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TWO CLASSES OF ALMOST GALOIS COVERINGS FOR ALGEBRAS

BY

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Abstract. We prove that for any representation-finite algebra A (in fact, finite locally bounded k-category), the universal covering $F : \tilde{A} \to A$ is either a Galois covering or an almost Galois covering of integral type, and F admits a degeneration to the standard Galois covering $\bar{F} : \tilde{A} \to \tilde{A}/G$, where $G = \Pi(\Gamma_A)$ is the fundamental group of Γ_A. It is shown that the class of almost Galois coverings $F : R \to R'$ of integral type, containing the series of examples from our earlier paper [Bol. Soc. Mat. Mexicana 17 (2011)], behaves much more regularly than usual with respect to the standard properties of the pair $(F_\lambda, F_\bullet)$ of adjoint functors associated to F.

Introduction. The covering technique has remained one of the most important and efficient tools of modern representation theory of finite-dimensional algebras over a field. They were invented over thirty years ago by Bongartz, Gabriel and Riedtmann in order to study and classify representation-finite algebras ([19, 23, 3, 18], see also [20, 26]).

Coverings are usually used to reduce a problem for modules over an algebra to an analogous one, often much simpler, for its cover category. This allows one to answer many interesting theoretical questions and to obtain classifications for various classes of algebras (or matrix problems). From general results, in some cases it is possible to determine the representation type of an algebra A (more precisely, a finite locally bounded k-category) and to construct all indecomposable A-modules of the first kind with respect to F by applying the push-down functor $F_\lambda : \operatorname{mod} R \to \operatorname{mod} A$, where $F : R \to A$ is a *Galois covering* (see [18, 14, 11, 15] and also [6, 7]).

The approach to describing the category $\operatorname{mod} A$ of all finite-dimensional A-modules by using Galois coverings is helpful in principle only if A is a standard algebra. It is clear that to study nonstandard algebras one should extend the class of coverings by admitting also some non-Galois ones, and next develop for this larger class a theory similar to that for Galois coverings.

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The series of our recent articles [8, 9, 10] realizes this idea. We have introduced and thoroughly examined the notion of an *almost Galois G-covering functor* $F : R \to R'$ (between locally bounded k-categories) *of type* L, where L is a totally ordered factor group of G, in particular, covering functors of *integral type* (see [8, 2.2]). These functors behave nicely as regards the preservation of the representation types and as regards the nice properties of the pair $(F_\lambda, F_\bullet)$ of the functors associated to F, where $F_\bullet : \mathrm{MOD}\, R' \to \mathrm{MOD}\, R$ is the pull-up functor, $F_\lambda : \mathrm{MOD}\, R \to \mathrm{MOD}\, R'$ is the left adjoint to $F_\bullet$, and $\mathrm{MOD}\, R$ (respectively $\mathrm{MOD}\, R'$) denotes the category of all right modules over R (respectively R'). To some extent they are similar to Galois coverings. Recall that if R is a locally bounded k-category equipped with a free action of a group $G \subseteq \mathrm{Aut}_{k\text{-cat}}(R)$ on $\mathrm{ob}\, R$ and $F : R \to R'$ is an almost Galois G-covering functor of integral type then

$$(*)\qquad F_\bullet F_\lambda(N) \cong \bigoplus_{g\in G} {}^gN \quad \text{and} \quad F_\lambda({}^gN) \cong F_\lambda(N) \quad \text{for every } g \in G,$$

whenever N is an indecomposable module in $\mathrm{mod}\, R$ with $\mathrm{Ext}^1_R(N, {}^hN) = 0$ for all $h \in G^{\prec e}$, where $\prec$ is an ordering on G induced in some natural way from the standard ordering $<$ on $\mathbb{Z}$. Properties of this kind are essential if one thinks about reconstructing indecomposable R'-modules from those over R. It is also shown that a G-covering functor F of integral type can always be "built into the triangle" of the form

$$(\triangle)\qquad \begin{tikzcd} & R \arrow[dl, "F"'] \arrow[dr, "\bar F"] & \\ R' & \rightsquigarrow & \bar R \end{tikzcd}$$

where $\bar R$ is the quotient (i.e. orbit category) R/G of R by G, $\bar F : R \to \bar R$ is the canonical Galois covering functor with group G, given by the natural projection, and $R' \rightsquigarrow \bar R$ indicates that $\bar R$ is a degeneration of R' (see [8, 2.5]). In fact this degeneration (of finite locally bounded categories) is a very special one, namely, it can be extended to a degeneration of covering functors, F' to $\bar F$, in the sense of [10, 2.4] (see [10, Theorem 2.6]).

