MINIMAL NON-INVERTIBLE TRANSFORMATIONS OF SOLENOIDS

BY

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Abstract. We construct a continuous non-invertible minimal transformation of an arbitrary solenoid. Since solenoids, as all other compact monothetic groups, also admit minimal homeomorphisms, our result allows one to classify solenoids among continua admitting both invertible and non-invertible continuous minimal maps.

1. Introduction. We consider discrete dynamical systems $(X, f)$, where $X$ is a compact metric space and $f : X \to X$ is a continuous map. A dynamical system is minimal if $X$ has no non-empty proper closed $f$-invariant subsets ($M \subset X$ is $f$-invariant if $f(M) \subset M$; see e.g. [2]).

There is an interesting classification of compact metric spaces with respect to the existence of minimal homeomorphisms and non-invertible maps. Let us say that a space $X$ is of type $(\xi_1, \xi_2)$, where $\xi_1, \xi_2 \in \{0, 1\}$, according to the following rule: if there exists a minimal homeomorphism on $X$, then we set $\xi_1 = 1$, otherwise we set $\xi_1 = 0$; and if there exists a minimal non-invertible continuous map on $X$, then we set $\xi_2 = 1$, otherwise we set $\xi_2 = 0$.

There are well known easy examples of all four types of spaces. Moreover, all four types have representatives among connected spaces (continua). The unit interval $[0, 1]$ has the fixed point property, so it admits neither minimal homeomorphisms, nor minimal non-invertible maps (type $(0, 0)$). The 2-torus $\mathbb{T}$ admits both invertible and non-invertible minimal maps (type $(1, 1)$; see [11]). The pinched 2-torus (the 2-torus with two points identified) admits non-invertible minimal maps, while it has the fixed point property for homeomorphisms, so it does not admit minimal homeomorphisms (type $(0, 1)$; see [3]). The simple closed curve $S^1$ (i.e., the circle) admits minimal homeomorphisms but it does not admit minimal non-invertible maps (type $(1, 0)$; see [1]).

2010 Mathematics Subject Classification: Primary 37B05; Secondary 54H20.
Key words and phrases: minimal dynamical system, discrete dynamical system, non-invertible transformation, solenoid, continuum.

DOI: 10.4064/cm127-2-7
There is no other known infinite continuum of type $(1, 0)$ and it is a long standing open problem whether $S^1$ is indeed the unique continuum of this type. In search for other such continua it is natural to examine spaces as similar to the circle as possible. Let us mention that in [3] the authors conjecture that the pseudo-circle is of this type. It is known that there exists a minimal homeomorphism of the pseudo-circle (see [10]), but the existence of a non-invertible minimal map on this space is undecided. More information about this problem can be found in [3] and [11].

In this paper we focus on solenoids. Every solenoid, like the circle, has the structure of a compact monothetic group, hence it admits a minimal homeomorphism (rotation by a topological generator). Moreover, each solenoid is connected, which makes it even more similar to the circle. The same two properties (being a compact connected monothetic group) also hold for multidimensional tori, but solenoids and the circle are the only one-dimensional connected compact monothetic groups. Solenoids can be obtained from the circle via inverse limit procedures, a property they share with the pseudo-circle. All these facts make solenoids natural candidates to be tested for their type, and, to our knowledge, they have not yet been examined in this respect.

We will prove the following

**Theorem 1.1.** Every solenoid admits a minimal non-invertible continuous map.

That is, all solenoids are of type $(1, 1)$. This result does not solve the general problem whether there exists a continuum of type $(1, 0)$ different from a simple closed curve. But at least we decide how to classify solenoids, so we solve the restricted problem for monothetic groups of topological dimension one, and we eliminate a natural class of candidate spaces to behave (with respect to the existence of minimal maps) like the circle.

**2. Construction.** A solenoid (here denoted by $\Sigma$) was orginally defined as a continuum homeomorphic to the inverse limit $\varprojlim (S^1, f_n)$, where $f_n : S^1 \to S^1$ is a transformation of the form $z^{q_n}$ and $q_n$ is a sequence of natural numbers greater than 1 (see [12], [6] for more information on solenoids).

