1}^{\infty}$ in the following way: $T_{\mathrm{D},-1}^{c} = a$, $T_{\mathrm{U},0}^{c} = T_{\mathrm{U}}^{c} f$, and for $k = 0, 1, 2, \ldots,$

$$T_{\mathrm{D},k}^{c} = \begin{cases} \inf\Big\{ s \in [T_{\mathrm{U},k}^{c}; b] : \sup_{t \in [T_{\mathrm{U},k}^{c}; s]} f(t) - f(s) \ge c \Big\} & \text{if } T_{\mathrm{U},k}^{c} < b, \\ \infty & \text{if } T_{\mathrm{U},k}^{c} \ge b, \end{cases}$$

$$T^c_{\mathrm{U},k+1} = \begin{cases} \inf\left\{s \in [T^c_{\mathrm{D},k}; b] : f(s) - \inf_{t\in[T^c_{\mathrm{D},k};s]} f(t) \geq c\right\} & \text{if } T^c_{\mathrm{D},k} < b, \\ \infty & \text{if } T^c_{\mathrm{D},k} \geq b. \end{cases}$$

REMARK 3.1. The times $T^c_{\mathrm{U},k}$ and $T^c_{\mathrm{D},k}$ may be seen as the consecutive times of "switching" between the two disjoint borders $\{(t, f(t) - c/2) : t \in [a; b]\}$ and $\{(t, f(t) + c/2) : t \in [a; b]\}$ of the graph of a lazy function, which changes its value only if necessary for the relation $\|f - f^c\|_\infty \leq c/2$ to hold.

Note that there exists $K < \infty$ such that $T^c_{\mathrm{U},K} = \infty$ or $T^c_{\mathrm{D},K} = \infty$. Otherwise we would obtain two infinite sequences $(s_k)_{k=1}^\infty, (S_k)_{k=1}^\infty$ such that $a \leq s_1 < S_1 < s_2 < S_2 < \cdots \leq b$ and $f(S_k) - f(s_k) \geq c/2$. But this is a contradiction, since f is a càdlàg function and $(f(s_k))_{k=1}^\infty, (f(S_k))_{k=1}^\infty$ have a common limit.

Now for k such that $T^c_{\mathrm{D},k-1} < \infty$ and $T^c_{\mathrm{U},k} < \infty$ let us define two sequences of non-decreasing functions $m^c_k : [T^c_{\mathrm{D},k-1}; T^c_{\mathrm{U},k}) \cap [a; b] \to \mathbb{R}$ and $M^c_k : [T^c_{\mathrm{U},k}; T^c_{\mathrm{D},k}) \cap [a; b] \to \mathbb{R}$ by the formulas

$$m^c_k(s) = \inf_{t\in[T^c_{\mathrm{D},k-1};s]} f(t), \qquad M^c_k(s) = \sup_{t\in[T^c_{\mathrm{U},k};s]} f(t).$$

Next we define two finite sequences of real numbers (m^c_k) and (M^c_k), for k such that $T^c_{\mathrm{D},k-1} < \infty$ and $T^c_{\mathrm{U},k} < \infty$ respectively, by the formulas

$$m^c_k = m^c_k(T^c_{\mathrm{U},k}-) = \inf_{t\in[T^c_{\mathrm{D},k-1};T^c_{\mathrm{U},k})\cap[a;b]} f(t),$$
$$M^c_k = M^c_k(T^c_{\mathrm{D},k}-) = \sup_{t\in[T^c_{\mathrm{U},k};T^c_{\mathrm{D},k})\cap[a;b]} f(t).$$

3.2. Solution of the first problem. In this subsection we will solve the first problem: *what is the smallest possible* (*or infimum of*) *total variation of functions from the ball* $\{g : \|f - g\|_\infty \leq c/2\}$?

To solve this problem we start with some results concerning càdlàg functions. We apply the definitions of the previous subsection to the function f and assume that $T^c_{\mathrm{D}} f \geq T^c_{\mathrm{U}} f$. Define $f^c : [a; b] \to \mathbb{R}$ by

$$f^c(s) = \begin{cases} m^c_0 + c/2 & \text{if } s \in [a; T^c_{\mathrm{U},0}), \\ M^c_k(s) - c/2 & \text{if } s \in [T^c_{\mathrm{U},k}; T^c_{\mathrm{D},k}),\ k = 0, 1, 2, \ldots, \\ m^c_{k+1}(s) + c/2 & \text{if } s \in [T^c_{\mathrm{D},k}; T^c_{\mathrm{U},k+1}),\ k = 0, 1, 2, \ldots. \end{cases}$$

REMARK 3.2. Note that due to Remark 3.1, b belongs to one of the intervals $[T^c_{\mathrm{U},k}; T^c_{\mathrm{D},k})$ or $[T^c_{\mathrm{D},k}; T^c_{\mathrm{U},k+1})$ for some $k = 0, 1, 2, \ldots$ and the function f^c is defined for every $s \in [a; b]$.

