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MODULES WITH SEMIREGULAR ENDOMORPHISM RINGS

BY

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Dedicated to Kanzo Masaike on the occasion of his sixty-fifth birthday

Abstract. We characterize the semiregularity of the endomorphism ring of a module with respect to the ideal of endomorphisms with large kernel, and show some new classes of modules with semiregular endomorphism rings.

Introduction. In this paper, a *ring* is an associative ring with an identity, and a *module* a unital right module. Let R be a ring and M an R-module. For the endomorphism ring $\Lambda = \mathrm{End}_R(M)$, we denote by $\mathrm{Rad}_M(\Lambda)$ or $\mathrm{Rad}(\Lambda)$ the Jacobson radical of Λ, and by $\mathrm{Lar}_M(\Lambda)$ or $\mathrm{Lar}(\Lambda)$ the ideal of Λ consisting of all endomorphisms of M with large kernel (see Section 2 for details).

A ring R is said to be *semiregular with respect to an ideal I* if the factor ring R/I is (von-Neumann) regular and any idempotent in R/I lifts to an idempotent in R. A ring semiregular with respect to the Jacobson radical is simply called *semiregular* [1]. It is well known that the endomorphism ring of an injective module is semiregular with respect to the Jacobson radical. This classical theorem is due to the work by R. E. Jonson, Y. Utumi, and J. Lambek (see [3, §4.4]). It has been slightly generalized to quasi-injective modules or continuous modules by Faith–Utumi [2] and Utumi [5]. Moreover, Utumi proved that the Jacobson radical of the endomorphism ring Λ of an injective module M coincides with $\mathrm{Lar}_M(\Lambda)$. Thus it has not been clear how the ideal $\mathrm{Lar}_M(\Lambda)$ relates to the semiregularity for injective modules.

This motivates the research in this paper. Our aim is to give a characterization for a module M having semiregular endomorphism ring with respect to the ideal $\mathrm{Lar}_M(\mathrm{End}(M))$, and as an application, we prove that a module M decomposable into a direct sum of indecomposable injective submodules has semiregular endomorphism ring with respect to $\mathrm{Lar}_M(\mathrm{End}(M))$.

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1. Preliminary results. Let R be a ring and M an R-module. A submodule N of M is said to be *large* in M, denoted by $N \trianglelefteq M$, if $N \cap K \neq 0$ for all non-zero submodules K of M. A submodule K of M is said to be a *semicomplement* of a submodule N in M if $K \cap N = 0$ and $K + N \trianglelefteq M$, and a *complement* of N in M if it is maximal in the set of all semicomplements of N in M. A submodule N of M is called a (*semi*)*complement* provided it is a (semi)complement of a submodule of M.

Let $\Lambda = \mathrm{End}_R(M)$ be the endomorphism ring of M. We denote by $\overline{\Lambda}$ the factor ring of Λ by $\mathrm{Lar}_M(\Lambda)$. For an element $u \in \Lambda$, $M^{\langle u \rangle}$ denotes the submodule of M of elements invariant under u, that is, $M^{\langle u \rangle} = \mathrm{Ker}(1-u)$.

LEMMA 1.1. *For an idempotent $\bar{u}$ of $\overline{\Lambda}$ and $I = \mathrm{Ker}(u^2 - u)$, the following conditions hold*:

(i) $(1-u)I \cap uI = 0$.
(ii) $uI \cap \mathrm{Ker}\, u = 0$.

Proof. (i) For $x \in (1-u)I \cap uI$, let $x = (1-u)a = ub$ for some $a, b \in I$. Then $ux = u(1-u)a = (u-u^2)a = 0$, and $(1-u)x = (1-u)ub = (u-u^2)b = 0$. Hence $x = ux + (1-u)x = 0$.

