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A NOTE ON OPTIMAL PROBABILITY LOWER BOUNDS FOR CENTERED RANDOM VARIABLES

BY

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Abstract. We obtain lower bounds for $\mathbb{P}(\xi \geq 0)$ and $\mathbb{P}(\xi > 0)$ under assumptions on the moments of a centered random variable ξ. The estimates obtained are shown to be optimal and improve results from the literature. They are then applied to obtain probability lower bounds for second order Rademacher chaos.

1. Introduction. In this note we obtain lower bounds for $\mathbb{P}(\xi \geq 0)$ and $\mathbb{P}(\xi > 0)$ under assumptions on the moments of ξ. Here ξ is a centered real-valued random variable. For instance, we consider the case where the first and pth moments are fixed, and the case where the second and pth moments are fixed. Such lower bounds are used in [2, 4, 6, 9] to estimate tail probabilities. They can be used to estimate $\mathbb{P}(\xi \leq E\xi)$ for certain random variables ξ. Let $c_p = (\mathbb{E}|\xi|^p)^{1/p}$ and $c_{p,q} = c_p/c_q$. Examples of known estimates that are often used for $p = 2$ and $p = 4$ are respectively

$$\mathbb{P}(\xi \geq 0) \geq \left(\frac{c_{1,p}}{2}\right)^{p/(p-1)} \quad \text{and} \quad \mathbb{P}(\xi \geq 0) \geq \frac{1}{4c_{p,2}^{2p/(p-2)}}.$$

A proof of the first estimate can be found in [9]. The second estimate is obtained in [3]. In this note we will improve both estimates and in several cases we will show that the results obtained are sharp.

In the last part we give some applications of our results. We improve an estimate for second order Rademacher chaos from [3]. This result has applications to certain quadratic optimization problems (cf. [1, 3]). Finally, we give applications to Hilbert-space-valued random variables. In particular, this improves a result from [2].

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2. Probability lower bounds. The following result is an improvement of [9, Proposition 3.3.7].

PROPOSITION 2.1. *Let ξ be a centered non-zero random variable and let $p \in (1, \infty)$. Then*

$$\mathbb{P}(\xi \geq 0) \geq \mathbb{P}(\xi > 0) \geq \left(\frac{c_{1,p}}{2}\right)^{p/(p-1)} (\psi^{-1}(c_{1,p}))^{-1/(p-1)}. \tag{2.1}$$

Here $\psi : [1/2, 1) \to (0, 1]$ is the strictly decreasing function defined by

$$\psi(x) = 2(x^{-1/(p-1)} + (1-x)^{-1/(p-1)})^{-(p-1)/p}.$$

The same lower bound holds for $\mathbb{P}(\xi < 0)$ and $\mathbb{P}(\xi \leq 0)$. Moreover, the estimate (2.1) for $\mathbb{P}(\xi \geq 0)$ and $\mathbb{P}(\xi \leq 0)$ is sharp.

For all $p \in (1, \infty)$ the following bound holds:

$$\begin{aligned} \mathbb{P}(\xi \geq 0) &\geq \mathbb{P}(\xi > 0) \\ &\geq \left(\frac{c_{1,p}}{2}\right)^{p/(p-1)} \left(1 - \left(\left(\frac{c_{1,p}}{2}\right)^{-p/(p-1)} - 1\right)^{-(p-1)}\right)^{-1/(p-1)}. \end{aligned} \tag{2.2}$$

The estimate (2.1) improves the well-known estimate

$$\mathbb{P}(\xi \geq 0) \geq (c_{1,p}/2)^{p/(p-1)}$$

(cf. [9, Proposition 3.3.7]) by a factor $(\psi^{-1}(c_{1,p}))^{-(p-1)}$. The lower bound (2.2) is not optimal, but in general it is more explicit than (2.1).

In the cases $p = 2$ and $p = 3$ one can calculate ψ^{-1} explicitly. For $p = 2$, the inverse is given by $\psi^{-1}(x) = \frac{1}{2} + \frac{1}{2}\sqrt{1 - x^2}$. Therefore, a straightforward calculation gives the following explicit lower bound, which is sharp as well.