As we will not use this definition, we refrain from providing the details of inverse limit constructions. Instead we will use a different characterization of solenoids relying on odometers (see [5] for more information on odometers). Odometers are characterized as infinite compact monothetic zero-dimensional groups. Since every odometer is homeomorphic to the Cantor set, we will denote odometers by $\mathcal{C}$. By fixing a topological generator $c_0$ and defining $h : \mathcal{C} \to \mathcal{C}$ by $h(c) = c \oplus c_0$ (where $\oplus$ denotes addition in the odometer) we obtain a minimal equicontinuous dynamical system $(\mathcal{C}, h)$. The hom-
ecomorphism $h$ has the following properties: there exists a strictly increasing sequence $(p_k)_{k \geq 0}$ such that $p_0 = 1$ and $p_k | p_{k+1}$, and a decreasing sequence of closed-and-open sets $V_k$ containing 0 such that $\{V_k, h(V_k), \ldots, h^{p_k-1}(V_k)\}$ is a partition of $\mathcal{C}$ ($V_0 = \mathcal{C}$). In fact, any subsequence of $(p_k)$ starting with $p_0$ (and the corresponding subsequence of sets $V_k$) has the same properties. The simplest dyadic odometer is shown in Figure 1.

![Fig. 1. The action on the dyadic odometer (first and second approximation)](image)

It is known (see [7]) that every solenoid is homeomorphic to a quotient space of the product $[0, 1] \times \mathcal{C}$ of the interval with the Cantor set, with respect to the relation identifying the points $(1, c)$ and $(0, h(c))$, where $h$ is as described above.

In this representation, we can equip the solenoid with the structure of an Abelian topological group, as follows:

$$(t, c) + (s, c') = \begin{cases} 
(t + s, c \oplus c') & \text{if } t + s < 1, \\
(t + s - 1, h(c \oplus c')) & \text{if } t + s \geq 1.
\end{cases}$$

Clearly, $(0, 0)$ is the neutral element of $\Sigma$.

Being compact, metric, connected and Abelian, every solenoid $\Sigma$ is necessarily a monothetic group (see [9]). It is not hard to see that any element of the form $(b, 0)$ with $b$ irrational is a topological generator (i.e., $\{n \cdot (b, 0) : n \in \mathbb{N}\}$ is dense in $\Sigma$). Translation by any topological generator (in particular by $(b, 0)$) is minimal (see e.g. [4]). From now on, we fix a solenoid $\Sigma$ and an irrational number $b \in (1/5, 1/4)$, and we denote by $\alpha$ the translation by $(b, 0)$. Clearly, $\alpha$ preserves arc lengths (in either representation of the solenoid).

Given a point $p$ in a continuum $X$, the composant of $X$ at $p$ is the set $\kappa(p) = \{q \in X : \text{there exists a proper subcontinuum } A \text{ of } X \text{ such that } p, q \in A\}$. 
Solenoids are indecomposable continua (i.e., cannot be represented as a
union of two proper subcontinua) and indecomposable continua have con-
tinuum many pairwise disjoint composants (see [12]).

We will construct a non-invertible, minimal map \( \Psi \) of the solenoid. We
are going to define \( \Psi \) as a limit of homeomorphisms \( \Psi_k: \Sigma \to \Sigma \) with \( \Psi_k = \alpha \circ \beta_1 \circ \cdots \circ \beta_k \), where \( \beta_i: \Sigma \to \Sigma \) will be referred to as the \( i \)th perturbation.

Before we start the construction we define two auxiliary families of func-
tions on the interval \([0, 1]\). For \( k \in \{1, 2, \ldots\} \) let \( F_k \) be the function defined by the following formula (see Figure 2):

\[
F_k(x) = \begin{cases} 
\frac{x}{k+1}, & x \in \left[0, \frac{k+1}{2k+1}\right], \\
2x - 1, & x \in \left(\frac{k+1}{2k+1}, 1\right].
\end{cases}
\]

For \( k \in \{2, 3, \ldots\} \) let us denote by \( f_k \) the function such that \( F_k^{-1} \circ f_k^{-1} = F_k \) (see Figure 3). It is not hard to calculate that

\[
f_{k-1}(x) = \begin{cases} 
\frac{k}{k+1}x, & x \in \left[0, \frac{k+1}{2k+1}\right], \\
2kx - k, & x \in \left(\frac{k+1}{2k+1}, \frac{k}{2k-1}\right], \\
x, & x \in \left(\frac{k}{2k-1}, 1\right].
\end{cases}
\]

Fig. 2. Graph of the function \( F_k \)

Since \( b < 1 \) and we have \( \alpha((0, 0)) = (b, 0) \), it is obvious that \( \alpha((0, 0)) \in [0, 1] \times \{0\} \). In that case \( \alpha \) preserves the composant of \((0, 0)\) and thus it preserves all composants.
Let $U_k = [0, b) \times V_k$. Let us observe that for each $k$ there exists a number $m_k$ such that the images of $U_k$ under the first $m_k$ iterates of $\alpha$, i.e., $U_k, \alpha(U_k), \ldots, \alpha^{m_k-1}(U_k)$, are pairwise disjoint, and together with $\alpha^{m_k}(U_k)$ they cover $\Sigma$ (in fact, $m_0 = 4$, $4p_k \leq m_k < 5p_k$, see Figures 4 and 5).