REMARK 3.3. One may think about f^c as the laziest function possible which changes its value only if necessary for the relation $\|f - f^c\|_\infty \leq c/2$ to hold. Its starting value is such that it stays in $[f(t) - c/2; f(t) + c/2]$ for the longest time possible.

REMARK 3.4. In the case $T_D^c f < T_U^c f$ we may apply the definitions of the previous subsection to the function $-f$ and simply define $f^c = -(-f)^c$. Thus we will assume that the mapping $f \mapsto f^c$ is defined for any càdlàg function. Similarly, in all the proofs of this section we assume $T_D^c f \le T_U^c f$, but all results (i.e. Lemma 3.5, Theorem 3.6, Corollary 3.8, Lemma 3.10, Theorem 3.11, Corollary 3.12 and Theorem 4.1) apply to any càdlàg function f. Obvious modifications are only necessary in the definition of $T_{U,k}^c$ and $T_{D,k}^c$, and of f_U^c and f_D^c of Theorem 3.6.

LEMMA 3.5. *The function f^c uniformly approximates the function f with accuracy $c/2$ and has finite total variation. Moreover f^c is a càdlàg function and every point of discontinuity of f^c is also a point of discontinuity of f.*

Proof. Let us fix $s \in [a; b]$. We have three possibilities.

- $s \in [a; T_{U,0}^c)$. In this case, since $a \le s < T_U^c f \le T_D^c f$,
$$f(s) - f^c(s) = f(s) - \inf_{t \in [a; T_{U,0}^c)} f(t) - c/2 \in [-c/2; c/2).$$
- $s \in [T_{U,k}^c; T_{D,k}^c)$ for some $k = 0, 1, 2, \ldots$. In this case $M_k^c(s) - f(s)$ is in $[0; c)$, hence
$$f(s) - f^c(s) = f(s) - M_k^c(s) + c/2 \in (-c/2; c/2].$$
- $s \in [T_{D,k}^c; T_{U,k+1}^c)$ for some $k = 0, 1, 2, \ldots$. In this case $f(s) - m_{k+1}^c(s)$ belongs to $[0, c)$, hence
$$f(s) - f^c(s) = f(s) - m_{k+1}^c(s) - c/2 \in [-c/2; c/2).$$

The function f^c has finite total variation since it is non-decreasing on $[T_{U,k}^c; T_{D,k}^c)$, $k = 0, 1, 2, \ldots$, and non-increasing on $[T_{D,k}^c; T_{U,k+1}^c)$, $k = 0, 1, 2, \ldots$, and it has a finite number of jumps between these intervals.

For a similar reason, f^c has left and right limits. To see that it is right-continuous, let us fix $s \in [a; b]$ and notice that by definition of f^c, for $t \in (s; b]$ sufficiently close to s,
$$f^c(t) = \inf_{u \in [s;t]} f^c(u) \quad \text{or} \quad f^c(t) = \sup_{u \in [s;t]} f^c(u),$$
and the assertion follows from the right-continuity of f.

A similar argument may be applied to prove that f^c is continuous at every point of continuity of f except $T_{U,0}^c, T_{D,0}^c, T_{U,1}^c, T_{D,1}^c, \ldots$; but if $s = T_{D,i}^c$ and f is continuous at s then $f(T_{U,i}^c-) = f(T_{U,i}^c) = \inf_{t \in [T_{D,i-1}^c; T_{U,i}^c)} f(t) + c$ and
$$f^c(T_{U,i}^c-) = \inf_{t \in [T_{D,i-1}^c; T_{U,i}^c)} f(t) + c/2 = f(T_{U,i}^c) - c/2 = f^c(T_{U,i}^c).$$
A similar argument applies when $s = T_{D,i}^c$. ∎

Since f^c is of finite total variation, we know that there exist two non-decreasing f^c_{U} and $f^c_{\mathrm{D}} : [a;b] \to [0;\infty)$ such that $f^c(t) = f^c(a) + f^c_{\mathrm{U}}(t) - f^c_{\mathrm{D}}(t)$.