COROLLARY 2.2. *Let ξ be a centered non-zero random variable. Then*

$$\mathbb{P}(\xi \geq 0) \geq \mathbb{P}(\xi > 0) \geq \frac{1}{2} - \frac{1}{2}\sqrt{1 - c_{1,2}^2}.$$

This result can be used to slightly improve certain probability lower bounds from [4], where the bound $c_{1,2}^2/4$ is used.

Proof of Proposition 2.1. By symmetry it suffices to consider $\mathbb{P}(\xi > 0)$. By normalization we may assume that $c_p = 1$, and therefore $c = c_1 = c_{1,p}$. Let $p_1 = \mathbb{P}(\xi > 0)$ and $p_2 = \mathbb{P}(\xi < 0)$. Let $\xi_+ = \max\{\xi, 0\}$ and $\xi_- = \max\{-\xi, 0\}$. Then $0 = \mathbb{E}\xi = \mathbb{E}\xi_+ - \mathbb{E}\xi_-$ and $c = \mathbb{E}|\xi| = \mathbb{E}\xi_+ + \mathbb{E}\xi_-$. It follows that $\mathbb{E}\xi^+ = \mathbb{E}\xi^- = c/2$. Let $u = \mathbb{E}\xi_+^p$. Then $1 - u = \mathbb{E}\xi_-^p$. By the Cauchy–Schwarz inequality we have

$$c^p/2^p = (\mathbb{E}\xi_+)^p = (\mathbb{E}\xi_+ \operatorname{sign}(\xi_+))^p \leq \mathbb{E}\xi_+^p (\mathbb{E} \operatorname{sign}(\xi_+))^{p-1} = u p_1^{p-1}.$$

Therefore, $p_1 \geq (c^p/2^p u)^{1/(p-1)}$. Similarly, $p_2 \geq (c^p/2^p(1-u))^{1/(p-1)}$. It follows that

$$p_1 = 1 - \mathbb{P}(\xi \le 0) \le 1 - p_2 \le 1 - \left(\frac{c^p}{2^p(1-u)}\right)^{1/(p-1)}.$$

Hence, to estimate p_1 from below, we only need to consider the $u \in (0,1)$ which satisfy

$$\left(\frac{c^p}{2^p u}\right)^{1/(p-1)} \le 1 - \left(\frac{c^p}{2^p(1-u)}\right)^{1/(p-1)}.$$

This is equivalent to

$$\frac{2^p}{c^p} \ge f(u) := \left(\frac{1}{u^{1/(p-1)}} + \frac{1}{(1-u)^{1/(p-1)}}\right)^{p-1},$$
$$c \le \phi(u) = 2(u^{-1/(p-1)} + (1-u)^{-1/(p-1)})^{-(p-1)/p}.$$

Notice that ϕ is strictly increasing on $(0,1/2]$ and strictly decreasing on $[1/2,1)$. One easily checks that there exists a unique $u_0 \in (0,1/2]$ and a unique $u_1 \in [1/2,1)$ such that $\phi(u_0) = \phi(u_1) = c$. Moreover, $c \le \phi(u)$ if and only if $u \in [u_0,u_1]$. It follows that $(c^p/2^p u)^{1/(p-1)}$ attains its minimum at u_1, and therefore

$$p_1 \ge (c/2)^{p/(p-1)} u_1^{-1/(p-1)}.$$

This completes the first part of the proof.

To prove (2.2), note that it suffices to estimate ψ^{-1} from above, or equivalently ψ from above. Clearly for all $x \in [1/2,1)$,

$$\psi(x) \le 2(1 + (1-x)^{-1/(p-1)})^{-(p-1)/p} =: \alpha(x).$$

Now $\alpha^{-1}(x) = 1 - ((x/2)^{-p/(p-1)} - 1)^{-(p-1)}$. This clearly implies the result.