Fig. 3. Graph of the function $f_{k-1}$

Fig. 4. The first $m_1+1$ (counting from 0 to $m_1$) iterates of $U_1$ under the transformation $\Psi_0$. The picture shows the case with $p_1 = 2$, and $m_1 = 8$. Note that the eighth image of $U_k$ completes the cover of $\Sigma$ (and is not disjoint from $U_k$). The 1-fender is covered by the last five of the nine images.
The sets \([0,1] \times h^{-1}(V_k)\) will be called the \(k\)-fenders. Note that they form a decreasing sequence of sets.

Let \(\Psi_0 = \alpha\) and set \(N_0 = 0\).

We turn to the construction of the first map \(\Psi_1\). We define a selfhomeomorphism \(\gamma_1\) of \(U_1\) by the formula

\[
\gamma_1(x,y) = (\hat{F}_1(x), y),
\]

where \(\hat{F}_1 : [0,b] \to [0,b]\) is rescaled \(F_1\). Next, we let \(\beta_1 = \gamma_1\) on \(U_1\), and on \(\Psi_0(U_1)\) we define

\[
\beta_1 = \alpha \gamma_1^{-1} \alpha^{-1}.
\]

On the rest of the space we let \(\beta_1\) be the identity.

Since the set \(V_1\) is closed and open, and the function \(\hat{F}_1\) matches the identity at 0 and \(b\), the perturbation \(\beta_1\) is easily seen to be continuous. Now we set

\[
\Psi_1 = \Psi_0 \circ \beta_1.
\]

Note that for any point \(p \in U_1\), \(\Psi_1^2(p) = \alpha^2(p)\). It is crucial that the action of \(\Psi_1\) is topologically conjugate to that of \(\alpha\). Indeed, the reader will easily verify that \(\Psi_1 = \phi_1^{-1} \circ \alpha \circ \phi_1\), where \(\phi_1\) equals the identity except on \(\Psi_1(U_1)\) where it matches \(\beta_1\).

Note that the perturbation \(\beta_1\) is the identity on the 1-fender. This is where the future “inverse modifications” will take place.

Assume that for some \(k \geq 1\) we have \(4p_k \geq m_{k-1} + N_{k-1} + 4\) (this was satisfied for \(k = 1\), because \(p_1 \geq 2\), \(m_0 = 4\) and \(N_0 = 0\)), and that we have defined the map \(\Psi_k\) as the composition \(\Psi_k = \alpha \circ \beta_1 \circ \cdots \circ \beta_k\) which is topologically conjugate to \(\alpha\) via a map \(\phi_k\) which is the identity on \(U_k\). Further, we assume that the perturbation \(\beta_k\) differs from the identity only on \(U_k\) and on the set \(\Psi_{k-1}^{m_{k-1}-3}(U_k)\) (this was also satisfied for \(k = 1\) because \(m_0 = 4\) and the modifications affected \(U_1\) and \(\Psi_1(U_1)\)), and that for any \(p \in U_k\), \(\Psi_k^{m_{k-1}-2}(p) = \alpha^{m_{k-1}-2}(p)\) (which also holds for \(k = 1\)).

At this point we note some properties implied by the inductive assumptions. The perturbation introduced in \(\Psi_{k-1}^{m_{k-1}-3}(U_k)\) “reverses” the perturbation made in \(U_k\), so that the orbit under \(\Psi_k\) of any given point \(p \in U_1\) returns to its original orbit under \(\alpha\). Now we look at the orbit of \(p\) through the iterates numbered \(m_{k-1} - 2, \ldots, m_k\). The orbit visits (many times) the set \(U_1\), where it is subject to a perturbation (depending on how “deep” inside \(U_1\) it falls, i.e., to which set \(U_j\) \((1 \leq j \leq k)\)), but then it always visits the set \(\Psi_{j-1}^{m_{j-1}-3}(U_j)\), where this perturbation is “undone”. Eventually, before reaching (for the first time) the \(k\)-fender, the orbit returns to its original bed of the orbit of \(\alpha\). Within the \(k\)-fender the orbit under \(\Psi_k\) matches that under \(\alpha\). This implies, among other things, that the \(k\)-fender is covered
by the “last five” of the observed images $\Psi_k^{m_k-4}(U_k), \ldots, \Psi_k^{m_k}(U_k)$ (because they coincide with $\alpha_k^{m_k-4}(U_k), \ldots, \alpha_k^{m_k}(U_k)$).