Let us examine the signs of the jumps of the function f^c between the intervals $[T^c_{\mathrm{U},k}; T^c_{\mathrm{D},k})$ and $[T^c_{\mathrm{D},k}; T^c_{\mathrm{U},k+1})$. Due to the càdlàg property we have

$$\begin{aligned}
f^c(T^c_{\mathrm{U},k}) - f^c(T^c_{\mathrm{U},k}-) &= f^c(T^c_{\mathrm{U},k}) - m^c_k - c/2 \\
&= f(T^c_{\mathrm{U},k}) - \inf_{t\in[T^c_{\mathrm{D},k-1};T^c_{\mathrm{U},k})} f(t) - c \geq 0, \\
f^c(T^c_{\mathrm{D},k}) - f^c(T^c_{\mathrm{D},k}-) &= f^c(T^c_{\mathrm{D},k}) - M^c_k + c/2 \\
&= f(T^c_{\mathrm{D},k}) - \sup_{t\in[T^c_{\mathrm{U},k};T^c_{\mathrm{D},k})} f(t) + c \leq 0.
\end{aligned}$$

Hence we may set $f^c_{\mathrm{U}}(s) = f^c_{\mathrm{D}}(s) = 0$ for $s \in [a; T^c_{\mathrm{U},0})$,

$$f^c_{\mathrm{U}}(s) = \begin{cases} \sum_{i=0}^{k-1}\{M^c_i - m^c_i - c\} + M^c_k(s) - m^c_k - c & \text{if } s \in [T^c_{\mathrm{U},k}; T^c_{\mathrm{D},k}), \\ \sum_{i=0}^{k}\{M^c_i - m^c_i - c\} & \text{if } s \in [T^c_{\mathrm{D},k}; T^c_{\mathrm{U},k+1}) \end{cases}$$

and

$$f^c_{\mathrm{D}}(s) = \begin{cases} \sum_{i=0}^{k-1}\{M^c_i - m^c_{i+1} - c\} & \text{if } s \in [T^c_{\mathrm{U},k}; T^c_{\mathrm{D},k}), \\ \sum_{i=0}^{k-1}\{M^c_i - m^c_{i+1} - c\} + M^c_k - m^c_{k+1}(s) - c & \text{if } s \in [T^c_{\mathrm{D},k}; T^c_{\mathrm{U},k+1}). \end{cases}$$

THEOREM 3.6. *If $g : [a;b] \to \mathbb{R}$ uniformly approximates f with accuracy $c/2$ and has finite total variation, and $g_{\mathrm{U}}, g_{\mathrm{D}} : [a;b] \to [0;\infty)$ are non-decreasing functions such that $g(t) = g(a) + g_{\mathrm{U}}(t) - g_{\mathrm{D}}(t)$, $t \in [a;b]$, then for any $s \in [a;b]$,*

$$g_{\mathrm{U}}(s) \geq f^c_{\mathrm{U}}(s) \quad \textit{and} \quad g_{\mathrm{D}}(s) \geq f^c_{\mathrm{D}}(s). \tag{3.1}$$

Proof. Again, we consider three cases.

- $s \in [a; T^c_{\mathrm{U},0})$. In this case $g_{\mathrm{U}}(s) \geq 0 = f^c_{\mathrm{U}}(s)$ as well as $g_{\mathrm{D}}(s) \geq 0 = f^c_{\mathrm{D}}(s)$.
- $s \in [T^c_{\mathrm{U},k}; T^c_{\mathrm{D},k})$ for some $k = 0, 1, 2, \ldots$. In this case, since g uniformly approximates f with accuracy $c/2$ and $g_{\mathrm{U}}, g_{\mathrm{D}}$ are non-decreasing, for $i = 0, 1, \ldots, k-1$ we get

$$\begin{aligned}
&\sup_{s_i\in[T^c_{\mathrm{U},i};T^c_{\mathrm{D},i})} g_{\mathrm{U}}(s_i) - \inf_{s_i\in[T^c_{\mathrm{D},i-1};T^c_{\mathrm{U},i})} g_{\mathrm{U}}(s_i) \\
&\qquad \geq \sup_{s_i\in[T^c_{\mathrm{U},i};T^c_{\mathrm{D},i})} (g_{\mathrm{U}} - g_{\mathrm{D}})(s_i) - \inf_{s_i\in[T^c_{\mathrm{D},i-1};T^c_{\mathrm{U},i})} (g_{\mathrm{U}} - g_{\mathrm{D}})(s_i)
\end{aligned}$$

