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ALGEBRAIC AND TOPOLOGICAL STRUCTURES ON THE SET OF MEAN FUNCTIONS AND GENERALIZATION OF THE AGM MEAN

BY

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Abstract. We present new structures and results on the set $\mathcal{M}_{\mathscr{D}}$ of mean functions on a given symmetric domain $\mathscr{D}$ in $\mathbb{R}^2$. First, we construct on $\mathcal{M}_{\mathscr{D}}$ a structure of abelian group in which the neutral element is the *arithmetic* mean; then we study some symmetries in that group. Next, we construct on $\mathcal{M}_{\mathscr{D}}$ a structure of metric space under which $\mathcal{M}_{\mathscr{D}}$ is the closed ball with center the *arithmetic* mean and radius $1/2$. We show in particular that the *geometric* and *harmonic* means lie on the boundary of $\mathcal{M}_{\mathscr{D}}$. Finally, we give two theorems generalizing the construction of the AGM mean. Roughly speaking, those theorems show that for any two given means M_1 and M_2, which satisfy some regularity conditions, there exists a unique mean M satisfying the functional equation $M(M_1, M_2) = M$.

1. Introduction.

Let $\mathscr{D}$ be a nonempty symmetric domain in $\mathbb{R}^2$. A *mean function* (or simply a *mean*) on $\mathscr{D}$ is a function $M : \mathscr{D} \to \mathbb{R}$ satisfying the following three axioms:

(i) M is symmetric, that is, $M(x, y) = M(y, x)$ for all $(x, y) \in \mathscr{D}$.
(ii) For all $(x, y) \in \mathscr{D}$, we have $\min(x, y) \le M(x, y) \le \max(x, y)$.
(iii) For all $(x, y) \in \mathscr{D}$, we have $M(x, y) = x \Rightarrow x = y$.

Note that because of (ii), the implication in (iii) is actually an equivalence.

Among the most known examples of mean functions, we cite:

- The arithmetic mean A defined on $\mathbb{R}^2$ by $\mathrm{A}(x, y) = \frac{x+y}{2}$.
- The geometric mean G defined on $(0, +\infty)^2$ by $\mathrm{G}(x, y) = \sqrt{xy}$.
- The harmonic mean H defined on $(0, +\infty)^2$ by $\mathrm{H}(x, y) = \frac{2xy}{x+y}$.
- The Gauss arithmetic-geometric mean AGM defined on $(0, +\infty)^2$ as follows: Given positive real numbers x, y, we let $\mathrm{AGM}(x, y)$ be the

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common limit of the two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ defined by

$$\begin{cases} x_0 = x, \quad y_0 = y, \\ x_{n+1} = \dfrac{x_n + y_n}{2} & (\forall n \in \mathbb{N}), \\ y_{n+1} = \sqrt{x_n y_n} & (\forall n \in \mathbb{N}). \end{cases}$$

For a survey on mean functions, we refer to Chapter 8 of the book [Bor], in which AGM takes the principal place. However, there are some differences between that reference and the present paper. In [Bor], only axiom (ii) is taken to define a mean function; (iii) is added to obtain the so called *strict* mean, while (i) is not considered. In this paper, we shall see that the three axioms (i), (ii) and (iii) are both necessary and sufficient to define a *good* mean or a *good* set of mean functions on a given domain. In particular, axiom (iii), absent in [Bor], is necessary for the foundation of our algebraic and topological structures (see Sections 2 and 3).

Given a nonempty symmetric domain $\mathscr{D}$ in $\mathbb{R}^2$, we denote by $\mathcal{M}_{\mathscr{D}}$ the set of mean functions on $\mathscr{D}$. The purpose of this paper is on the one hand to establish some algebraic and topological structures on $\mathcal{M}_{\mathscr{D}}$ and to study some of their properties, and on the other hand to generalize in a natural way the arithmetic-geometric mean AGM.