Almost Galois coverings of integral type can be constructed (in the scheme $(\triangle)$) for many classical examples of nonstandard algebras in the representation-finite as well as in the tame case (see [9, Example 2.3 and Theorem 4.2]; there R is a universal cover of the standard form $\bar R$ of R'). They seem to be a proper tool for understanding the structure of algebras of this kind and the way of constructing them.

In the present paper we study the properties of coverings for finite categories from two classes, containing the examples mentioned above. The first class consists of square-free categories and is closely related to representation-finite algebras. The second one is defined in a purely combinatorial way, referring to the notion of s-flower (see 3.2). In each case we are mainly interested in fact in the same general questions: whether a G-covering F of the category from the class considered is an almost Galois covering of integral type, and (if this is the case) whether the pair $(F_\lambda, F_\bullet)$ consisting of the push-down and pull-up functors associated to F behaves more regularly than in the general situation, in the context of the isomorphisms $(*)$ described above.

The paper is divided into three sections. Section 1 contains preliminaries, i.e. the notation and basic definitions and facts.

In Section 2 we consider the class of square-free finite locally bounded categories. Recall that it contains all Auslander categories for representation-finite k-algebras. We show that if $F : R \to R'$ is a G-covering functor between square-free categories and $R = R(Q, I)$ satisfies certain natural conditions, then F is either a Galois covering or an almost Galois covering of integral type, with respect to some slightly modified action of G on R; moreover, the isomorphisms $(*)$ hold for F with respect to the original action (see Theorem 2.1(A)). As a consequence, we infer that for any finite representation-finite locally bounded category A, the universal covering $F : \tilde{A} \to A$, where $\tilde{A}$ is a common universal cover for A and for its standard form $\bar{A}$ (see [18, 4] for definition), is either a Galois covering or an almost Galois covering of integral type. In this situation F also admits a degeneration to the canonical Galois covering $\bar{F} : \tilde{A} \to \bar{A} = \tilde{A}/G$, where G is the fundamental group of the Auslander–Reiten quiver $\mathit{\Gamma}_A$ (see Theorem 2.1(B)). Additionally, we discuss in detail the scope of validity of the isomorphisms $(*)$ for the case when R is a special biserial string category (Corollary 2.6).

Section 3 is devoted to properties of series of natural important examples of almost Galois G-coverings $F : R \to R'$ of integral type, with G being an infinite cyclic group, introduced in [8, 4.1]. It occurs that they fit into a more general combinatorial scheme, which can be axiomatized (see 3.2). We show that the pairs $(F_\lambda, F_\bullet)$ of adjoint functors associated to F behave in this case in a much more regular way with respect to the isomorphisms $(*)$ than in the general situation of coverings of integral type (Theorem 3.8; announced in [8, 4.3]). In particular, the set of testing elements $h \in G^{\prec e}$ in the Ext-vanishing condition is finite, very concrete and does not depend on N in $\operatorname{ind} R$. (Note that in the general situation this set is always finite, but strongly depends on the support of N.) Theorem 3.8 is a specialization of the main result of this part, Theorem 3.2, which says that in the abstract situation described by the system of combinatorial axioms, the push-down and pull-up functors $(F_\lambda, F_\bullet)$ associated to the almost Galois covering F, with an infinite cyclic group G, have just the nice properties mentioned above. The proof of this result requires a deep and detailed analysis of the shape of R-modules

$F_\bullet F_\lambda(N)$, which refers to the construction of the functors Φ_r and $\Phi_{\underline{n}}$ (3.2 and 3.4, respectively), and their nontrivial properties (Proposition 3.4 and Corollary 3.6).