$$
\begin{aligned}
&= \sup_{s_i \in [T^c_{\mathrm{U},i}; T^c_{\mathrm{D},i})} g(s_i) - \inf_{s_i \in [T^c_{\mathrm{D},i-1}; T^c_{\mathrm{U},i})} g(s_i) \\
&\geq \sup_{s_i \in [T^c_{\mathrm{U},i}; T^c_{\mathrm{D},i})} \{f(s_i) - c/2\} - \inf_{s_i \in [T^c_{\mathrm{D},i-1}; T^c_{\mathrm{U},i})} \{f(s_i) + c/2\} \\
&= M^c_i - m^c_i - c.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
g_{\mathrm{U}}(s) - \inf_{s_k \in [T^c_{\mathrm{D},k-1}; T^c_{\mathrm{U},k})} g_{\mathrm{U}}(s_k) \\
&= \sup_{t \in [T^c_{\mathrm{U},k}; s]} g_{\mathrm{U}}(t) - \inf_{s_k \in [T^c_{\mathrm{D},k-1}; T^c_{\mathrm{U},k})} g_{\mathrm{U}}(s_k) \\
&\geq \sup_{t \in [T^c_{\mathrm{U},k}; s]} (g_{\mathrm{U}} - g_{\mathrm{D}})(t) - \inf_{s_k \in [T^c_{\mathrm{D},k-1}; T^c_{\mathrm{U},k})} (g_{\mathrm{U}} - g_{\mathrm{D}})(s_k) \\
&= \sup_{t \in [T^c_{\mathrm{U},k}; s]} g(t) - \inf_{s_k \in [T^c_{\mathrm{D},k-1}; T^c_{\mathrm{U},k})} g(s_k) \\
&\geq \sup_{t \in [T^c_{\mathrm{U},k}; s]} \{f(t) - c/2\} - \inf_{s_k \in [T^c_{\mathrm{D},k-1}; T^c_{\mathrm{U},k})} \{f(s_k) + c/2\} \\
&= M^c_k(s) - m^c_k - c.
\end{aligned}
$$

Summing up and using the monotonicity of g_{U} we get

$$
g_{\mathrm{U}}(s) \geq \sum_{i=0}^{k-1} \{M^c_i - m^c_i - c\} + M^c_k(s) - m^c_k - c = f^c_{\mathrm{U}}(s).
$$

The proof of the corresponding inequality for g_{D} follows similarly and we get

$$
g_{\mathrm{D}}(s) \geq \sum_{i=0}^{k-1} \{M^c_i - m^c_{i+1} - c\} = f^c_{\mathrm{D}}(s).
$$

- $s \in [T^c_{\mathrm{D},k}; T^c_{\mathrm{U},k+1})$. The proof is similar to the previous case. ∎

From Theorem 3.6 we immediately see that the decomposition

$$
f^c(s) = f^c(a) + f^c_{\mathrm{U}}(s) - f^c_{\mathrm{D}}(s) \tag{3.2}
$$

is minimal (cf. [RY, p. 5]), thus the total variation of f^c on $[a; s]$ equals $f^c_{\mathrm{U}}(s) + f^c_{\mathrm{D}}(s)$.

REMARK 3.7. From Lemma 3.5 and the minimality of the decomposition (3.2) it follows that f^c_{U} and f^c_{U} are also càdlàg functions and that every point of discontinuity of f^c_{U} or f^c_{U} is also a point of discontinuity of f. Moreover, due to the minimality of the variation of f^c, jumps of f^c are no greater than jumps of f.

COROLLARY 3.8. *The function f^c is optimal, i.e. if $g : [a; b] \to \mathbb{R}$ is such that $\|f - g\|_\infty \le c/2$ and has finite total variation, then for every $s \in [a; b]$,*

$$\mathrm{TV}(g, [a; s]) \ge \mathrm{TV}(f^c, [a; s]).$$

Moreover, it is unique in the sense that if for every $s \in [a; b]$ the opposite inequality holds,

$$\mathrm{TV}(g, [a; s]) \le \mathrm{TV}(f^c, [a; s]),$$

and $c \le \sup_{s,u\in[a;b]} |f(s) - f(u)|$, then $g = f^c$.

Proof. Let $g_{\mathrm{U}}, g_{\mathrm{D}} : [a; b] \to [0; \infty)$ be non-decreasing functions such that for $s \in [a; b]$,

$$g(s) = g(a) + g_{\mathrm{U}}(s) - g_{\mathrm{D}}(s) \quad \text{and} \quad \mathrm{TV}(g, [a; s]) = g_{\mathrm{U}}(s) + g_{\mathrm{D}}(s).$$

The first assertion follows directly from Theorem 3.6 and the fact that $\mathrm{TV}(g, [a; s]) = g_{\mathrm{U}}(s) + g_{\mathrm{D}}(s)$.