In Section 2, we define on $\mathcal{M}_{\mathscr{D}}$ a structure of abelian group in which the neutral element is the arithmetic mean. The study of this group reveals that the arithmetic, geometric and harmonic means lie in a particular class of mean functions that we call *normal mean functions*. We then study symmetries on $\mathcal{M}_{\mathscr{D}}$ and we discover that the symmetry with respect to each of the three means A, G and H coincides with another type of symmetry (with respect to the same means) which we call *functional symmetry*. The problem of describing the set of all means realizing that curious coincidence is still open.

In Section 3, we define on $\mathcal{M}_{\mathscr{D}}$ a structure of metric space which turns out to be a closed ball with center A and radius $1/2$. We then use the group structure to calculate the distance between two means on $\mathscr{D}$; this permits us in particular to establish a simple characterization of the boundary of $\mathcal{M}_{\mathscr{D}}$.

In Section 4, we introduce the concept of *functional middle* of two mean functions on $\mathscr{D}$ which generalizes in a natural way the arithmetic-geometric mean, so that the latter is the functional middle of the arithmetic and geometric means. We establish two sufficient conditions for the existence and uniqueness of the functional middle of two means. The first one uses the metric space structure of $\mathcal{M}_{\mathscr{D}}$ by imposing on the two means in question the condition that the distance between them is less than 1. The second requires the two means to be continuous on $\mathscr{D}$. In the proof of the latter, axiom (iii) plays a vital role.

2. An abelian group structure on $\mathcal{M}_{\mathscr{D}}$. Given a nonempty symmetric domain $\mathscr{D}$ in $\mathbb{R}^2$, we denote by $\mathcal{A}_{\mathscr{D}}$ the set of asymmetric maps on $\mathscr{D}$, that is, maps $f : \mathscr{D} \to \mathbb{R}$ satisfying

$$f(x,y) = -f(y,x) \quad (\forall (x,y) \in \mathscr{D}).$$

It is clear that $(\mathcal{A}_{\mathscr{D}}, +)$ (where $+$ is the usual addition of maps from $\mathscr{D}$ into $\mathbb{R}$) is an abelian group with neutral element the null map.

Now, consider $\widetilde{\varphi} : \mathcal{M}_{\mathscr{D}} \to \mathbb{R}^{\mathscr{D}}$ defined by:

$$\forall M \in \mathcal{M}_{\mathscr{D}}, \forall (x,y) \in \mathscr{D} : \quad \widetilde{\varphi}(M)(x,y) := \begin{cases} \log\left(-\frac{M(x,y)-x}{M(x,y)-y}\right) & \text{if } x \neq y, \\ 0 & \text{if } x = y. \end{cases}$$

The axioms (i)–(iii) ensure that $-\frac{M(x,y)-x}{M(x,y)-y}$ (for $x \neq y$) is well-defined and positive.

THEOREM 2.1. *We have $\widetilde{\varphi}(\mathcal{M}_{\mathscr{D}}) = \mathcal{A}_{\mathscr{D}}$. In addition, the map $\varphi : M \mapsto \widetilde{\varphi}(M)$ is a bijection from $\mathcal{M}_{\mathscr{D}}$ to $\mathcal{A}_{\mathscr{D}}$ and its inverse is given by*

$$\forall f \in \mathcal{A}_D, \forall (x,y) \in \mathscr{D} : \quad \varphi^{-1}(f)(x,y) = \frac{x + ye^{f(x,y)}}{e^{f(x,y)} + 1}. \tag{2.1}$$

Proof. Axiom (i) ensures that for all $M \in \mathcal{M}_{\mathscr{D}}$, we have $\widetilde{\varphi}(M) \in \mathcal{A}_D$. Next, if f is an asymmetric map on $\mathscr{D}$, we easily verify that $M : \mathscr{D} \to \mathbb{R}$ defined by

$$M(x,y) := \frac{x + ye^{f(x,y)}}{e^{f(x,y)} + 1} \quad (\forall (x,y) \in \mathscr{D})$$

is a mean on $\mathscr{D}$ and $\widetilde{\varphi}(M) = f$. Since obviously $\widetilde{\varphi}$ is injective, the proof is finished. ∎