1. Preliminaries and notation. In the paper we use standard definitions and notation, e.g. we denote by $\mathbb{N}$ (resp., by $\mathbb{Z}$) the set of all natural numbers with 0 (resp., the set of all integers). For a positive $n \in \mathbb{N}$, we set $[n] := \{1, \ldots, n\}$ and $\mathrm{Sub}(n) := \{\underline{e} = (e_1, \ldots, e_p) \in \mathbb{N}^p : 1 \leq e_1 < \cdots < e_p \leq n,\ p \geq 1\}$.

Below, we briefly recall the most important notions and facts used in the next sections. For basic information concerning representation theory of algebras we refer to [1].

1.1. Let k be an algebraically closed field. Following [3] a k-category R (each set $R(x,y)$ of morphisms from x to y in R, $x, y \in \mathrm{ob}\, R$, is a k-linear space and composition of morphisms in R is k-bilinear) is called *locally bounded* if all objects of R have local endomorphism rings, different objects are nonisomorphic, and the sums $\sum_{y \in R} \dim_k R(x,y)$ and $\sum_{y \in R} \dim_k R(y,x)$ are finite for each $x \in \mathrm{ob}\, R$. The Jacobson radical of R is always denoted by $J(R)$. Recall that R is *square-free* if $\dim_k (J/J^2)(x,y) \leq 1$ for all $x, y \in \mathrm{ob}\, R$, where $J = J(R)$. By $\underline{\dim}\, R$ we denote the family $(\dim_k J(x,y))_{x,y \in \mathrm{ob}\, R}$, and if S is a subcategory of R, then $\widehat{S}$ is the full subcategory of R formed by all $x \in \mathrm{ob}\, R$ such that $R(x,y) \neq 0$ or $R(y,x) \neq 0$, for some $y \in \mathrm{ob}\, S$. Note that for locally bounded categories R and R', any full, faithful and dense functor $F : R \to R'$ is always invertible. Consequently, R and R' are equivalent if and only if they are isomorphic.

For a locally bounded k-category R, by an *R-module* we mean a contravariant k-linear functor from R to the category of all k-vector spaces. An R-module M is *locally finite-dimensional* (resp. *finite-dimensional*) if $\dim_k M(x)$ is finite for each $x \in \mathrm{ob}\, R$ (resp. $\dim_k M = \sum_{x \in \mathrm{ob}\, R} \dim_k M(x)$, the *dimension* of M, is finite). We denote by $\mathrm{MOD}\, R$ the category of all R-modules, by $\mathrm{Mod}\, R$ (resp. $\mathrm{mod}\, R$) the full subcategory of $\mathrm{MOD}\, R$ formed by all locally finite-dimensional (resp. finite-dimensional) R-modules and by $\mathrm{ind}\, R$ the full subcategory of $\mathrm{mod}\, R$ formed by all indecomposable objects.

For any M from $\mathrm{Mod}\, R$, by the *dimension vector* of M we mean the collection $\underline{\dim}_k M = (\dim_k M(x))_{x \in \mathrm{ob}\, R} \in \mathbb{N}^{\mathrm{ob}\, R}$ and by the *support* of M the set $\mathrm{supp}\, M = \{x \in \mathrm{ob}\, R : M(x) \neq 0\}$. The category R is called *locally representation-finite* (resp. *locally support-finite*) if for every $x \in \mathrm{ob}\, R$, there exist only a finite number of pairwise nonisomorphic modules M in $\mathrm{ind}\, R$ with $M(x) \neq 0$ (resp. the union $R(x)$ of all $\mathrm{supp}\, M$, for M in $\mathrm{ind}\, R$ with $M(x) \neq 0$, is a finite set).