(ii) For $x \in uI \cap \mathrm{Ker}\, u$, let $x = ua$ for some $a \in I$. Then $x = ua = u^2a = ux$, and hence $x = 0$. ∎

LEMMA 1.2. *For $u \in \Lambda$, $\overline{u} \in \overline{\Lambda}$ is an idempotent if and only if $M^{\langle u \rangle} \oplus \mathrm{Ker}\, u \trianglelefteq M$.*

Proof. Assume that $\overline{u}^2 = \overline{u}$, and let $N = \mathrm{Ker}(u^2 - u)$. Then $N \trianglelefteq M$ and $uN \subseteq \mathrm{Ker}(1-u)$ obviously, and $\mathrm{Ker}(1-u) \subseteq u\,\mathrm{Ker}(1-u) \subseteq uN$, which implies that $uN = u\,\mathrm{Ker}(1-u)$. It follows that $N + \mathrm{Ker}\, u = \mathrm{Ker}\,(1-u) + \mathrm{Ker}\, u$, so $N = \mathrm{Ker}(1-u) + \mathrm{Ker}\, u$, because $\mathrm{Ker}\, u \subseteq N$. Thus we have $M^{\langle u \rangle} \oplus \mathrm{Ker}\, u \trianglelefteq M$.

Conversely, assume that $L = M^{\langle u \rangle} \oplus \mathrm{Ker}\, u \trianglelefteq M$. For any $x \in L$, let $x = a + b$ for some $a \in \mathrm{Ker}(1-u)$ and $b \in \mathrm{Ker}\, u$. Then

$$u^2x = u^2a + u^2b = u^2a = ua = ua + ub = u(a+b) = ux,$$

which implies that $L \subseteq \mathrm{Ker}(u^2 - u)$. Hence $\mathrm{Ker}(u^2-u) \trianglelefteq M$, that is, $\overline{u}^2 = \overline{u}$, because $L \trianglelefteq M$. ∎

2. Lifting idempotents and regular rings. In this section we prove some properties of lifting idempotents and regularity. As before, $\Lambda = \mathrm{End}_R(M)$ denotes the endomorphism ring of an R-module M, and $\overline{\Lambda} = \Lambda/\mathrm{Lar}(\Lambda)$.

Proposition 2.1. *For an idempotent $\overline{u}$ in $\overline{\Lambda}$, the following conditions are equivalent*:

(i) *$\overline{u}$ lifts to an idempotent in Λ.*

(ii) *There is a semicomplement N of* $\operatorname{Ker} u$ *in M such that uN is large in a direct summand of M.*

Proof. We may assume that $\overline{u} \neq 0$, because the conditions hold trivially for $\overline{u} = 0$.

(i)⇒(ii). Let e be an idempotent of Λ with $\overline{e} = \overline{u}$. We have to find a submodule N of M such that $N + \operatorname{Ker} u$ is a direct sum and large in M and $uN \trianglelefteq eM$. Now, there are large submodules L_1, L_2 of M such that

$$(u^2 - u)L_1 = 0, \quad (e - u)L_2 = 0,$$

where we can take L_1 including $\operatorname{Ker} u$. Let X be a complement of $\operatorname{Ker} u$ in M and let

$$N = L_1 \cap L_2 \cap X.$$

Then $X \neq 0$ and $N \neq 0$, because $\overline{u} \neq 0$ and $L_1 \cap L_2 \trianglelefteq M$. Since $L_2 \trianglelefteq M$, we have

$$N = L_2 \cap (L_1 \cap X) \trianglelefteq L_1 \cap X,$$

and since $\operatorname{Ker} u \subseteq L_1$, the modular law yields

$$(L_1 \cap X) \oplus \operatorname{Ker} u = L_1 \cap (X \oplus \operatorname{Ker} u) \trianglelefteq M,$$

because $L_1 \trianglelefteq M$ and $X \oplus \operatorname{Ker} u \trianglelefteq M$. Thus we have $N \oplus \operatorname{Ker} u \trianglelefteq M$.

Next we claim that $uN \trianglelefteq eM$. Clearly, $uN \subseteq eM$ and $u(L_1 \cap L_2) = e(L_1 \cap L_2)$, because $ux = ex$ for all $x \in L_2$. On the other hand, $e(L_1 \cap L_2) \trianglelefteq eM$, because $L_1 \cap L_2 \trianglelefteq M$. Thus it suffices to show that $uN \trianglelefteq u(L_1 \cap L_2)$. Take any non-zero element ux of $u(L_1 \cap L_2)$ with $x \in L_1 \cap L_2$. Then, since $N \oplus \operatorname{Ker} u \trianglelefteq M$, there is an element r of R such that $0 \neq uxr \in N \oplus \operatorname{Ker} u$. Let $uxr = a + b$ for some $a \in N$ and $b \in \operatorname{Ker} u$. Then $uxr = u^2xr$, because $xr \in L_1$, and hence

$$0 \neq uxr = u^2xr = ua + ub = ua \in uN,$$

which shows that uN is large in $u(L_1 \cap L_2)$.