We now transport, by φ, the abelian group structure $(\mathcal{A}_{\mathscr{D}}, +)$ onto $\mathcal{M}_{\mathscr{D}}$, that is, we define on $\mathcal{M}_{\mathscr{D}}$ the following composition law $*$:

$$\forall M_1, M_2 \in \mathcal{M}_{\mathscr{D}} : \quad M_1 * M_2 = \varphi^{-1}(\varphi(M_1) + \varphi(M_2)).$$

So $(\mathcal{M}_{\mathscr{D}}, *)$ is an abelian group and φ is a group isomorphism from $(\mathcal{M}_{\mathscr{D}}, *)$ to $(\mathcal{A}_D, +)$. Furthermore, since the null map on $\mathscr{D}$ is the neutral element of $(\mathcal{A}_{\mathscr{D}}, +)$ and $\varphi^{-1}(0) = \mathrm{A}$, the arithmetic mean A is the neutral element of $(\mathcal{M}_{\mathscr{D}}, *)$.

By explicitly calculating $M_1 * M_2$ (for $M_1, M_2 \in \mathcal{M}_{\mathscr{D}}$), we obtain:

PROPOSITION 2.2. *The composition law $*$ on $\mathcal{M}_{\mathscr{D}}$ is defined by*

$$(M_1 * M_2)(x,y) := \begin{cases} \frac{x(M_1(x,y)-y)(M_2(x,y)-y)+y(M_1(x,y)-x)(M_2(x,y)-x)}{(M_1(x,y)-x)(M_2(x,y)-x)+(M_1(x,y)-y)(M_2(x,y)-y)} & \text{if } x \neq y, \\ x & \text{if } x = y, \end{cases}$$

for $M_1, M_2 \in \mathcal{M}_{\mathscr{D}}$ and $(x,y) \in \mathscr{D}$. ∎

Now, it is easy to verify that the images of the geometric and harmonic means under the isomorphism φ are given by

$$\varphi(\mathrm{G})(x,y) = \tfrac{1}{2}\log x - \tfrac{1}{2}\log y \qquad (\forall (x,y) \in (0,+\infty)^2), \tag{2.2}$$

$$\varphi(\mathrm{H})(x,y) = \log x - \log y \qquad (\forall (x,y) \in (0,+\infty)^2). \tag{2.3}$$

From (2.2) and (2.3), we see that $\varphi(\mathrm{G})$ and $\varphi(\mathrm{H})$ (and trivially also $\varphi(\mathrm{A})$) have a particular form: each can be written as $h(x) - h(y)$, where h is a real function of one variable.

To generalize, we define a *normal mean* as a mean function $M : I^2 \to \mathbb{R}$ ($I \subset \mathbb{R}$) such that $\varphi(M)$ has the form $h(x) - h(y)$ for some map $h : I \to \mathbb{R}$. Equivalently, a *normal mean function* is a function $M : I^2 \to \mathbb{R}$ ($I \subset \mathbb{R}$) which can be written as

$$M(x,y) = \frac{xP(x) + yP(y)}{P(x) + P(y)} \qquad (\forall x, y \in I),$$

where $P : I \to \mathbb{R}$ is a positive function on I.

Some symmetries on $(\mathcal{M}_{\mathscr{D}}, *)$. We are now interested in the symmetric image of a given mean M_1 with respect to another mean M_0 via the group structure $(\mathcal{M}_{\mathscr{D}}, *)$. Denote by S_{M_0} the symmetry with respect to M_0 in the group $(\mathcal{M}_{\mathscr{D}}, *)$, defined by

$$\forall M_1, M_2 \in \mathcal{M}_{\mathscr{D}} : \quad S_{M_0}(M_1) = M_2 \;\Leftrightarrow\; M_1 * M_2 = M_0 * M_0.$$

Using the group isomorphism φ, we obtain by a simple calculation an explicit expression of $S_{M_0}(M_1)$:

Proposition 2.3. *For any $M_0, M_1 \in \mathcal{M}_{\mathscr{D}}$,*

$$S_{M_0}(M_1) = \frac{x(M_1 - x)(M_0 - y)^2 - y(M_0 - x)^2(M_1 - y)}{(M_1 - x)(M_0 - y)^2 - (M_0 - x)^2(M_1 - y)},$$

where, for simplicity, we have written M_0 for $M_0(x,y)$, M_1 for $M_1(x,y)$ and $S_{M_0}(M_1)$ for $S_{M_0}(M_1)(x,y)$. ■

As an application, we get the following immediate corollary:

Corollary 2.4. *For any $M \in \mathcal{M}_{\mathscr{D}}$, we have:*

(1) $S_{\mathrm{A}}(M) = x + y - M = 2\mathrm{A} - M$.