Locally bounded categories are used in a natural way to study modules over algebras. With any finite locally bounded category R we can associate a finite-dimensional basic algebra $A(R) = \bigoplus_{x,y\in \mathrm{ob}\, R} R(x,y)$, with multiplication induced by composition in R. It is clear that we always have an equivalence $\mathrm{mod}\, A(R) \simeq \mathrm{mod}\, R$ of module categories, where $\mathrm{mod}\, A$ denotes the category of all finite-dimensional A-modules for any algebra A. Conversely, if A is finite-dimensional algebra, $\mathrm{mod}\, A$ can be interpreted as $\mathrm{mod}\, R(A)$, where $R(A)$ is a full subcategory of $\mathrm{mod}\, A$ formed by a fixed set of representatives of isomorphism classes of indecomposable projective A-modules.

For locally bounded categories we have the notions of representation types: finite, tame and wild; moreover, just as for algebras, the tame-wild dichotomy holds true (see [5, 17, 16]).

A specially important role in representation theory of algebras is played by algebras and categories of quivers with relations.

Let $Q = (Q_0, Q_1)$ be a quiver, where Q_0 is the set of vertices and Q_1 the set of arrows, together with the functions $s, t : Q_1 \to Q_0$ attaching to each arrow its source and sink, respectively. Recall that by a *walk* of length $n \geq 0$ in Q starting at x and ending at y, for $x, y \in Q_0$, we mean a sequence $w = \beta_1 \dots \beta_n$ consisting of arrows and their formal inverses such that $s(\beta_i) = x_{i-1}$ and $t(\beta_i) = x_i$ if $\beta_i \in Q_1$, and $s(\beta_i^{-1}) = x_i$ and $t(\beta_i^{-1}) = x_{i-1}$ if $\beta_i^{-1} \in Q_1$, for some $x_0, x_1, \dots, x_n \in Q_0$, $x_0 = x$, $x_n = y$. A trivial walk from x to x is always denoted by ε_x. For a walk w as above we set $Q_0(w) = \{x_0, \dots, x_n\}$ and $Q_1(w) := \{\beta_1, \dots, \beta_n, \beta_1^{-1}, \dots, \beta_n^{-1}\} \cap Q_1$. It is clear that $Q(w) := (Q_0(w), Q_1(w))$ is a subquiver of Q. We say that a walk $w = \beta_1 \dots \beta_n$ is *reduced* if $\beta_{i+1} \neq \beta_i^{-1}$ for every i, and is an (oriented) *path* if $\beta_i \in Q_1$ for every i. Note that if Q is a tree then for each reduced walk $w = \beta_1 \dots \beta_n$ the subquiver $Q(w)$ is a line of type $\mathbb{A}_{n+1}$. (A quiver $\mathcal{L}$ is called a *line* if its diagram, i.e. the associated unoriented graph, is of the type $\mathbb{A}_m$ for $m \in \mathbb{N}$, $\mathbb{A}_\infty$ or ${}_\infty\mathbb{A}_\infty$). The set of all walks (resp. paths, arrows) from x to y is denoted by $\mathcal{W}(x,y) = \mathcal{W}_Q(x,y)$ (resp. $\mathcal{P}(x,y) = \mathcal{P}_Q(x,y)$, $Q_1(x,y)$). For walks $w \in \beta_1 \dots \beta_n \in \mathcal{W}(x,y)$ and $w' = \beta'_1 \dots \beta'_m \in \mathcal{W}(y,z)$, the walk $ww' = \beta_1 \dots \beta_n \beta'_1 \dots \beta'_m \in \mathcal{W}(x,z)$ is called the composition of w and w'.

The *path algebra* of a finite quiver Q is the k-algebra

$$A(Q) = \Big(\bigoplus_{w\in\mathcal{P}} kw, \cdot \Big)$$

with unit $1 = \sum_{x\in Q_0} \varepsilon_x$, where $\mathcal{P} = \bigcup_{x,y\in Q_0} \mathcal{P}(x,y)$ and $\cdot$ is induced by composition of paths. If I is an admissible ideal in $A(Q)$, i.e. (Q, I) is a finite *bounded quiver*, then we set $A(Q,I) = A(Q)/I$. The algebra $A(Q,I)$ is finite-dimensional and called the *algebra of the bounded quiver* (Q,I). It is well known that $\mathrm{mod}\, A(Q) \simeq \mathrm{rep}_k(Q,I)$, where $\mathrm{rep}_k(Q,I)$ denotes the category of all finite-dimensional representations of the quiver Q over k, satisfying the relations from I (see [1]).