To prove the sharpness of (2.1) let $c \in (0,1]$ be arbitrary and let $\mu = (c/2)^{p/(p-1)} u_1^{-1/(p-1)}$, where $u_1 = \psi^{-1}(c)$. It suffices to construct a centered random variable ξ with $\mathbb{E}|\xi|^p = 1$, $\mathbb{E}|\xi| = c$ and $\mathbb{P}(\xi \le 0) = \mu$. Let $x_1 = c/2\mu$ and $x_2 = c/2(1-\mu)$, and let $\xi = x_1$ with probability μ and $\xi = x_2$ with probability $1-\mu$. Then $\mathbb{E}|\xi| = c$ and

$$\begin{aligned}
\mathbb{E}|\xi|^p &= \frac{c^p}{2^p}\,(\mu^{1-p} + (1-\mu)^{1-p}) \\
&= \frac{c^p}{2^p}\left(\frac{2^p}{c^p}\,u_1 + \left(1 - \left(\frac{c}{2}\right)^{p/(p-1)} u_1^{-1/(p-1)}\right)^{1-p}\right) \\
&= \frac{c^p}{2^p}\left(\frac{2^p}{c^p}\,u_1 + \left(\left(\frac{c}{2}\right)^{p/(p-1)} (1-u_1)^{-1/(p-1)}\right)^{1-p}\right) \\
&= \frac{c^p}{2^p}\left(\frac{2^p}{c^p}\,u_1 + \frac{2^p}{c^p}\,(1-u_1)\right) = 1. \ \blacksquare
\end{aligned}$$

In [3] it is shown that if ξ satisfies $\mathbb{E}\xi = 0$, $E\xi^2 = 1$, $\mathbb{E}\xi^4 \leq \tau$, then $\mathbb{P}(\xi \geq 0)$ and $\mathbb{P}(\xi \leq 0)$ are both greater than or equal to $(2\sqrt{3}-3)/\tau$. Below we will improve that result. More precisely, we obtain sharp lower bounds for $\mathbb{P}(\xi \leq 0)$, $\mathbb{P}(\xi \geq 0)$, $\mathbb{P}(\xi < 0)$ and $\mathbb{P}(\xi > 0)$.

Proposition 2.3. *Let ξ be a centered non-zero random variable. Then $\mathbb{P}(\xi \geq 0) \geq \mathbb{P}(\xi > 0) \geq f(c_{4,2}^4)$, where*

$$f(x) := \begin{cases} \frac{1}{2} - \frac{1}{2}\sqrt{\frac{x-1}{x+3}} & \text{if } x \in \left[1, \frac{3\sqrt{3}}{2} - \frac{3}{2}\right), \\ \frac{2\sqrt{3}-3}{x} & \text{if } x \geq \frac{3\sqrt{3}}{2} - \frac{3}{2}. \end{cases} \tag{2.3}$$

The same lower bound holds for $\mathbb{P}(\xi < 0)$ and $\mathbb{P}(\xi \leq 0)$. Moreover, the estimates are already sharp for $\mathbb{P}(\xi \geq 0)$ and $\mathbb{P}(\xi \leq 0)$.

Proof. By symmetry we only need to consider $\mathbb{P}(\xi > 0)$. By normalization we may assume that $c_2 = 1$ and therefore $c := c_4^4 = c_{4,2}^4$. The proof of the first part is a slight modification of the argument in [3]. Let $p_1 = \mathbb{P}(\xi > 0)$ and $p_2 = \mathbb{P}(\xi < 0)$. Let $\xi_+ = \max\{\xi, 0\}$ and $\xi_- = \max\{-\xi, 0\}$. Then $0 = \mathbb{E}\xi = \mathbb{E}\xi_+ - \mathbb{E}\xi_-$. Let $s = \mathbb{E}\xi^+ = \mathbb{E}\xi^-$. By Hölder's inequality we have $\mathbb{E}\xi_+^2 \leq (\mathbb{E}\xi_+^4)^{1/3}s^{2/3}$ and $\mathbb{E}\xi_-^2 \leq (\mathbb{E}\xi_-^4)^{1/3}s^{2/3}$. From this and $1 = \mathbb{E}\xi^2 = \mathbb{E}\xi_+^2 + \mathbb{E}\xi_-^2$ we deduce that

$$c \geq \mathbb{E}\xi_+^4 + \mathbb{E}\xi_-^4 \geq (\mathbb{E}\xi_+^2)^3 s^{-2} + (\mathbb{E}\xi_-^2)^3 s^{-2} = (u^3 + (1-u)^3)s^{-2},$$

where $u = \mathbb{E}\xi_+^2$. On the other hand, by the Cauchy–Schwarz inequality we have

$$s^2 = (\mathbb{E}\xi_+)^2 = (\mathbb{E}\xi_+ \operatorname{sign}(\xi_+))^2 \leq \mathbb{E}\xi_+^2(\mathbb{E}\operatorname{sign}(\xi_+)) = up_1.$$