The opposite inequality, $\mathrm{TV}(g, [a; s]) \le \mathrm{TV}(f^c, [a; s])$, holds for every $s \in [a; b]$ iff $g_{\mathrm{U}}(s) = f^c_{\mathrm{U}}(s)$ and $g_{\mathrm{D}}(s) = f^c_{\mathrm{D}}(s)$. Thus in that case we get $g(s) - f^c(s) = g(a) - f^c(a)$ and

$$\begin{aligned} c/2 &\ge \sup_{s\in[a;T^c_{\mathrm{U},0})} \{g(s) - f(s)\} \\ &= \sup_{s\in[a;T^c_{\mathrm{U},0})} \{g(a) - f^c(a) + f^c(s) - f(s)\} \\ &= g(a) - f^c(a) + c/2 \end{aligned} \tag{3.3}$$

(notice that $T^c_{\mathrm{U},0} \le b$ since $c \le \sup_{s,u\in[a;b]} |f(s) - f(u)|$ and $T^c_{\mathrm{U},0} \le T^c_{\mathrm{D},0}$, and that $f^c(T^c_{\mathrm{U},0}-) - \inf_{s\in[a;T^c_{\mathrm{U},0})} f(s) = c/2$). On the other hand,

$$\begin{aligned} -c/2 \le g(T^c_{\mathrm{U},0}) - f(T^c_{\mathrm{U},0}) &= g(a) - f^c(a) + f^c(T^c_{\mathrm{U},0}) - f(T^c_{\mathrm{U},0}) \\ &= g(a) - f^c(a) - c/2. \end{aligned} \tag{3.4}$$

From (3.3) and (3.4) we get $g(a) = f^c(a)$. This together with the equalities $g_{\mathrm{U}}(s) = f^c_{\mathrm{U}}(s)$ and $g_{\mathrm{D}}(s) = f^c_{\mathrm{D}}(s)$ gives $g = f^c$. ∎

REMARK 3.9. The formula obtained for the smallest possible total variation of a function from the ball $\{g : \|f - g\|_\infty \le c/2\}$ reads

$$f^c_{\mathrm{U}}(b) + f^c_{\mathrm{D}}(b)$$

and does not resemble formula (1.2). In Section 4 we will show that these values coincide.

3.3. Solution of the second problem. In this subsection we will solve the second problem: *for a càdlàg function $f : [a; b] \to \mathbb{R}$ and $c > 0$ find*

$$\inf\{\mathrm{TV}(f + h, [a; b]) : \|h\|_{\mathrm{osc}} \le c\},$$

where $h : [a; b] \to \mathbb{R}$, $\|h\|_{\mathrm{osc}} := \sup_{s,u\in[a;b]} |h(s) - h(u)|$.

We will show that

$$\inf\{\mathrm{TV}(f+h,[a;b]) : \|h\|_{\mathrm{osc}} \le c\} = f_{\mathrm{U}}^{c}(b) + f_{\mathrm{D}}^{c}(b),$$

where f_{U}^{c} and f_{D}^{c} were defined in the previous subsection. To do it let us simply define

$$f^{0,c} = f_{\mathrm{U}}^{c} - f_{\mathrm{D}}^{c}.$$

LEMMA 3.10. *The increments of $f^{0,c}$ uniformly approximate the increments of f with accuracy c, and the function $f^{0,c}$ has finite total variation.*

Proof. Since $f^{c} - f^{0,c}$ is constant, the first and the second assertions follow immediately from Lemma 3.5 and from a simple calculation that for any $s, u \in [a;b]$,

$$\begin{aligned}\{f^{0,c}(s) - f^{0,c}(u)\} &- \{f(s) - f(u)\}\\ &= \{f^{c}(s) - f(s)\} - \{f^{c}(u) - f(u)\} \in [-c;c]. \ \blacksquare\end{aligned}$$

Now we will prove the analog of Theorem 3.6.

THEOREM 3.11. *If the increments of $g : [a;b] \to \mathbb{R}$ uniformly approximate the increments of f with accuracy c, g has finite total variation and $g_{\mathrm{U}}, g_{\mathrm{D}} : [a;b] \to [0;\infty)$ are non-decreasing functions such that $g(t) = g(a) + g_{\mathrm{U}}(t) - g_{\mathrm{D}}(t)$, $t \in [a;b]$, then for any $s \in [a;b]$,*

$$g_{\mathrm{U}}(s) \ge f_{\mathrm{U}}^{c}(s) \quad \text{and} \quad g_{\mathrm{D}}(s) \ge f_{\mathrm{D}}^{c}(s).$$

Proof. It is enough to see that for $h = g - f$, $\|h\|_{\mathrm{osc}} \le c$, thus for

$$\alpha = -\frac{1}{2}\Big\{\inf_{s\in[a;b]} h(s) + \sup_{s\in[a;b]} h(s)\Big\},$$

we have $\|\alpha + h\|_{\infty} \le \frac{1}{2}c$, and the function $g_{\alpha} = \alpha + g$ belongs to the ball $\{g : \|f-g\|_{\infty} \le \frac{1}{2}c\}$. Application of Theorem 3.6 to the function g_{α} concludes the proof. $\blacksquare$