(2) $S_{\mathrm{G}}(M) = \dfrac{xy}{M} = \dfrac{\mathrm{G}^2}{M}$ (*when* $\mathscr{D} \subset (0,+\infty)^2$).

(3) $S_{\mathrm{H}}(M) = \dfrac{xyM}{(x+y)M - xy} = \dfrac{\mathrm{H}M}{2M - \mathrm{H}}$ (*when* $\mathscr{D} \subset (0,+\infty)^2$).

(4) $S_{\mathrm{H}} = S_{\mathrm{G}} \circ S_{\mathrm{A}} \circ S_{\mathrm{G}}$. ■

Now, we are going to define another symmetry on $\mathcal{M}_{\mathscr{D}}$ (for $\mathscr{D}$ of a certain form), independent of the group structure $(\mathcal{M}_{\mathscr{D}}, *)$. This new symmetry is determined by solving a functional equation but it curiously coincides, in many cases, with the symmetry defined above.

Definition 2.5. Let I be a nonempty interval of $\mathbb{R}$, $\mathscr{D} = I^2$ and M_0, M_1 and M_2 be three mean functions on $\mathscr{D}$ such that M_1 and M_2 take their values in I. We say that M_2 is the *functional symmetric mean* of M_1 with respect to M_0 if the following functional equation is satisfied:

$$M_0(M_1(x,y), M_2(x,y)) = M_0(x,y) \quad (\forall (x,y) \in \mathscr{D}).$$

Equivalently, we say that M_0 is the *functional middle* of M_1 and M_2.

According to axiom (iii), it is immediate that if the functional symmetric mean exists then it is unique. This justifies the following notation:

Notation 2.6. Given two mean functions M_0 and M_1 on $\mathscr{D} = I^2$ with values in I (where I is an interval of $\mathbb{R}$), we denote by $\sigma_{M_0}(M_1)$ the functional symmetric mean (if it exists) of M_1 with respect to M_0.

A simple calculation establishes the following:

Proposition 2.7. *Let M be a mean function on a suitable symmetric domain $\mathscr{D}$ of $\mathbb{R}^2$. Then*

$$\begin{aligned} \sigma_{\mathrm{A}}(M) &= x + y - M, \\ \sigma_{\mathrm{G}}(M) &= \frac{xy}{M} && (\textit{for } \mathscr{D} \subset (0,+\infty)^2), \\ \sigma_{\mathrm{H}}(M) &= \frac{xyM}{(x+y)M - xy} && (\textit{for } \mathscr{D} \subset (0,+\infty)^2). \ \blacksquare \end{aligned}$$

The remarkable phenomenon of the coincidence of the two symmetries defined on $\mathcal{M}_{\mathscr{D}}$ in the particular cases of the means A, G and H leads to the following question:

Open question. For which mean functions M on $\mathscr{D} = (0,+\infty)^2$ the two symmetries with respect to M (in the sense of the group law introduced on $\mathcal{M}_{\mathscr{D}}$ and in the functional sense) coincide?

Example. Using the definition of AGM (see Section 1), it is easy to show that A and G are symmetric in the functional sense with respect to AGM.