Therefore, $p_1 \geq (u^3 + (1-u)^3)/uc$. Minimization over $u \in (0,1)$ gives $u = 1/\sqrt{3}$ and $p_1 \geq (2\sqrt{3}-3)/c$.

Next we improve the estimate for $c \in [1, 3\sqrt{3}/2 - 3/2)$. In the same way as for p_1, one can show that $p_2 \geq (u^3 + (1-u)^3)/(1-u)c$. Therefore,

$$p_1 = 1 - \mathbb{P}(\xi < 0) \leq 1 - p_2 \leq \frac{u^3 + (1-u)^3}{(1-u)c}.$$

If we combine this with the lower estimate for p_1, the only $u \in (0,1)$ which have to be considered are those for which

$$\frac{u^3 + (1-u)^3}{uc} \leq 1 - \frac{u^3 + (1-u)^3}{(1-u)c}.$$

One easily checks that this happens if and only if

$$u_0 = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{c-1}{c+3}} \leq u \leq \frac{1}{2} + \frac{1}{2}\sqrt{\frac{c-1}{c+3}} = u_1.$$

For the c's we consider one may check that $1/\sqrt{3} \notin (u_0, u_1)$. Therefore, the minimum is attained at the boundary. Since $g(u_0) = u_1$ and $g(u_1) = u_0$, u_0 is the minimum of g on $[u_0, u_1]$. This shows that $p_1 \geq u_0$.

To show this estimate is sharp for $x \geq 3\sqrt{3}/2 - 3/2$ we will construct a certain family $(\xi_\varepsilon)_{\varepsilon \geq 0}$ of random variables. Let $\varepsilon \geq 0$ be not too large. Let ξ_ε be equal to $x_i(\varepsilon)$ with probability λ_i, for $i = 1, 2, 3$. Let

$$\lambda_1 = \left(\frac{3}{2} - \frac{\sqrt{3}}{2}\right)/c, \qquad \lambda_2 = 1 - \left(\frac{3\sqrt{3}}{2} - \frac{3}{2}\right)/c, \qquad \lambda_3 = (2\sqrt{3} - 3)/c.$$

Let $x_2(\varepsilon) = -\varepsilon$, and let $x_1(\varepsilon) < 0$ and $x_3(\varepsilon) > 0$ be the solution of

$$\mathbb{E}\xi = \lambda_1 x_1 + \lambda_2 \varepsilon + \lambda_3 x_3 = 0, \qquad \mathbb{E}\xi^2 = \lambda_1 x_1^2 + \lambda_2 \varepsilon^2 + \lambda_3 x_3^2 = 1.$$

Notice that

$$x_1(0) = -\frac{1 - \frac{1}{3}\sqrt{3}}{\sqrt{2 - \sqrt{3}}}\sqrt{c}, \qquad x_2 = 0, \qquad x_3(0) = \frac{\frac{1}{3}\sqrt{3}}{\sqrt{2 - \sqrt{3}}}\sqrt{c}.$$

For $\varepsilon > 0$ small enough one may check that $x_1(\varepsilon) < x_2(\varepsilon) < 0 < x_3(\varepsilon)$ and $P(\xi_\varepsilon \geq 0) = \lambda_3$. Moreover,

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}\xi_\varepsilon^4 = \lim_{\varepsilon \downarrow 0} \lambda_1 x_1^4(\varepsilon) + \lambda_2 x_2^4(\varepsilon) + \lambda_3 x_3^4(\varepsilon) = \lambda_1 x_1^4(0) + \lambda_2 x_2^4(0) + \lambda_3 x_3^4(0) = c.$$

This part of the proof is complete.