(ii)⇒(i). Let $L = \operatorname{Ker}(u^2 - u)$. Since $\overline{u}^2 = \overline{u}$, we have $\operatorname{Ker} u \subseteq L \trianglelefteq M$. Let $K = L \cap (N \oplus \operatorname{Ker} u)$, which is large in M. First, we show that $uK \trianglelefteq eM$. For this, it is enough to prove that $uK \trianglelefteq uN$, because $uK \subseteq uN$ and $uN \trianglelefteq eM$, by assumption. Take any non-zero element ux of uN with $x \in N$. Since $K \trianglelefteq M$, there is an $r \in R$ with $0 \neq xr \in K$. Hence $0 \neq uxr \in uK$, because $0 \neq xr \in N$ and $N \cap \operatorname{Ker} u = 0$, which implies the claim, and it follows that

$$(1 - e)K \oplus uK \trianglelefteq (1 - e)M \oplus eM = M.$$

Now, following the idea in the proof of [3, §4.4, Proposition 1], let

$$f = e + eu(1 - e),$$

which is clearly idempotent in Λ. We claim that $\overline{u} = \bar{f}$. Since $uN \subseteq eM$, we have $ux = eux$ for any $x \in N$, and hence for any $x \in N + \operatorname{Ker} u$. Thus $\overline{eu} = \overline{u}$ in $\overline{\Lambda}$, so that to prove our claim it suffices to show that $\bar{f} = \overline{eu}$. For this, by the fact observed above that $(1-e)K \oplus uK \trianglelefteq M$, it is enough to prove the following equalities:

$$(f - eu)((1-e)K) = 0, \qquad (f - eu)(uK) = 0.$$

The first equality follows from the following one, for any $x \in K$:

$$f(1-e)x = (e + eu(1-e))(1-e)x = eu(1-e)x.$$

For the second equality, note that $fux = eux$ for any $x \in K$, which follows from that fact that $ux = eux$, because $ux \in uK \subseteq eM$. Hence we have $fux = eux = eu^2x$ for all $x \in K$, which proves the second equality, and completes the proof. ■

It should be noted that the restriction of an element u of Λ to a semicomplement N of $\operatorname{Ker} u$ in M is a monomorphism and induces an isomorphism $u|_N : N \xrightarrow{\sim} uN$. Hence the inverse $(u|_N)^{-1}$ is defined.

PROPOSITION 2.2. *The factor ring $\overline{\Lambda}$ is regular if and only if, for any $u \in \Lambda$, there is a semicomplement N of* $\operatorname{Ker} u$ *in M such that the inverse $(u|_N)^{-1} : uN \to N$ extends to an endomorphism of M, or equivalently, there is an element v of Λ with $vux = x$ for all $x \in N$.*

Proof. Assume that $\overline{\Lambda}$ is regular. We will show that, for any $u \in \Lambda$ and a semicomplement N of $\operatorname{Ker} u$ in M, the inverse $u|_N^{-1} : uN \xrightarrow{\sim} N$ extends to an endomorphism of M. We may assume $\overline{u} \neq 0$. Since $\overline{\Lambda}$ is regular, there is a $v \in \Lambda$ with $\overline{u}\,\overline{v}\,\overline{u} = \overline{u}$, and hence $\overline{w}^2 = \overline{w}$ for $w = vu$. For a large submodule L of M annihilated by $uvu - u$, we have $(w^2 - w)L = 0$, so that $wL \cap \operatorname{Ker} w = 0$ by Lemma 1.1(ii). Now we claim that $N = wL$ is a complement of $\operatorname{Ker} u$ in M. First, notice that $N \cap \operatorname{Ker} u = 0$, because $\operatorname{Ker} u \subseteq \operatorname{Ker} w$ and $N \subseteq M^{\langle w \rangle}$. Since $uN = uwL = uvuL = uL$, we have $N + \operatorname{Ker} u = L + \operatorname{Ker} u$. Thus $N + \operatorname{Ker} u$ is large in M, because $L + \operatorname{Ker} u \trianglelefteq M$. Next, we claim that v is an extension of the inverse of $u|_N : N \xrightarrow{\sim} uN$, that is, vu is the identity on N. In fact, for any $x \in N$ with $x = wy$ for some $y \in L$, we have $x = w^2y = vuwy = vux$ for all $x \in N$.