3. A metric space structure on $\mathcal{M}_{\mathscr{D}}$. Throughout this section, we fix a nonempty symmetric domain $\mathscr{D}$ in $\mathbb{R}^2$. We suppose that $\mathscr{D}$ contains at least one point (x_0, y_0) of $\mathbb{R}^2$ such that $x_0 \neq y_0$ (otherwise $\mathcal{M}_{\mathscr{D}}$ reduces to a unique element). For all couples (M_1, M_2) of mean functions on $\mathscr{D}$, define

$$\mathrm{d}(M_1, M_2) := \sup_{(x,y)\in\mathscr{D},\, x\neq y} \left| \frac{M_1(x,y) - M_2(x,y)}{x - y} \right|.$$

Proposition 3.1. *The map* d *of* $\mathcal{M}_{\mathscr{D}}^2$ *into* $[0, +\infty]$ *is a distance on* $\mathcal{M}_{\mathscr{D}}$. *In addition, the metric space* $(\mathcal{M}_{\mathscr{D}}, \mathrm{d})$ *is the closed ball with center* A (*the arithmetic mean*) *and radius* $1/2$.

Proof. First let us show that $\mathrm{d}(M_1, M_2)$ is finite for all M_1, M_2. For all $(x, y) \in \mathscr{D}$, $x \neq y$, the numbers $M_1(x, y)$ and $M_2(x, y)$ lie in the interval $[\min(x, y), \max(x, y)]$, so

$$|M_1(x, y) - M_2(x, y)| \leq \max(x, y) - \min(x, y) = |x - y|.$$

Hence

$$\sup_{(x,y)\in\mathscr{D},\, x\neq y} \left| \frac{M_1(x, y) - M_2(x, y)}{x - y} \right| \leq 1,$$

that is, $\mathrm{d}(M_1, M_2) \leq 1$. Further, since the three axioms of a distance are trivially satisfied, d is a distance on $\mathcal{M}_{\mathscr{D}}$.

Now, given $M \in \mathcal{M}_{\mathscr{D}}$, let us show that $\mathrm{d}(M, \mathrm{A}) \leq 1/2$. For all $(x, y) \in \mathscr{D}$, $x \neq y$, the number $M(x, y)$ lies in the closed interval with endpoints x and y, so

$$\begin{aligned}|M(x, y) - \mathrm{A}(x, y)| &\leq \max(x - \mathrm{A}(x, y), y - \mathrm{A}(x, y)) \\ &= \max\left(x - \frac{x + y}{2}, y - \frac{x + y}{2}\right) = \max\left(\frac{x - y}{2}, \frac{y - x}{2}\right) = \frac{1}{2}|x - y|.\end{aligned}$$

It follows that

$$\sup_{(x,y)\in\mathscr{D},\, x\neq y} \left| \frac{M(x, y) - \mathrm{A}(x, y)}{x - y} \right| \leq \frac{1}{2},$$

that is, $\mathrm{d}(M, \mathrm{A}) \leq 1/2$, as required. ■

Remark 3.2. Given $M_1, M_2 \in \mathcal{M}_{\mathscr{D}}$, since the map

$$(x, y) \mapsto \frac{M_1(x, y) - M_2(x, y)}{x - y}$$

is obviously asymmetric (on the set $\{(x, y) \in \mathscr{D} : x \neq y\}$), we also have

$$\mathrm{d}(M_1, M_2) = \sup_{(x,y)\in\mathscr{D},\, x\neq y} \frac{M_1(x, y) - M_2(x, y)}{x - y}.$$

We now establish a practical formula for the distance between two mean functions on $\mathscr{D}$.

Proposition 3.3. *Let* M_1 *and* M_2 *be two mean functions on* $\mathscr{D}$. *Set* $f_1 = \varphi(M_1)$ *and* $f_2 = \varphi(M_2)$. *Then*

$$\mathrm{d}(M_1, M_2) = \sup_{(x,y)\in\mathscr{D}} \frac{e^{f_1} - e^{f_2}}{(e^{f_1} + 1)(e^{f_2} + 1)} = \sup_{(x,y)\in\mathscr{D}} \left(\frac{1}{e^{f_1} + 1} - \frac{1}{e^{f_2} + 1}\right).$$

Proof. Using (2.1), for all $(x,y) \in \mathscr{D}$ we have

$$M_1(x,y) = \varphi^{-1}(f_1)(x,y) = \frac{x + ye^{f_1(x,y)}}{e^{f_1(x,y)} + 1},$$
$$M_2(x,y) = \varphi^{-1}(f_2)(x,y) = \frac{x + ye^{f_2(x,y)}}{e^{f_2(x,y)} + 1}.$$