The sharpness of the result for $x \in [1, 3\sqrt{3}/2 - 3/2)$ follows if we take for ξ a random variable with two values. Indeed, let

$$x_2 = \frac{1}{2}\sqrt{2 + 2c + 2\sqrt{(c-1)(c+3)}}, \qquad x_1 = -1/x_2,$$

$\lambda_1 = x_2/(x_2 - x_1)$ and $\lambda_2 = -x_1/(x_2 - x_1)$. One easily checks that $\mathbb{E}\xi = 0$, $\mathbb{E}\xi^2 = 1$, $\mathbb{E}\xi^4 = c$ and

$$\lambda_1 = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{c-1}{c+3}}. \ \blacksquare$$

In [3] also a lower bound is obtained if one uses the pth moment instead of the fourth moment. It is shown that $\mathbb{P}(\xi \geq 0) \geq \frac{1}{4} c_{p,2}^{-2p/(p-2)}$. In the next remark we improve the factor $\frac{1}{4}$.

Remark 2.4. Let ξ be a centered non-zero random variable and let $p \in (2, \infty)$. Then

$$\begin{aligned}\mathbb{P}(\xi \geq 0) \geq \mathbb{P}(\xi > 0) &\geq \frac{1}{4}\, c_{p,2}^{-2p/(p-2)}((3 - 4/p)^{-1/(p-2)} + 1) \\ &\geq \frac{e^{-1} + 1}{4}\, c_{p,2}^{-2p/(p-2)}.\end{aligned}$$

Proof. It follows from the proof in [3] that

$$\mathbb{P}(\xi > 0) \geq \min_{u \in (0,1)} c_{p,2}^{-2p/(p-2)} f(u), \qquad f(u) = \frac{1}{u}\,(u^{p-1} + (1-u)^{p-1})^{2/(p-2)}.$$

The function f has a minimum at $u = u_0$ in $[1/2, 1)$. Moreover, it satisfies $f'(u_0) = 0$.

Indeed, if $u_0 \in (0, 1/2)$ were a minimum point of f, then $f(1-u_0) < f(u_0)$, which is impossible. That a minimum u exists on $[1/2, 1)$ and that it satisfies $f'(u) = 0$ is clear. A calculation shows that $f'(u) = \alpha(u)g(u)$, where $\alpha(u) > 0$ and

$$g(u) = pu^{p-1} - p(1-u)^{p-2}u - p(1-u)^{p-2} + 2(1-u)^{p-2}.$$

Therefore, $f'(u) = 0$ if and only if $g(u) = 0$. Let us estimate u_0 from above. Since $g(u_0) = 0$, we have

$$(1-u_0)^{p-2}\left(1 - \frac{2}{p}\right) = u_0(u_0^{p-2} - (1-u_0)^{p-2}).$$

As $u_0 \geq 1/2$, we obtain

$$(1-u_0)^{p-2}\left(1 - \frac{2}{p}\right) \geq \frac{1}{2}\,(u_0^{p-2} - (1-u_0)^{p-2}),$$

and therefore

$$\frac{1}{u_0} \geq (3 - 4/p)^{-1/(p-2)} + 1.$$

We conclude that

$$\begin{aligned} f(u) &\geq ((3-4/p)^{-1/(p-2)} + 1)(u^{p-1} + (1-u)^{p-1})^{2/(p-2)} \\ &\geq ((3-4/p)^{-1/(p-2)} + 1) \cdot \frac{1}{4}. \end{aligned}$$

The final estimate follows from $(3-4/p)^{1/(p-2)} \downarrow e$ as $p \downarrow 2$. ∎

3. Applications. We will need the following estimate for second order chaoses. It is well-known to experts. For a random variable ξ and $p \in [1, \infty)$, let $\|\xi\|_p = (\mathbb{E}|\xi|^p)^{1/p}$.