Conversely, for any $x \in N$ and $y \in \operatorname{Ker} u$, there is a $v \in \Lambda$ with $vux = x$, and hence $uvu(x + y) = uvux = ux = u(x + y)$. It therefore follows that $(uvu - u)(N + \operatorname{Ker} u) = 0$, which implies that $\overline{u}\,\overline{v}\,\overline{u} = \overline{u}$ in $\overline{\Lambda}$. ■

It is well known that $\operatorname{Lar}_M(\Lambda) = \operatorname{Rad}_M(\Lambda)$ for an injective module M. One inclusion between these ideals of Λ comes from a property of endomorphisms of M—see the proposition below, where it should be noted that an endomorphism u of M is monomorphic if $M^{\langle u \rangle} \trianglelefteq M$.

PROPOSITION 2.3. *Let M be an R-module and $\Lambda = \mathrm{End}_R(M)$. Then* $\mathrm{Lar}_M(\Lambda) \subseteq \mathrm{Rad}_M(\Lambda)$ *if and only if any endomorphism u of M with $M^{\langle u\rangle} \trianglelefteq M$ is bijective.*

Proof. Suppose that $\mathrm{Lar}(\Lambda) \subseteq \mathrm{Rad}(\Lambda)$, and take a monomorphism $u \in \Lambda$ with $M^{\langle u\rangle} \trianglelefteq M$. Then $1-u \in \mathrm{Lar}(\Lambda)$ and hence $1-u \in \mathrm{Rad}(\Lambda)$, which implies that u is an isomorphism.

Conversely, take any $v \in \mathrm{Lar}(\Lambda)$. Since $\mathrm{Ker}\, v \trianglelefteq M$ and $\mathrm{Ker}\, v \cap \mathrm{Ker}\,(1-v) = 0$, we have $\mathrm{Ker}(1-v) = 0$, that is, $1-v$ is a monomorphism. Let $u = 1-v$. Then $\mathrm{Ker}(1-u) \trianglelefteq M$, that is, $M^{\langle u\rangle} \trianglelefteq M$. Therefore, by assumption, $1-v$ is invertible in Λ for all $v \in \mathrm{Lar}(\Lambda)$ and hence $\mathrm{Lar}(\Lambda) \subseteq \mathrm{Rad}(\Lambda)$ (see [3, §3.2, Proposition 5]). ∎

COROLLARY 2.4. $\mathrm{Lar}_M(\Lambda) \subseteq \mathrm{Rad}_M(\Lambda)$ *for any artinian module M.*

Proof. Let $u : M \to M$ be a monomorphism with $M^{\langle u\rangle} \trianglelefteq M$. We show that u is an epimorphism. Since M is artinian, there is an integer m with $u^m M = u^{2m} M$. Let $v = u^m$, and let $f : vM \to vM$ be the composition of the inclusion $vM \hookrightarrow M$ and the canonical morphism $M \to vM$ induced by v. Then f is an isomorphism, because f is clearly a monomorphism and $vM = v^2 M = f(vM)$, which implies that f is an epimorphism. Thus the inclusion $vM \hookrightarrow M$ is splittable. On the other hand, obviously $M^{\langle u\rangle} = uM^{\langle u\rangle}$, and hence $M^{\langle u\rangle} = vM^{\langle u\rangle} \subseteq vM$, so that $vM \trianglelefteq M$, because $M^{\langle u\rangle} \trianglelefteq M$. Therefore we have $vM = M$, which obviously implies that u is an epimorphism. ∎

3. Main theorems. A semiregular endomorphism ring $\Lambda = \mathrm{End}_R(M)$ with respect to the ideal $\mathrm{Lar}(\Lambda)$ of Λ is simply said to be *L-semiregular*.