On the other hand, any future perturbations affect only (subsets of) $U_k$ and the $k$-fender. Thus, the set (which we will call the $k$-buffer)

$$\Psi_k^{m_k-1-2}(U_k) \cup \ldots \cup \Psi_k^{m_k-5}(U_k)$$

is not affected by any perturbation in the future, which means that on this set we have $\Psi_l = \Psi_{k-1}$ for all $l \geq k$. Note that, since $m_k \geq 4p_k \geq m_k-1 + N_k-1 + 4$, there are at least $N_k-1 + 2$ items in the union defining the $k$-buffer. All these objects (the sets $U_k$, the $k$-fender, the $k$-buffer, the places affected by the perturbations, etc.) are shown for $k = 1$ and 2 in Figures 4 and 5.

Fig. 5. The first $m_2$ images of $U_2$ under the transformation $\Psi_1$. (In this picture $p_2 = 4$ and $m_2 = 17$. Also, we have ignored the perturbation $\beta_1$ which slightly shifts some of the vertical division lines.) The perturbation $\beta_2$ affects $U_2$ and $\Psi_2(U_2)$. The 2-buffer consists of $\Psi_1^{13}(U_2), \ldots, \Psi_1^{17}(U_2)$, while $\Psi_1^{13}(U_2), \ldots, \Psi_1^{17}(U_2)$ cover the 2-fender. We have also marked the set $\Psi_2^{15}(U_3)$ where the perturbation $\beta_3$ will take place (to show that this set falls within the 2-fender).

We are in a position to construct $\Psi_{k+1}$. We cover the space by finitely many open $1/k$-balls. By minimality of $\Psi_k$, there exists a number $N_k$ such that every orbit under $\Psi_k$ visits every selected ball before $N_k$ iterates.
By passing to a subsequence of \((p_k)\) we can assume that \(4p_{k+1} \geq m_k + N_k + 4\). Since \(\Psi_k\) is conjugate to \(\alpha\) via a map \(\phi_k\) which is the identity on \(U_k\) (and hence also on \(U_{k+1}\)), the iterates \(\Psi_k^n(U_{k+1})\) for \(0 \leq n \leq m_k - 1\) are pairwise disjoint, and together with \(\Psi_k^{m_k+1}(U_{k+1})\) they cover \(\Sigma\).

We define a selfhomeomorphism \(\gamma_{k+1}\) of \(U_{k+1}\) by the formula
\[
\gamma_{k+1}(x, y) = (\hat{f}_k(x), y),
\]
where \(\hat{f}_k : [0, b] \to [0, b]\) is rescaled \(f_k\). Next, we let \(\beta_{k+1} = \gamma_{k+1}\) on \(U_{k+1}\). Note that \(\Psi_k^{m_k - 3}(U_{k+1}) \subset \Psi_k^{m_k - 3}(U_k)\) is contained in the \(k\)-fender. On \(\Psi_k^{m_k - 3}(U_{k+1})\), we define
\[
\beta_{k+1} = \alpha^{m_k - 3} \gamma_{k+1}^{-1} \alpha^{-m_k + 3},
\]
and we let \(\beta_{k+1}\) be the identity on the remainder of the space.

At this point we set
\[
\Psi_{k+1} = \Psi_k \circ \beta_{k+1} = \alpha \circ \beta_1 \circ \beta_2 \circ \cdots \circ \beta_{k+1}.
\]
The verification of the condition that \(\Psi_{k+1}^{m_k - 2}(p) = \alpha^{m_k - 2}(p)\) for any \(p \in U_{k+1}\) is straightforward, by multiple substitution.

It is fairly easy to check that \(\Psi_{k+1}\) is topologically conjugate to \(\Psi_k\) (and hence to \(\alpha\)), where the conjugating map \(\phi_{k+1}\) is the identity on \(U_{k+1}\) (\(\phi_{k+1}\) differs from the identity only on \(\Psi_k^n(U_{k+1})\) with \(n = 1, \ldots, m_k - 3\); we skip the details).