Similarly, given a quiver Q we construct the *path k-category* $R(Q)$ of Q. We set $\operatorname{ob} R(Q) = Q_0$ and $R(Q)(x,y) = \bigoplus_{w\in\mathcal{P}(y,x)} kw$ for $x, y \in Q_0$. The composition $\circ$ of morphisms in $R(Q)$ is again defined as the k-bilinear map given on the basis by composition of paths in Q. If I is an admissible ideal in the category $R(Q)$ (in the sense of [3]) then the factor category $R(Q,I) = R(Q)/I$ is locally bounded and called the *category of the bounded quiver* (Q,I). It is easy to check that as above we have $\operatorname{mod} R(Q) \simeq \operatorname{rep}_k(Q,I)$. It is easily seen that if Q is finite then $A(R(Q,I)) \cong A(Q,I')$ and $R(A(Q,I')) \cong R(Q,I)$, where $I' = \bigoplus_{x,y\in Q_0} I(x,y)$.

We will often not distinguish between the category and the algebra of a finite bounded quiver (Q,I).

Assume that (Q,I) is a bounded quiver such that Q is a tree. A reduced walk w in Q is called a *V-sequence* if w does not contain any subwalk u such that u or u^{-1} is a path $\alpha_1\ldots\alpha_n \in I$. Any V-sequence w defines an indecomposable module $N(w)$ over $R = R(Q,I)$. The module $N(w)$ as a representation of Q has one-dimensional spaces k at all vertices "visited" by w and zero spaces otherwise; the structure maps are given by identities for arrows belonging to w and zero maps otherwise. Note that $N(w) = N(w^{-1})$, and $N(w') \cong N(w)$ if and only if $w' = w$ or $w' = w^{-1}$.

1.2. Let R and R' be a pair of locally bounded k-categories. Recall from [3, 18] that a functor $F : R \to R'$ is called a *covering functor* if F is dense and for any $x \in \operatorname{ob} R$ and $a \in \operatorname{ob} R'$, F induces k-isomorphisms

$$(*)\qquad \bigoplus_{y\in F^{-1}(a)} R(x,y) \cong R'(F(x),a) \quad\text{and}\quad \bigoplus_{y\in F^{-1}(a)} R(y,x) \cong R'(a,F(x)).$$

Given a covering functor $F : R \to R'$ one can study interrelations between the module categories $\operatorname{MOD} R$ and $\operatorname{MOD} R'$ by using the pair of functors

$$\operatorname{MOD} R \underset{F_\bullet}{\overset{F_\lambda}{\rightleftarrows}} \operatorname{MOD} R'$$

where $F_\bullet : \operatorname{MOD} R' \to \operatorname{MOD} R$ is the pull-up functor associated with the functor F, assigning to each X in $\operatorname{MOD} R'$ the R-module $X\circ F$, and the push-down functor $F_\lambda : \operatorname{MOD} R \to \operatorname{MOD} R'$ is the left adjoint to $F_\bullet$ (see [21]). The R-module $F_\lambda(N)$, for N in $\operatorname{MOD} R$, is defined as follows: $F_\lambda(N)(a) = \bigoplus_{x\in F^{-1}(a)} N(x)$ for $a \in \operatorname{ob} R'$, and $F_\lambda(N)(\alpha) = [N({}_x\dot{\alpha}_y)] : \bigoplus_{x\in F^{-1}(a)} N(x) \to \bigoplus_{y\in F^{-1}(b)} N(y)$ for $\alpha \in R'(b,a)$, where $\sum_{y\in F^{-1}(b)} F({}_x\dot{\alpha}_y) = \alpha$ for $x \in F^{-1}(a)$. We also have the right adjoint $F_\rho : \operatorname{MOD} R \to \operatorname{MOD} R'$ to $F_\bullet$, where the R-module $F_\rho(N)$ is given by $F_\rho(N)(b) = \prod_{y\in F^{-1}(b)} N(y)$ for

$b \in \operatorname{ob} R'$, and $F_\rho(N)(\alpha) = [N({}_x\alpha_y^{\cdot})] : \prod_{x\in F^{-1}(a)} N(x) \to \prod_{y\in F^{-1}(b)} N(y)$ for $\alpha \in R'(b,a)$, where $\sum_{x\in F^{-1}(a)} F({}_x\alpha_y^{\cdot}) = \alpha$ for any $y \in F^{-1}(b)$.