Since the decomposition $f^{0,c}(s) = f_{\mathrm{U}}^{c}(s) - f_{\mathrm{D}}^{c}(s)$ is minimal and $f^{0,c}(a) = 0$ we immediately obtain

COROLLARY 3.12. *The function $f^{0,c}$ is optimal, i.e. if $g : [a;b] \to \mathbb{R}$ is such that*

$$\sup_{a\le u<s\le b} |\{g(s) - g(u)\} - \{f(s) - f(u)\}| \le c$$

and g has finite total variation, then for every $s \in [a;b]$,

$$\mathrm{TV}(g,[a;s]) \ge \mathrm{TV}(f^{0,c},[a;s]).$$

Moreover, it is unique in the sense that if $g(a) = 0$ and for every $s \in [a;b]$ the opposite inequality holds,

$$\mathrm{TV}(g,[a;s]) \le \mathrm{TV}(f^{0,c},[a;s]),$$

then $g = f^{0,c}$.

From Corollary 3.12 it immediately follows that

$$\inf\{\mathrm{TV}(f+h,[a;b]) : \|h\|_{\mathrm{osc}} \le c\} = f_{\mathrm{U}}^c(b) + f_{\mathrm{D}}^c(b).$$

Indeed, for any h such that $\|h\|_{\mathrm{osc}} \le c$ we put $g = f + h$ and if g has finite total variation then it satisfies the assumptions of Corollary 3.12 and we get

$$\mathrm{TV}(g,[a;b]) \ge \mathrm{TV}(f^{0,c},[a;b]) = f_{\mathrm{U}}^c(b) + f_{\mathrm{D}}^c(b).$$

4. Relation of the solutions of the first and second problems to truncated variations. In order to prove (1.2), (1.4) and (1.8), where $\mathrm{UTV}^c(f,[a;s])$ and $\mathrm{DTV}^c(f,[a;s])$ are defined by (1.6) and (1.7) respectively, it is enough to prove

THEOREM 4.1. *For a given càdlàg function $f : [a;b] \to \mathbb{R}$ and for any $s \in (a;b]$ the following equalities hold:*

$$\mathrm{UTV}^c(f,[a;s]) = f_{\mathrm{U}}^c(s), \tag{4.1}$$

$$\mathrm{DTV}^c(f,[a;s]) = f_{\mathrm{D}}^c(s), \tag{4.2}$$

$$\mathrm{TV}^c(f,[a;s]) = f_{\mathrm{U}}^c(s) + f_{\mathrm{D}}^c(s). \tag{4.3}$$

Proof. Examining the proof of Lemma 3 from [L2], we see that it may be applied (with obvious modifications) to the càdlàg (but not necessarily continuous) function f and we obtain

$$\begin{aligned}\mathrm{UTV}^c(f,[a;s]) \\ = \sup_{a\le t<u\le (T_{\mathrm{D}}^c f)\wedge s} (f(u)-f(t)-c)_+ + \mathrm{UTV}^c(f,[(T_{\mathrm{D}}^c f)\wedge s; s]).\end{aligned} \tag{4.4}$$

Now, from the assumption $T_{\mathrm{D}}^c f \ge T_{\mathrm{U}}^c f$ we get $T_{\mathrm{D}}^c f = T_{\mathrm{D},0}^c$ and we have

$$\sup_{a\le t<u\le (T_{\mathrm{D}}^c f)\wedge s} (f(u)-f(t)-c)_+ = \begin{cases} 0 & \text{if } s \in [a; T_{\mathrm{U},0}^c), \\ M_0^c(s) - m_0^c - c & \text{if } s \in [T_{\mathrm{U},0}^c; T_{\mathrm{D},0}^c), \\ M_0^c - m_0^c - c & \text{if } s \ge T_{\mathrm{D},0}^c. \end{cases}$$

Iterating equality (4.4) we obtain

$$\begin{aligned}&\mathrm{UTV}^c(f,[a;s]) \\ &\quad = \begin{cases} 0 & \text{if } s \in [a; T_{\mathrm{U},0}^c), \\ \displaystyle\sum_{i=0}^{k-1}(M_i^c - m_i^c - c) + M_k^c(s) - m_k^c - c & \text{if } s \in [T_{\mathrm{U},k}^c; T_{\mathrm{D},k}^c), \\ \displaystyle\sum_{i=0}^{k}(M_i^c - m_i^c - c) & \text{if } s \in [T_{\mathrm{D},k}^c; T_{\mathrm{U},k+1}^c) \end{cases} \\ &\quad = f_{\mathrm{U}}^c(s).\end{aligned}$$