The rest is a simple calculation. ■

As an application, we get the following immediate corollary:

COROLLARY 3.4. *Let M be a mean function on $\mathscr{D}$ and $f := \varphi(M)$. Then, setting $s := \sup_{\mathscr{D}} f \in [0, +\infty]$, we have*

$$\mathrm{d}(M, \mathrm{A}) = \frac{1}{2} \cdot \frac{e^s - 1}{e^s + 1}.$$

(We naturally suppose that $\frac{e^s-1}{e^s+1} = 1$ when $s = +\infty$.) In particular, the mean M lies on the boundary of $\mathcal{M}_{\mathscr{D}}$ (that is, on the circle with center A *and radius* 1/2*) if and only if* $\sup_{\mathscr{D}} f = +\infty$. ■

EXAMPLE. The two means G and H lie on the boundary of $\mathcal{M}_{\mathscr{D}}$.

4. Construction of a functional middle of two means. Let $I \subset \mathbb{R}$ ($I \neq \emptyset$) and let $\mathscr{D} = I^2$. The aim of this section is to prove, under some *regularity* conditions, the existence and uniqueness of the *functional middle* of two given means M_1 and M_2 on $\mathscr{D}$, that is, the existence and uniqueness of a new mean M on $\mathscr{D}$ satisfying the functional equation

$$M(M_1, M_2) = M.$$

In this context, we obtain two results which only differ in the condition imposed on M_1 and M_2. The first one requires $\mathrm{d}(M_1, M_2) \neq 1$ (where d is the distance defined in Section 3) while the second requires M_1 and M_2 to be continuous on $\mathscr{D}$ (by taking I an interval of $\mathbb{R}$). Notice further that our way of establishing the existence of the functional middle is constructive and generalizes the idea of the AGM mean. Our first result is the following:

THEOREM 4.1. *Let M_1 and M_2 be two mean functions on $\mathscr{D} = I^2$, with values in I and such that $\mathrm{d}(M_1, M_2) < 1$. Then there exists a unique mean function M on $\mathscr{D}$ satisfying the functional equation*

$$M(M_1, M_2) = M.$$

Moreover, for all $(x,y) \in \mathscr{D}$, $M(x,y)$ is the common limit of the two real sequences $(x_n)_n$ and $(y_n)_n$ defined as follows:

$$\begin{cases} x_0 = x, \quad y_0 = y, \\ x_{n+1} = M_1(x_n, y_n) \quad (\forall n \in \mathbb{N}), \\ y_{n+1} = M_2(x_n, y_n) \quad (\forall n \in \mathbb{N}). \end{cases}$$

Proof. Let $k := \mathrm{d}(M_1, M_2)$. By hypothesis, $k < 1$. Let $(x_n)_n$ and $(y_n)_n$ be as in the statement and let $(u_n)_n$ and $(v_n)_n$ be defined by

$$u_n := \min(x_n, y_n) \quad \text{and} \quad v_n := \max(x_n, y_n) \qquad (\forall n \in \mathbb{N}).$$

For all $n \in \mathbb{N}$, we have

$$u_{n+1} = \min(x_{n+1}, y_{n+1}) = \min(M_1(x_n, y_n), M_2(x_n, y_n)) \geq \min(x_n, y_n) = u_n$$

(because $M_1(x_n, y_n) \geq \min(x_n, y_n)$ and $M_2(x_n, y_n) \geq \min(x_n, y_n)$). Similarly, for all $n \in \mathbb{N}$,

$$\begin{aligned} v_{n+1} &= \max(x_{n+1}, y_{n+1}) = \max(M_1(x_n, y_n), M_2(x_n, y_n)) \\ &\leq \max(x_n, y_n) = v_n. \end{aligned}$$

Next, for all $n \in \mathbb{N}$,

$$\begin{aligned} |v_{n+1} - u_{n+1}| &= |\max(x_{n+1}, y_{n+1}) - \min(x_{n+1}, y_{n+1})| \\ &= |x_{n+1} - y_{n+1}| = |M_1(x_n, y_n) - M_2(x_n, y_n)| \\ &\leq k|x_n - y_n| \quad \text{(by definition of } k) \\ &= k|v_n - u_n|. \end{aligned}$$

By induction on n, we get

$$|v_n - u_n| \leq k^n |v_0 - u_0| \qquad (\forall n \in \mathbb{N}).$$

It follows (since $k \in [0, 1)$) that $(v_n - u_n)$ tends to 0 as n tends to infinity.