LEMMA 3.1. *Let $(\xi_i)_{i\geq 1}$ be an i.i.d. sequence of symmetric random variables with $\mathbb{E}|\xi_i|^2 = 1$ and $\mathbb{E}|\xi_i|^4 \leq 3$. Then for any real numbers $(a_{i,j})_{1\leq i<j\leq n}$,*

$$\Big\|\sum_{1\leq i<j\leq n} \xi_i\xi_j a_{ij}\Big\|_4 \leq \sqrt[4]{15}\Big\|\sum_{1\leq i<j\leq n} \xi_i\xi_j a_{ij}\Big\|_2. \tag{3.1}$$

Moreover, in the case $(\xi_i)_{i\geq 1}$ is a Rademacher sequence or a Gaussian sequence the inequality (3.1) *is sharp.*

Proof. For $j > i$ let $a_{ij} = a_{ji}$ and let $a_{ii} = 0$. By homogeneity we may assume that

$$\Big\|\sum_{1\leq i<j\leq n} \xi_i\xi_j a_{ij}\Big\|_2^2 = \sum_{1\leq i<j\leq n} a_{ij}^2 = \frac{1}{2}. \tag{3.2}$$

Let $(\gamma_i)_{i\geq 1}$ be a sequence of independent standard Gaussian random variables. Since $\mathbb{E}|\xi_i|^2 \leq \mathbb{E}|\gamma_i|^2$ and $\mathbb{E}|\xi_i|^4 \leq \mathbb{E}|\gamma_i|^4$, we have

$$\Big\| \sum_{1\le i<j\le n} \xi_i\xi_j a_{ij} \Big\|_4 \le \Big\| \sum_{1\le i<j\le n} \gamma_i\gamma_j a_{ij} \Big\|_4. \tag{3.3}$$

Denote by A the matrix $(a_{ij})_{1\le i,j\le n}$. By diagonalization we may write $A = PDP^T$, where $D = (\lambda_i)$ is a diagonal matrix and P is an orthogonal matrix. Clearly, $\langle A\gamma, \gamma\rangle = \langle D\gamma', \gamma'\rangle$, where $\gamma = (\gamma_1, \dots, \gamma_n)$ and $\gamma' = P^T\gamma$. Since P is orthogonal, γ' has the same distribution as γ. Therefore,

$$0 = \mathbb{E}\langle A\gamma, \gamma\rangle = \mathbb{E}\langle D\gamma', \gamma'\rangle = \sum_{i=1}^n \lambda_i.$$

Similarly one may check that $\sum_{i=1}^n \lambda_i^2 = 1$. It follows that

$$\begin{aligned}\mathbb{E}\langle A\gamma, \gamma\rangle^4 = \mathbb{E}|\langle D\gamma', \gamma'\rangle|^4 = \mathbb{E}\Big| \sum_{i=1}^n \lambda_i(\gamma_i^2 - 1)\Big|^4 &= 36\sum_{i=1}^n \lambda_i^4 + 24\sum_{i=1}^n \lambda_i^2 \\ &\le 36\Big(\sum_{i=1}^n \lambda_i^2\Big)^2 + 24\sum_{i=1}^n \lambda_i^2 = 60.\end{aligned}$$

Therefore,

$$\mathbb{E}\Big| \sum_{1\le i<j\le n} \gamma_i\gamma_j a_{ij}\Big|^4 = \frac{1}{16}\,\mathbb{E}\langle A\gamma, \gamma\rangle^4 \le \frac{15}{4}.$$

By (3.2) and (3.3) this implies the result.

To show that the inequality (3.1) is sharp it suffices to consider the case where the $(\xi_i)_{i\ge 1}$ are standard Gaussian random variables. Indeed, if (3.1) holds for a Rademacher sequence $(\xi_i)_{i\ge1}$, then the central limit theorem implies (3.1) for the Gaussian case. Now assume $(\xi_i)_{i\ge1}$ are standard Gaussian random variables. Let $a_{ij} = 1$ for all $i \ne j$ and $a_{ii} = 0$. Notice that $\sum_{1\le i<j\le n} \xi_i\xi_j a_{ij} = \frac{1}{2}\langle A\xi, \xi\rangle$, where $\xi = (\xi_i)_{i=1}^n$. For the right-hand side of (3.1) we have

$$\Big\| \sum_{1\le i<j\le n} \xi_i\xi_j a_{ij}\Big\|_2^2 = \sum_{1\le i<j\le n} a_{ij}^2 = \frac{n(n-1)}{2}.$$