3. Properties of the limit transformation \(\Psi\)

**Lemma 3.1.** The transformations \(\Psi_k\) converge uniformly.

**Proof.** It suffices to show that the sequence \(\Psi_k\) is Cauchy in the supremum metric for continuous maps. For given two natural numbers \(k\) and \(l\) note that the transformations \(\Psi_k\) and \(\Psi_{k+l}\) differ on the disjoint sets \(U_{k+1}\) and \(\Psi_k^{m_k - 3}(U_{k+1}), \ldots, \Psi_k^{m_k - l - 3}(U_{k+l})\). On \(U_{k+1}\), \(\Psi_k\) is the composition of \(\alpha\) with \(\hat{F}_k\), while \(\Psi_{k+1}\) is piecewise the composition of \(\alpha\) with \(\hat{F}_{k+j}\) for \(j = 1, \ldots, l\). Since \(\hat{F}_k\) converges uniformly, it is obvious that, on this set, \(\Psi_k\) is uniformly close to \(\Psi_{k+l}\) for large \(k\) (we also use the fact that \(\alpha\) is uniformly continuous). On each of the sets \(\Psi_k^{m_k - j - 1}(U_{k+j})\) \((j = 1, \ldots, l)\) the map \(\Psi_k\) is just \(\alpha\) while \(\Psi_{k+l}\) equals the composition of \(\alpha\) with \(\hat{F}_{k+j}\). Since the functions \(\hat{F}_{k+j}\) converge uniformly to the identity map, the uniform Cauchy condition follows on these sets as well. \(\Box\)

We denote by \(\Psi\) the resulting limit transformation, \(\Psi = \lim_{k \to \infty} \Psi_k\). Let us observe that \(\Psi\) is not invertible since for any point \(x \in [0, b/2]\) we have \(\lim_{k \to \infty} \hat{F}_k(x) = 0\), hence \(\Psi((x, 0)) = \alpha((0, 0)) = (b, 0)\).
It remains to show that the transformation $\Psi$ is minimal. Before we proceed to the next theorem we recall one fact concerning the composants of solenoids. Given two points $p$ and $q$ belonging to the same composant, the ray starting at $p$ and containing $q$ is by definition the union of all arcs containing $q$ and having an endpoint at $p$. Every composant of a solenoid is the union of two rays starting at the same point; moreover, every ray is a dense subset of the solenoid (see [8]).

**Theorem 3.2.** The transformation $\Psi$ is minimal.

**Proof.** Using the same arguments as in Lemma 3.1 it is not hard to show that $d(\alpha(p), \Psi_k(p)) < b/2$ for any $p \in \Sigma$ and $k$ (where $d$ is the distance on $\Sigma$ measured along the composant). Every orbit under $\alpha$ “walks forward” along a ray with steps of length $b$, hence every orbit under $\Psi_k$ “walks forward” along a ray with steps of length strictly between $b/2$ and $3b/2$, while every orbit under $\Psi$ has steps between $b/2$ and $3b/2$ (inclusive). The intersection of the set $D_k = \Psi_k^{m_k-1}(U_k) \cup \Psi_k^{m_k-1-1}(U_k) \cup \Psi_k^{m_k-1}(U_k)$ with any ray is an arc bounded by some point $p$ and $\Psi_k^3(p)$, hence has length strictly larger than $3b/2$. This implies that every orbit under $\Psi$ must visit the set $D_k$.

Note that the $k$-buffer consists of $D_k$ together with at least $N_k - 1$ further images (under the iterates of $\Psi_k$) of $U_k$, and that on the $k$-buffer we have $\Psi_k = \Psi$. This implies that every orbit under $\Psi$ visits the $k$-buffer at least $N_k$ consecutive times, during which it visits every ball in the selected cover by $1/k$-balls. This ends the proof of minimality. ■

Questions:

- It seems that the simplified construction with only the perturbations defined on $U_k$ (without inverses and fenders) also leads to a minimal limit transformation. Is it indeed so?

We repeat the main long standing open question:

- Is it true that the simple closed curve is the only infinite continuum of type $(1, 0)$ (i.e., such that it admits a minimal homeomorphism but does not admit non-invertible minimal maps)?

One can ask more specific versions of the above question:

- If we drop connectedness, then the only known infinite compact metric spaces of type $(1, 0)$ are disjoint unions of finitely many circles. Every such space can be equipped with the structure of a compact monothetic group (the circle times a finite cyclic group). Is it true that among infinite compact monothetic groups these are the only ones of type $(1, 0)$? (We have just eliminated solenoids.)

- Is it true that the pseudo-circle admits non-invertible minimal maps? (See [12] for general information concerning the pseudo-circle.)
Acknowledgements. The author thanks Tomasz Downarowicz for enlightening discussions, remarks and great help with the final editing of this paper.

The author’s research was supported by grant MENII N N201 394537, Poland, for the years 2009–2012.

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Received 28 August 2011;
revised 1 July 2012 (5537)