Clearly, $F_\bullet(\operatorname{mod} R') \subset \operatorname{Mod} R$ and $F_\lambda(\operatorname{mod} R), F_\rho(\operatorname{mod} R) \subset \operatorname{mod} R'$.

Let $G \subseteq \operatorname{Aut}_{k\text{-cat}}(R)$ be a subgroup of the group $\operatorname{Aut}_{k\text{-cat}}(R)$ of all k-automorphisms of a locally bounded k-category R. Then G also acts on the category $\operatorname{MOD} R$ by translations ${}^g(-)$, which assign to each M in $\operatorname{MOD} R$ the R-module ${}^gM = M \circ g^{-1}$. Given M in $\operatorname{MOD} R$, we set $G_M = \{g \in G : {}^gM \simeq M\}$. We say that G *acts freely* on $\operatorname{ind} R$ if $G_M = \{\mathrm{id}_R\}$ for every indecomposable M from $\operatorname{mod} R$.

Assume now that G acts freely on objects of R (i.e. $G_x = \{\mathrm{id}_R\}$ for every $x \in \operatorname{ob} R$). Then the covering functor $F : R \to R'$ is called a *G-covering* if the set $F^{-1}(a)$ is G-invariant and the action of G on $F^{-1}(a)$ is transitive, for every $a \in \operatorname{ob} R'$. Note that in this situation we can identify $\operatorname{ob} R'$ with a fixed set $(\operatorname{ob} R)_0$ of representatives of G-orbits from $(\operatorname{ob} R)/G$. Moreover, the isomorphisms $(*)$ then have the form $\bigoplus_{g\in G} R(\bar{x}, g\bar{y}) \cong R'(\bar{x},\bar{y})$ and $\bigoplus_{g\in G} R(g\bar{y},\bar{x}) \cong R'(\bar{y},\bar{x})$, respectively, where $\bar{x},\bar{y} \in (\operatorname{ob} R)_0$. For any $g \in G$, we denote by $R'(\bar{x},\bar{y})_g$ the image of $R(\bar{x}, g^{-1}\bar{y})$ via the first of them.

Recall that for G as above there exists one distinguished G-covering functor. Namely, we can always form the quotient (orbit category) $\bar{R} = R/G$, which is again locally bounded (we set $\operatorname{ob}\bar{R} = (\operatorname{ob} R)/G$, and the morphism spaces are defined in terms of G-orbits of morphisms in R; see [3, 18] for the precise definition). Then the natural projection yields a G-covering functor $\bar{F} : R \to \bar{R}$ such that $\bar{F}g = \bar{F}$ for all $g \in G$, called a *Galois covering*.

Galois covering functors have nice properties and are well understood (see [3, 18]). In particular, the restrictions of the functors $\bar{F}_\lambda$ and $\bar{F}_\rho$ to $\operatorname{mod} R$ are naturally isomorphic; for any N in $\operatorname{MOD} R$ we have the natural isomorphisms $\bar{F}_\lambda(N) \cong \bar{F}_\lambda({}^gN)$ and $\bar{F}_\bullet\bar{F}_\lambda N \cong \bigoplus_{g\in G} {}^gN$; for any indecomposable N, N' in $\operatorname{mod} R$, $\bar{F}_\lambda(N) \cong \bar{F}_\lambda(N')$ yields an isomorphism $N' \cong {}^gN$ for some $g \in G$. Moreover, if G acts freely on indecomposables, i.e. $G_M = \{\mathrm{id}_R\}$ for all indecomposable objects M in $\operatorname{mod} R$ (always in case G is torsion free), then $\bar{F}_\lambda$ preserves indecomposability. There exist many results concerning nice behavior of Galois coverings with respect to preserving representation types in specific situations (see [18, 14, 11, 15], also [6, 7]).