THEOREM 3.1. *For an R-module M and $\Lambda = \mathrm{End}_R(M)$, the following conditions are equivalent*:

(i) *Λ is L-semiregular.*
(ii) *For any $u \in \Lambda$, there are semicomplements N_1, N_2 of* $\mathrm{Ker}\, u$ *in M such that*
  (a) *$(u|_{N_1})^{-1} : uN_1 \to N_1$ extends to an endomorphism of M,*
  (b) *uN_2 is large in a direct summand of M if $u^2 - u \in \mathrm{Lar}(\Lambda)$.*
(iii) *For any $u \in \Lambda$, there is a semicomplement N of* $\mathrm{Ker}\, u$ *in M such that*
  (a) *$(u|_N)^{-1} : uN \to N$ extends to an endomorphism of M,*
  (b) *uN is large in a direct summand of M if $u^2 - u \in \mathrm{Lar}(\Lambda)$.*

Proof. The implications (i)⇒(ii) and (iii)⇒(i) follow from Propositions 2.1 and 2.2.

(ii)⇒(iii). If $\mathrm{Ker}\, u \trianglelefteq M$, it is enough to take $N = 0$. Hence we assume that $\mathrm{Ker}\, u$ is not large in M.

For $u \in \Lambda$ let N_1, N_2 be the non-zero semicomplements of $\operatorname{Ker} u$ in M given in (ii) and $uN_2 \trianglelefteq eM$ for some idempotent e of Λ. Assume that $\overline{u}^2 = \overline{u}$. It suffices to show that we can take a common submodule N as N_1 and N_2. Let $\varphi : N_1 \oplus \operatorname{Ker} u \to N_1$ be a projection and

$$N' = (N_1 \oplus \operatorname{Ker} u) \cap N_2, \quad N = \varphi(N').$$

Then, since $N \subseteq N_1$, we have $N \cap \operatorname{Ker} u = 0$ and the restriction of vu to N is the identity. Hence, to show that N satisfies (iii)(b), it suffices to prove that

$$N \oplus \operatorname{Ker} u \trianglelefteq M, \quad uN \trianglelefteq eM.$$

However, $N' + \operatorname{Ker} u = N + \operatorname{Ker} u$ because $uN' = uN$, and therefore we will show that $N' \oplus \operatorname{Ker} u \trianglelefteq M$ and $uN' \trianglelefteq eM$. Now, by the modular law,

$$\begin{aligned} N' \oplus \operatorname{Ker} u &= ((N_1 \oplus \operatorname{Ker} u) \cap N_2) \oplus \operatorname{Ker} u \\ &= (N_1 \oplus \operatorname{Ker} u) \cap (N_2 \oplus \operatorname{Ker} u), \end{aligned}$$

which implies that $N' \oplus \operatorname{Ker} u \trianglelefteq M$, because $N_i \oplus \operatorname{Ker} u \trianglelefteq M$ by the choice of N_i for $i = 1, 2$. Next, to show that $uN' \trianglelefteq eM$, we show that $uN' \trianglelefteq uN_2$, because $uN' \subseteq uN_2 \trianglelefteq eM$. Take a non-zero $x \in uN_2$ and let $x = uy$ for some $y \in N_2$. Since $N_1 \oplus \operatorname{Ker} u \trianglelefteq M$, there is an $r \in R$ with $0 \neq yr \in N_1 \oplus \operatorname{Ker} u$, so that $0 \neq yr \in N'$. It therefore follows that $0 \neq xr \in N'$, because $N' \cap \operatorname{Ker} u = 0$. ∎

Proposition 3.2. *For an R-module M and $\Lambda = \operatorname{End}_R(M)$, the following conditions are equivalent*:

(i) *$\Lambda/\operatorname{Rad}(\Lambda)$ is regular and $\operatorname{Rad}(\Lambda) = \operatorname{Lar}(\Lambda)$.*
(ii) *$\Lambda/\operatorname{Lar}(\Lambda)$ is regular and any $u \in \Lambda$ with $M^{\langle u \rangle} \trianglelefteq M$ is an isomorphism.*

Proof. The implication (i)⇒(ii) follows from Proposition 2.3.