Thus the bounded monotonic sequences $(u_n)_n$ and $(v_n)_n$ converge to the same limit. Since

$$u_n \leq x_n \leq v_n \quad \text{and} \quad u_n \leq y_n \leq v_n \qquad (\forall n \in \mathbb{N}),$$

the sequences $(x_n)_n$ and $(y_n)_n$ also converge to the above limit. Denote the common limit of the four sequences by $M(x, y)$.

Now we show that the map $M : \mathscr{D} \to \mathbb{R}$ just defined is a mean function on $\mathscr{D}$ and satisfies $M(M_1, M_2) = M$. First we check the three axioms of a mean function.

(i) Given $(x, y) \in \mathscr{D}$, on changing (x, y) to (y, x) in the definition of the sequences $(x_n)_n$ and $(y_n)_n$, they remain unchanged except their first terms (since M_1 and M_2 are symmetric). So,

$$M(x, y) = M(y, x) \qquad (\forall (x, y) \in \mathscr{D}).$$

(ii) Given $(x, y) \in \mathscr{D}$, since the corresponding sequences $(u_n)_n$ and $(v_n)_n$ are respectively nondecreasing and nonincreasing and since $M(x, y)$ is their common limit, we have $u_0 \leq M(x, y) \leq v_0$, that is,

$$\min(x, y) \leq M(x, y) \leq \max(x, y).$$

(iii) Fix $(x, y) \in \mathscr{D}$. Suppose that $M(x, y) = x$ and, towards a contradiction, $x \neq y$. Since M_1 and M_2 are means, we have (by axiom (iii))

$$M_1(x, y) \neq x \quad \text{and} \quad M_2(x, y) \neq x. \tag{4.1}$$

We distinguish two cases:

CASE 1: $x < y$. Then $M(x, y) = x = \min(x, y) = u_0$. So $(u_n)_n$ is nondecreasing and converges to u_0. It follows that $(u_n)_n$ is necessarily constant and in particular $u_1 = u_0$, that is,

$$\min(M_1(x, y), M_2(x, y)) = x,$$

which contradicts (4.1).

CASE 2: $x > y$. Then $M(x, y) = x = \max(x, y) = v_0$. So $(v_n)_n$ is nonincreasing and converges to v_0. It follows that $(v_n)_n$ is constant and in particular $v_1 = v_0$, that is,

$$\max(M_1(x, y), M_2(x, y)) = x,$$

which again contradicts (4.1), proving (iii).

To prove $M(M_1, M_2) = M$, note that changing in the definition of $(x_n)_n$ and $(y_n)_n$ the couple (x, y) of $\mathscr{D}$ to $(M_1(x, y), M_2(x, y))$ just amounts to shifting these sequences (namely we obtain $(x_{n+1})_n$ instead of $(x_n)_n$, and $(y_{n+1})_n$ instead of $(y_n)_n$). Consequently, the common limit (which is $M(x, y)$) remains the same:

$$M(M_1(x, y), M_2(x, y)) = M(x, y).$$

It remains to show that M is the unique mean satisfying the functional equation $M(M_1, M_2) = M$. Let M' be any mean function satisfying $M'(M_1, M_2) = M'$ and fix $(x, y) \in \mathscr{D}$. We associate to (x, y) the sequence $(x_n, y_n)_{n\in\mathbb{N}}$ as in the statement of the theorem. Using the relation $M'(M_1, M_2) = M'$, we have

$$M'(x, y) = M'(x_1, y_1) = M'(x_2, y_2) = \cdots = M'(x_n, y_n) = \cdots .$$

But since M' is a mean, it follows that for all $n \in \mathbb{N}$,

$$\min(x_n, y_n) \leq M'(x, y) \leq \max(x_n, y_n),$$

and letting $n \to \infty$ yields

$$M'(x, y) = M(x, y),$$

as required. ∎

From Theorem 4.1, we derive the following corollary:

COROLLARY 4.2. *Let $\mathfrak{M}$ be a mean function on $\mathscr{D} = I^2$ with values in I. Then there exists a unique mean on $\mathscr{D}$ satisfying the functional equation*

$$M\left(\frac{x+y}{2}, \mathfrak{M}(x, y)\right) = M(x, y) \qquad (\forall (x, y) \in \mathscr{D}).$$

In addition, for all $(x, y) \in \mathscr{D}$, $M(x, y)$ *is the common limit of the two real sequences* $(x_n)_n$ *and* $(y_n)_n$ *defined by*

$$\begin{cases} x_0 = x, \quad y_0 = y, \\ x_{n+1} = \dfrac{x_n + y_n}{2} \qquad (\forall n \in \mathbb{N}), \\ y_{n+1} = \mathfrak{M}(x_n, y_n) \quad (\forall n \in \mathbb{N}). \end{cases}$$

Proof. Since the metric space $(\mathcal{M}_{\mathscr{D}}, \mathrm{d})$ is the closed ball with center A and radius 1/2 (see Proposition 3.1), we have $\mathrm{d}(\mathfrak{M}, \mathrm{A}) \leq 1/2 < 1$. The corollary then immediately follows from Theorem 4.1. ∎

In the following theorem, we establish another sufficient condition for the existence and uniqueness of the functional middle of two means.

THEOREM 4.3. *Suppose that* I *is an interval of* $\mathbb{R}$ *and let* M_1 *and* M_2 *be two mean functions on* $\mathscr{D} = I^2$ *with values in* I. *Suppose that* M_1 *and* M_2 *are continuous on* $\mathscr{D}$. *Then there exists a unique mean function* M *on* $\mathscr{D}$ *satisfying the functional equation*

$$M(M_1, M_2) = M.$$

In addition, for all $(x, y) \in \mathscr{D}$, $M(x, y)$ *is the common limit of the two real sequences* $(x_n)_n$ *and* $(y_n)_n$ *defined as in Theorem* 4.1.

Proof. Fix $(x, y) \in \mathscr{D}$ and define $(u_n)_n$ and $(v_n)_n$ as in the proof of Theorem 4.1. Again $(u_n)_n$ and $(v_n)_n$ are convergent. Let $u = u(x, y)$ and $v = v(x, y)$ denote their respective limits (so u and v lie in $[u_0, v_0] = [\min(x, y), \max(x, y)] \subset I$).

Now, since M_1 and M_2 are symmetric on $\mathscr{D}$, we have, for all $n \in \mathbb{N}$,

$$x_{n+1} = M_1(u_n, v_n) \quad \text{and} \quad y_{n+1} = M_2(u_n, v_n).$$

By continuity, the sequences $(x_n)_n$ and $(y_n)_n$ are also convergent and their respective limits are $M_1(u, v)$ and $M_2(u, v)$. Letting $n \to \infty$ in $x_{n+1} = M_1(x_n, y_n)$, we obtain

$$M_1(u, v) = M_1(M_1(u, v), M_2(u, v)),$$

which implies (by axiom (iii)) that

$$M_1(u, v) = M_2(u, v).$$

Thus $(x_n)_n$ and $(y_n)_n$ converge to the same limit. Denoting by $M(x, y)$ this common limit, we show as in the proof of Theorem 4.1 that M is a mean function on $\mathscr{D}$ and that it is the unique mean on $\mathscr{D}$ which satisfies the functional equation $M(M_1, M_2) = M$. ∎

REFERENCES

[Bor] J. M. Borwein and P. B. Borwein, *Pi and the AGM. A Study in Analytic Number Theory and Computational Complexity*, Wiley, New York, 1987.

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