Remark 4.2. Iterating (4.4) we obtain an equality a bit different from $\mathrm{UTV}^c(f,[a;s]) = f_{\mathrm{U}}^c$, but equivalent to it. To see this let us define the following sequence of times. We set $\tilde{T}_{\mathrm{D},-1}^c = a$, and for $k = 0,1,2,\dots$,

$$\tilde{T}_{\mathrm{D},k+1}^c = \inf\Big\{s > \tilde{T}_{\mathrm{D},k}^c : \sup_{t\in[\tilde{T}_{\mathrm{D},k}^c;s]} f(t) - f(s) \ge c\Big\}.$$

Let us fix $s_0 \in [a;b]$ and define $k_0 = \max\{k : \tilde{T}_{\mathrm{D},k}^c \le s_0\}$. Iterating (4.4) we obtain

$$\begin{aligned}&\mathrm{UTV}^c(f,[a;s_0])\\ &\quad= \sum_{k=1}^{k_0-1} \sup_{\tilde{T}_{\mathrm{D},k}^c\le s<u\le\tilde{T}_{\mathrm{D},k+1}^c} (f(u)-f(s)-c)_+ + \mathrm{UTV}^c(f,[\tilde{T}_{\mathrm{D},k_0}^c;s_0]),\end{aligned}$$

whose right hand side looks different from $f_{\mathrm{U}}^c(s_0)$. But it is easy to notice that for all $k \ge 1$ such that $\tilde{T}_{\mathrm{D},k+1}^c < T_{\mathrm{U},1}^c f$ the summand $\sup_{\tilde{T}_{\mathrm{D},k}^c\le s<u\le\tilde{T}_{\mathrm{D},k+1}^c}(f(u) - f(s) - c)_+$ is zero. Thus in fact the quantities $\mathrm{UTV}^c(f,[a;s_0])$ and $f_{\mathrm{U}}^c(s_0)$ coincide.

In the same way we prove that $\mathrm{DTV}^c(f)[a;s] = f_{\mathrm{D}}^c(s)$.

Now, to prove (4.3) simply notice that $\mathrm{TV}^c(f,[a;s]) \ge 0$ and if $s \in [T_{\mathrm{U},k}^c;T_{\mathrm{D},k}^c)$,

$$\begin{aligned}\mathrm{TV}^c(f,[a;s]) &\ge \sum_{i=0}^{k-1}(M_i^c - m_i^c - c) + \sum_{i=0}^{k-1}(M_i^c - m_{i+1}^c - c) + M_k^c(s) - m_k^c - c\\ &= f_{\mathrm{U}}^c(s) + f_{\mathrm{D}}^c(s).\end{aligned}$$

Analogously, if $s \in [T_{\mathrm{D},k}^c;T_{\mathrm{U},k+1}^c)$,

$$\begin{aligned}\mathrm{TV}^c(f,[a;s]) \ge& \sum_{i=0}^{k-1}(M_i^c - m_i^c - c)\\ &+ \sum_{i=0}^{k-1}(M_i^c - m_{i+1}^c - c) + M_k^c - m_{k+1}^c(s) - c\\ =& f_{\mathrm{U}}^c(s) + f_{\mathrm{D}}^c(s).\end{aligned}$$

Hence for all $s \in [a;b]$,

$$\mathrm{TV}^c(f,[a;s]) \ge f_{\mathrm{U}}^c(s) + f_{\mathrm{D}}^c(s).$$

So

$$\mathrm{TV}^c(f,[a;s]) \ge \mathrm{UTV}^c(f,[a;s]) + \mathrm{DTV}^c(f,[a;s]).$$

Since the opposite inequality is obvious, we finally get (4.3). ∎

Now we see that by Corollaries 3.8 and 3.12, the functions $h^c = f^c - f$ and $h^{0,c} = f(a) + f^{0,c} - f = f(a) + \mathrm{UTV}^c(f,[a;.]) - \mathrm{DTV}^c(f,[a;.]) - f$ are optimal and such that for any $s \in (a;b]$,

$$\begin{aligned}\inf\{\mathrm{TV}(f+h,[a;s]) : \|h\|_\infty \le c/2\} &= \mathrm{TV}(f+h^c,[a;s])\\ &= \mathrm{TV}^c(f,[a;s]),\\ \inf\{\mathrm{TV}(f+h,[a;s]) : \|h\|_{\mathrm{osc}} \le c\} &= \mathrm{TV}(f+h^{0,c},[a;s])\\ &= \mathrm{TV}^c(f,[a;s]).\end{aligned}$$

Moreover, by Remark 3.7, h^c and $h^{0,c}$ are also càdlàg functions and every point of discontinuity of h^c or $h^{0,c}$ is also a point of discontinuity of f.