(ii)⇒(i). It is enough to show that $\operatorname{Rad}(\Lambda) = \operatorname{Lar}(\Lambda)$. In fact, we have $\operatorname{Lar}(\Lambda) \subseteq \operatorname{Rad}(\Lambda)$, by Proposition 2.3. On the other hand, $\overline{\Lambda} = \Lambda/\operatorname{Lar}(\Lambda)$ is regular by assumption, and hence $\operatorname{Rad}(\overline{\Lambda}) = 0$, which implies that $\operatorname{Rad}(\Lambda) \subseteq \operatorname{Lar}(\Lambda)$. Therefore we have $\operatorname{Rad}(\Lambda) = \operatorname{Lar}(\Lambda)$. ∎

Proposition 3.2 can be restated as follows.

Theorem 3.3. *Let M be an R-module and $\Lambda = \operatorname{End}_R(M)$, and assume that any monomorphism $u \in \Lambda$ with $M^{\langle u \rangle} \trianglelefteq M$ is an isomorphism. Then Λ is L-semiregular if and only if Λ is semiregular and $\operatorname{Rad}(\Lambda) = \operatorname{Lar}(\Lambda)$.*

The following is an immediate consequence of Theorem 3.3 and Corollary 2.4.

Corollary 3.4. *The endomorphism ring $\Lambda = \operatorname{End}_R(M)$ of an artinian R-module M is L-semiregular if and only if Λ is semiregular and $\operatorname{Rad}(\Lambda) = \operatorname{Lar}(\Lambda)$.*

An R-module M is said to be *uniform* if any non-zero submodule of M is large in M.

COROLLARY 3.5. *The endomorphism ring* $\Lambda = \mathrm{End}_R(M)$ *of an artinian uniform R-module M is semiregular and* $\mathrm{Rad}(\Lambda) = \mathrm{Lar}(\Lambda)$.

Proof. Notice that either any endomorphism u of M is injective or $\mathrm{Ker}\, u \trianglelefteq M$. In the first case, it is easy to see that u is an isomorphism, because M is artinian. Hence, condition (ii) in Theorem 3.1 clearly holds for any endomorphism u of M, so that $\Lambda = \mathrm{End}_R(M)$ is L-semiregular. The semiregularity of Λ then follows from Theorem 3.3. ∎

The following corollary mentioned in the introduction now follows from Theorems 3.1 and 3.3, where a module M is said to be *continuous* in the sense of Utumi provided a submodule of M is a direct summand of M if it is isomorphic to a complement in M. Notice that any complement of a continuous module M is a direct summand of M. Obviously, injective modules and quasi-injective modules are continuous.

COROLLARY 3.6. *The endomorphism ring* $\Lambda = \mathrm{End}_R(M)$ *of a continuous R-module M is semiregular and* $\mathrm{Rad}(\Lambda) = \mathrm{Lar}(\Lambda)$.

Proof. Assume that M is continuous, and for any $u \in \Lambda$ take a complement N of $\mathrm{Ker}\, u$. Then N and uN ($\simeq N$) are direct summands of M by assumption. Hence condition (ii) in Theorem 3.1 holds and hence Λ is L-semiregular. On the other hand, for a monomorphism $u : M \to M$ with $M^{\langle u\rangle} \trianglelefteq M$, we clearly have $M^{\langle u\rangle} \subseteq uM \subseteq M$, so that $uM \trianglelefteq M$. It follows that $uM = M$, because uM is isomorphic to M and hence is a direct summand of M by continuity of M. Thus the corollary follows from Theorem 3.3. ∎

4. Direct sum of injective modules. As a generalization of the theorem for injective modules, we consider the semiregularity of the endomorphism ring of a module which is decomposable into a direct sum of injective submodules. The aim of this section is to show the semiregularity for direct sums of indecomposable injective submodules.

The following well known lemma is useful to check the decomposability of a module.

LEMMA 4.1. *Let M be a direct sum of submodules X and Y, and let $p_X : M \to X$, $p_Y : M \to Y$ be the projections. Then a submodule N of M is a direct summand with complement Y, $M = N \oplus Y$, if the restriction of p_X to N is isomorphic.*

LEMMA 4.2. *Let M be a direct sum of submodules X and Y, and N be a submodule of M such that $p_X|_N : N \to X$ is monomorphic and there is an injective hull of $p_Y(N)$ in Y. Then*

(i) *There is a submodule L such that $N \subseteq L \subseteq M$ and $M = L \oplus Y$.*
(ii) *If $p_X(N)$ has an injective hull X_0 in X, then there is an injective hull I of N in M with $p_X(I) = X_0$.*