As before, we may write $A = PDP^T$, where D is the diagonal matrix with eigenvalues $(\lambda_i)_{i=1}^n$ of A and P is orthogonal. It is easy to see that the eigenvalues of A are $n-1$ and -1, where the latter has multiplicity $n-1$. By the same calculation as before it follows that

$$\mathbb{E}\langle A\xi, \xi\rangle^4 = 60\sum_{i=1}^n \lambda_i^4 + 24\sum_{i\ne j}^n \lambda_i^2\lambda_j^2 = 36((n-1)^4 + n) + 24((n-1)^2 + n)^2.$$

Letting C denote the best constant in (3.1) gives

$$\frac{36}{16}\,((n-1)^4 + n) + \frac{24}{16}\,((n-1)^2 + n)^2 \le C^4\,\frac{n^2(n-1)^2}{4}.$$

Dividing by $n^4/4$ and letting n tend to infinity yields $9+6 \le C^4$, as required. ∎

By standard arguments (cf. [9, Chapter 3]) using Hölder's inequality one also deduces from Lemma 3.1 that

$$\Big\|\sum_{1\le i<j\le n}\xi_i\xi_j a_{ij}\Big\|_p \le 15^{(p-2)/2p}\Big\|\sum_{1\le i<j\le n}\xi_i\xi_j a_{ij}\Big\|_2 \quad \text{for } p\in(2,4), \tag{3.4}$$

$$\Big\|\sum_{1\le i<j\le n}\xi_i\xi_j a_{ij}\Big\|_2 \le 15^{(2-p)/2p}\Big\|\sum_{1\le i<j\le n}\xi_i\xi_j a_{ij}\Big\|_p \quad \text{for } p\in(0,2). \tag{3.5}$$

As an immediate consequence of Proposition 2.3 and Lemma 3.1 we obtain the following result. We state it for Rademacher random variables, but the same result holds for random variables $(\xi_n)_{n\ge1}$ as in Lemma 3.1.

PROPOSITION 3.2. *Let $(r_i)_{i\ge1}$ be a Rademacher sequence. For any real numbers $(a_{ij})_{i,j=1}^n$,*

$$\mathbb{P}\Big(\sum_{1\le i<j\le n} r_i r_j a_{ij}\ge 0\Big) \ge \frac{2\sqrt{3}-3}{15} > \frac{3}{100}.$$

If not all a_{ij} are identically zero then

$$\mathbb{P}\Big(\sum_{1\le i<j\le n} r_i r_j a_{ij} > 0\Big) \ge \frac{2\sqrt{3}-3}{15} > \frac{3}{100}.$$

This result has applications to certain quadratic optimization problems (cf. [1] and [3, Theorem 4.2]). It improves the known result with $1/87$ from [3, Lemma 4.1].

A conjecture (see [1]) is that the estimate in Proposition 3.2 holds with $1/4$. The methods we have described will probably never give such a bound, and a more sophisticated argument will be needed. However, another conjecture is that for a Rademacher sequence $(r_i)_{i\ge1}$ and $p=1$, (3.5) holds with constant 2, i.e.

$$\Big\|\sum_{1\le i<j\le n} r_i r_j a_{ij}\Big\|_2 \le 2\Big\|\sum_{1\le i<j\le n} r_i r_j a_{ij}\Big\|_1.$$

If this were true, then Corollary 2.2 implies that

$$\mathbb{P}\Big(\sum_{1\le i<j\le n} r_i r_j a_{ij}\ge 0\Big) \ge \frac{1}{2}-\frac{1}{4}\sqrt{3} > \frac{1}{15},$$

which is better than $\frac{3}{100}$.