5. Further properties of truncated variations. In this section we summarize the basic properties of the functionals defined. We start with

5.1. Algebraic properties. For any $c > 0$ we have

$$\mathrm{DTV}^c(f,[a;b]) = \mathrm{UTV}^c(-f,[a;b]), \tag{5.1}$$

$$\mathrm{TV}^c(f,[a;b]) = \mathrm{UTV}^c(f,[a;b]) + \mathrm{DTV}^c(f,[a;b]). \tag{5.2}$$

Property (5.1) follows simply from the definitions (1.6) and (1.7). Property (5.2) is a consequence of Theorem 4.1.

5.2. $\mathrm{UTV}^c(f,[a;b]), \mathrm{DTV}^c(f,[a;b])$ **and** $\mathrm{TV}^c(f,[a;b])$ **as functions of** c

FACT 5.1. *For any càdlàg function f the functions*

$$(0;\infty) \ni c \mapsto \mathrm{UTV}^c(f,[a;b]) \in [0;\infty),$$
$$(0;\infty) \ni c \mapsto \mathrm{DTV}^c(f,[a;b]) \in [0;\infty), (0;\infty) \ni c \mapsto \mathrm{TV}^c(f,[a;b]) \in [0;\infty)$$

are non-increasing, continuous, and convex functions of c. Moreover, $\lim_{c\downarrow 0} \mathrm{TV}^c(f,[a;b]) = \mathrm{TV}(f,[a;b])$ and for any $c \ge \|f\|_{\mathrm{osc}}$, $\mathrm{TV}^c(f,[a;b]) = 0$.

Proof. The finiteness of TV, UTV and DTV follows from Lemma 3.5 and Theorem 4.1. Monotonicity is obvious.

To prove convexity, let us fix $c, \varepsilon > 0$ and consider a partition $a \le t_0 < t_1 < \cdots < t_n \le b$ such that

$$\mathrm{UTV}^c(f,[a;b]) \le \sum_{i=0}^{n-1} \max\{f(t_{i+1}) - f(t_i) - c, 0\} + \varepsilon.$$

Taking $\alpha \in [0;1]$ and $c_1, c_2 > 0$ such that $c = \alpha c_1 + (1-\alpha)c_2$ we have

$$\begin{aligned}&\max\{f(t_{i+1}) - f(t_i) - \alpha c_1 - (1-\alpha)c_2, 0\}\\ &\quad = \max\{\alpha(f(t_{i+1}) - f(t_i) - c_1) + (1-\alpha)(f(t_{i+1}) - f(t_i) - c_2), 0\}\\ &\quad \le \alpha \max\{f(t_{i+1}) - f(t_i) - c_1, 0\} + (1-\alpha)\max\{f(t_{i+1}) - f(t_i) - c_2, 0\}.\end{aligned}$$

Now

$$
\begin{aligned}
\mathrm{UTV}^c(f,[a;b]) &\le \sum_{i=0}^{n-1} \max\{f(t_{i+1}) - f(t_i) - c, 0\} + \varepsilon \\
&\le \alpha \sum_{i=0}^{n-1} \max\{f(t_{i+1}) - f(t_i) - c_1, 0\} \\
&\quad + (1-\alpha) \sum_{i=0}^{n-1} \max\{f(t_{i+1}) - f(t_i) - c_2, 0\} + \varepsilon \\
&\le \alpha \mathrm{UTV}^{c_1}(f,[a;b]) + (1-\alpha)\mathrm{UTV}^{c_2}(f,[a;b]) + \varepsilon.
\end{aligned}
$$

Since ε may be arbitrarily small, we obtain the convexity assertion. From convexity and monotonicity we obtain the continuity assertion.

The same properties of DTV and TV follow immediately from (5.1) and (5.2).

The fact that $\mathrm{TV}^c(f,[a;b]) = 0$ for $c \ge \|f\|_{\mathrm{osc}}$ follows easily from the equality

$$
\max\{|f(t_{i+1}) - f(t_i)| - c, 0\} = 0
$$

satisfied for any such c and $t_i, t_{i+1} \in [a;b]$. ■

REMARK 5.2. [TV, Theorem 17] gives some estimates for the rate of convergence of $\mathrm{TV}^c(f,[a;b])$ to ∞ when $c \downarrow 0$ and f has finite p-variation with $p > 1$.

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