Proof. (i) Since $p_X|_N : N \to X$ is a monomorphism, $p_X(x) = 0$ implies $p_Y(x) = 0$, for $x \in N$. Hence the correspondence $\varphi_0 : p_X(N) \to p_Y(N)$ with $\varphi_0(p_X(x)) = p_Y(x)$ is well defined. Let Y_0 be an injective hull of $p_Y(N)$ in Y, and $\varphi_1 : p_X(N) \to Y_0$ be the composition of φ_0 with the inclusion $p_Y(N) \hookrightarrow Y_0$. Then φ_1 extends to a homomorphism $\varphi_2 : X \to Y_0$, by injectivity of Y_0, and we get the commutative diagram

$$\begin{array}{ccc} p_X(N) & \hookrightarrow & X \\ {\scriptstyle \varphi_1}\downarrow & \swarrow{\scriptstyle \varphi_2} & \downarrow{\scriptstyle \varphi} \\ Y_0 & \hookrightarrow & Y \end{array}$$

Let φ be the composition of φ_2 with the inclusion $Y_0 \hookrightarrow Y$, and let $L = \{(x, \varphi(x)) \mid x \in X\}$. Then $N \subseteq L$ and the restriction $p_X|_L : L \to X$ is obviously an isomorphism. It follows from Lemma 4.1 that $M = L \oplus Y$. ∎

Now we are able to prove the main result of this section.

THEOREM 4.3. *Let M be an R-module decomposable into a direct sum of indecomposable injective submodules. Then* $\mathrm{End}_R(M)$ *is L-semiregular.*

Proof. Let $M = \bigoplus_{i\in\Omega} M_i$, where all M_i are indecomposable injective submodules of M, and for a subset Ω' of Ω, denote by $M_{\Omega'}$ the direct summand $\bigoplus_{i\in\Omega'} M_i$ of M. For an endomorphism u of M, we will show that there is a semicomplement N of $\mathrm{Ker}\, u$ in M such that $N \trianglelefteq eM$ for some $e = e^2 \in \mathrm{End}_R(M)$, and $(u|_N)^{-1} : uN \to N$ lifts to an endomorphism of M. Then the theorem follows from Theorem 3.1.

Let Ω_1 be a maximal subset of Ω with $M_{\Omega_1} \cap \mathrm{Ker}\, u = 0$, and Ω_2 be a maximal subset of Ω with $uM_{\Omega_1} \cap M_{\Omega_2} = 0$. Then $M_{\Omega_1} \oplus \mathrm{Ker}\, u$ and $uM_{\Omega_1} \oplus M_{\Omega_2}$ are large in M. Let $X = M_{\Omega-\Omega_2}$ and $Y = M_{\Omega_2}$, and let p_X, p_Y be the projections of $M = X \oplus Y$ to X and Y, respectively. Since $p_X(uM_{\Omega_1}) \trianglelefteq X$, we have $p_X(uM_{\Omega_1}) \cap M_i \neq 0$ for any $i \in \Omega - \Omega_2$. Take a non-zero finitely generated submodule $S'_i \subseteq p_X(uM_{\Omega_1}) \cap M_{\Omega_i}$. Then M_i is an injective hull of S'_i, because M_i is indecomposable injective. Let $S_i = p_X^{-1}(S'_i) \cap uM_{\Omega_1}$ for $i \in \Omega - \Omega_2$. Clearly $p_X|_{S_i} : S_i \to S'_i$ is an isomorphism, which implies that S_i is finitely generated and so is $p_Y(S_i)$. Hence $p_Y(S_i)$ is contained in a direct sum of finitely many summands Y_j ($j \in \Omega_2$), and so $p_Y(S_i)$ has an injective hull in Y. It therefore follows from Lemma 4.2(ii) that there is an injective hull E_i of S_i of M such that $p_X(E_i) = M_i$, where

E_i is indecomposable, because of the uniformity of S_i. Now, if

$$E = \bigoplus_{i\in\Omega-\Omega_2} E_i \quad \text{and} \quad S = \bigoplus_{i\in\Omega-\Omega_2} S_i,$$

then $S \trianglelefteq E$. Since E_i and M_i are indecomposable injective and $p_X|_{S_i} : S_i \to S_i'$ is an isomorphism, $p_X|_{E_i} : E_i \to M_i$ is also an isomorphism for all $i \in \Omega - \Omega_2$ and so $p_X|_E : E \to X$ is an isomorphism. Hence, by Lemma 4.1, there is an idempotent e of $\operatorname{End}_R(M)$ with $E = eM$. This shows that $S \trianglelefteq eM$.