REMARK 3.3. Let $(\eta_i)_{i\ge1}$ be independent exponentially distributed random variables with $\mathbb{E}\eta_i = 1$ and let $\xi = \sum_{i=1}^n a_i(\eta_i-1)$ for real numbers $(a_i)_{i\ge1}$. In [3] the estimate $\mathbb{P}(\xi\ge0) > 1/20$ has been obtained. It follows from Proposition 2.3 and (see [3])

$$(\mathbb{E}|\xi|^4)^{1/4} \le 9(\mathbb{E}|\xi|^2)^{1/2}. \tag{3.6}$$

The inequality (3.6) is optimal. As in (3.5) we see that (3.6) implies

$$(\mathbb{E}|\xi|^2)^{1/2} \le C\mathbb{E}|\xi|)$$

for a certain constant $C \le 3$. One the other hand, taking $n = 2$ and $a_1 = 1$, $a_2 = -1$ gives $C \ge \sqrt{2}$. It is interesting to find the optimal value of C. If this value is small enough, then Proposition 2.1 will give a better result than 1/20.

A similar situation can be considered if one replaces η_i by γ_i^2.

Next we prove another probability bound. A uniform bound can already be found in [2].

Corollary 3.4. *Let $(r_i)_{i\ge1}$ be a Rademacher sequence. Let $(H, \langle\cdot,\cdot\rangle)$ be a Hilbert space. For any vectors $(a_i)_{i=1}^n$ from H,*

$$\mathbb{P}\Big(\Big\|\sum_{i=1}^n r_i a_i\Big\| \le \Big(\sum_{i=1}^n \|a_i\|^2\Big)^{1/2}\Big) \ge \frac{2\sqrt{3}-3}{15} > \frac{3}{100}, \tag{3.7}$$

$$\mathbb{P}\Big(\Big\|\sum_{i=1}^n r_i a_i\Big\| \ge \Big(\sum_{i=1}^n \|a_i\|^2\Big)^{1/2}\Big) \ge \frac{2\sqrt{3}-3}{15} > \frac{3}{100}. \tag{3.8}$$

For real numbers $(a_i)_{i=1}^n$, (3.7) holds with constant 3/8 (see [5]). A well-known conjecture is that it holds with 1/2. Again, for real numbers $(a_i)_{i=1}^n$, (3.8) holds with constant 1/10 (see [8]). A conjecture (see [4]) is that it holds with constant 7/64.

Poof of Corollary 3.4. As in [2] one can show that

$$\mathbb{P}\Big(\Big\|\sum_{i=1}^n r_i a_i\Big\| \ge \Big(\sum_{i=1}^n \|a_i\|^2\Big)^{1/2}\Big) = \mathbb{P}\Big(\sum_{1\le i<j\le n} r_i r_j a_{ij} \ge 0\Big),$$

where $a_{ij} = 2\mathrm{Re}(\langle a_i, a_j\rangle)$. Therefore, (3.8) follows from Proposition 3.2. The proof of (3.7) is the same. ∎

In the next result we obtain a probability bound for Gaussian random variables with values in a Hilbert space.

Proposition 3.5. *Let H be a real separable Hilbert space and let $G : \Omega \to H$ be a non-zero centered Gaussian random variable. Then*

$$\frac{2\sqrt{3}-3}{15} \le \mathbb{P}(\|G\| > (\mathbb{E}\|G\|^2)^{1/2}) \le \frac{1}{2}. \tag{3.9}$$

By [7] the upper bound $\frac{1}{2}$ is actually valid for Gaussian random variables with values in a real separable Banach space. We also refer to [10] for related results on Gaussian quadratic forms.

Proof of Proposition 3.5. It is well-known that we can find independent standard Gaussian random variables $(\gamma_n)_{n\geq 1}$, orthonormal vectors $(a_n)_{n\geq 1}$ in H and positive numbers $(\lambda_n)_{n\geq 1}$ such that $G = \sum_{n\geq 1} \sqrt{\lambda_n}\,\gamma_n a_n$, where the series converges almost surely in H. The convergence also holds in $L^2(\Omega; H)$. Notice that

$$\xi := \|G\|^2 - \mathbb{E}\|G\|^2 = \sum_{n\geq 1} \lambda_k(\gamma_k^2 - 1),$$

so that as in Lemma 3.1, $\mathbb{E}\xi^2 = 2\sum_{n\geq 1} \lambda_k^2$ and $\mathbb{E}\xi^4 \leq 60\sum_{n\geq 1} \lambda_k^2$. Therefore the lower estimate follows from Proposition 2.3. ∎

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