Let $N_i = u^{-1}(S_i) \cap M_{\Omega_1}$ and $N = \bigoplus_{i\in\Omega-\Omega_2} N_i$. Then $uN = S \trianglelefteq eM$. Since N_i is isomorphic to S_i by u, it is finitely generated, and hence there is an injective hull F_i of N_i in M. Let $\theta_i : F_i \hookrightarrow M$ be the inclusion, and let $v_i : E_i \to F_i$ be an extension of $(u|_{S_i})^{-1} : S_i \xrightarrow{\sim} N_i$, and $v' = \sum_i \theta_i v_i : E = \bigoplus_i E_i \to M$, that is, there is a commutative diagram

$$\begin{array}{ccccccc}
S_i & \hookrightarrow & E_i & \hookrightarrow & E & \hookrightarrow & M \\
{\scriptstyle (u|_{S_i})^{-1}}\downarrow \wr & & \downarrow {\scriptstyle v_i} & & \downarrow {\scriptstyle v'} & \swarrow {\scriptstyle v} & \\
N_i & \hookrightarrow & F_i & \overset{\theta_i}{\hookrightarrow} & M & &
\end{array}$$

Since E is a direct summand of M, the homomorphism v' naturally extends to an endomorphism v of M. It is clear that the restriction of vu to N is the identity. Moreover, $N \cap \operatorname{Ker} u = 0$ because $N \subseteq M_{\Omega_1}$, and $N \trianglelefteq M_{\Omega_1}$ because $S = \bigoplus S_i \trianglelefteq uM_{\Omega_1}$ and $u|_{M_{\Omega_1}}$ is a monomorphism. Thus

$$N \oplus \operatorname{Ker} u \trianglelefteq M_{\Omega_1} \oplus \operatorname{Ker} u \trianglelefteq M,$$

and therefore $N \oplus \operatorname{Ker} u \trianglelefteq M$. ∎

The endomorphism ring of a direct sum $M = \bigoplus_{i\in\Omega} M_i$ of indecomposable injective submodules is not necessarily semiregular. In fact, if $\operatorname{End}_R(M)$ is semiregular, then $\operatorname{End}_R(M)$ is an exchange ring and hence M has the finite exchange property by a theorem of Warfield. See [1, Corollaries 11.21 and 11.17]. Then the system $\{M_i\}_{i\in\Omega}$ is locally semi-T-nilpotent by [6] (see [1, Corollary 12.14] for a general result). But, in general, the family of indecomposable injective modules does not form a locally semi-T-nilpotent system. An example is obtained by making use of the following ring constructed by Osofsky.

Let $\mathbb{Z}_{(p)}$ denote the ring of p-adic integers for some prime p, and R be the trivial extension ring $\mathbb{Z}_{(p)} \ltimes \mathbb{Z}_{p^\infty}$, where $\mathbb{Z}_{p^\infty}$ is considered as a $\mathbb{Z}_{(p)}$-bimodule by the canonical isomorphism $\mathbb{Z}_{(p)} \simeq \operatorname{End}_{\mathbb{Z}}(\mathbb{Z}_{p^\infty})$. Then R is a commutative local ring with the maximal ideal generated by $\overline{p} = (p, 0)$ and simple socle. Moreover, R is an indecomposable injective cogenerator as an R-module. See [4, Example 1]. Now let M_n $(n \in \mathbb{N})$ be a copy of the R-module R and let $f_n : M_n \to M_{n+1}$ be the multiplication map $f_n(x) = x\overline{p}$ $(x \in M_n)$. It is clear that $f_n \dots f_1(1,0) = (p^n, 0) \neq 0$ for any $n \in \mathbb{N}$, which shows that the system $\{M_n, f_n\}_{n\in\mathbb{N}}$ is not locally semi-T-nilpotent.

We finish the paper by stating an open problem which was one of the motivation of this work.

Problem. Is the ring $\mathrm{End}_R(M)$ L-semiregular for an R-module M that decomposes into a direct sum of injective submodules?

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