1.3. The most significant example of a Galois covering functor is related to the combinatorial construction of the universal covering $(\tilde{Q},\tilde{I})$ and the fundamental group $\Pi_1(Q,I)$ of a quiver with relations (Q,I) given in [22]. This construction is based on the notion of a minimal relation.

Let (Q,I) be a bounded quiver, where $Q = (Q_0,Q_1)$ is a connected quiver and I an admissible ideal in the path k-category $R(Q)$ of Q. Recall that an element $\rho = \sum_{j=1}^m t^{(j)}\delta^{(j)} \in I(b,a)$, where $a,b \in Q_0$, $m \geq 2$ and $t^{(j)} \in k$, is called a *minimal linear relation* if $\sum_{j\in\Omega} t^{(j)}\delta^{(j)} \notin I(b,a)$ for any nonempty $\Omega \subsetneq \{1,\ldots,m_i\}$. Notice that the ideal I always has a set of generators consisting of zero relations and minimal relations.

Following the idea from [3] one can define $\Pi_1(Q,I) = \Pi_1((Q,I),a_0)$ and $\tilde{Q} = \tilde{Q}(a_0)$, for a fixed $a_0 \in Q_0$, as follows (see also [8] for details). Set $\mathcal{W} = \bigcup_{a,b\in Q_0} \mathcal{W}(a,b)$, $\mathcal{P} = \bigcup_{a,b\in Q_0} \mathcal{P}(a,b)$ and consider the equivalence relation $\sim\subseteq \mathcal{W}\times\mathcal{W}$ generated by the relations of the following two types:

$$\text{(i) } u\alpha\alpha^{-1}v \sim uv \quad \text{and} \quad \text{(ii) } u\delta v \sim u\delta' v,$$

where $u,v\in\mathcal{W}$, $\alpha\in Q_1$ or $\alpha^{-1}\in Q_1$, and $\delta,\delta'\in\mathcal{P}$ are such that $\delta=\delta_{j_1}$ and $\delta'=\delta_{j_2}$, with $1\le j_1,j_2\le m$, $j_1\ne j_2$, for some minimal linear relation $\rho\in I(b,a)$ as above. The relation $\sim$ depends only on the set of all minimal linear relations in I and it is a congruence with respect to composition of walks. Then the set $\Pi_1(Q,I) := \mathcal{W}(a_0,a_0)/\sim$ carries the structure of a group. We define a quiver $\tilde{Q}$ similarly, setting

$$\tilde{Q}_0 := \Big(\bigcup_{a\in Q_0} \mathcal{W}_Q(a_0,a)\Big)/\sim$$

and

$$\tilde{Q}_1([v_a],[v_b]) := \{([v_a],\alpha) : \alpha\in Q_1(a,b),\ v_a\alpha\sim v_b\}$$

for any $[v_a]\in\mathcal{W}_Q(a_0,a)/\sim$, $[v_b]\in\mathcal{W}_Q(a_0,b)/\sim$, where $[v]:=[v]_\sim$ for $v\in\mathcal{W}$. (Then $\mathcal{P}_{\tilde{Q}}([v_a],[v_b]) = \{([v_a],\delta) : \delta\in\mathcal{P}_Q(a,b),\ v_a\delta\sim v_b\}$.) Notice that $\tilde{Q}_1([v_a],[v_b])$ and $\mathcal{P}_{\tilde{Q}}([v_a],[v_b])$ can be alternatively written in the form $\tilde{Q}_1([v_a],[v_b]) = \{(\alpha,[v_b]) : \alpha\in Q_1(a,b), v_a\alpha\sim v_b\})$ and $\mathcal{P}_{\tilde{Q}}([v_a],[v_b]) = \{(\delta,[v_b]) : \delta\in\mathcal{P}_Q(a,b),\ v_a\delta\sim v_b\}$. The pairs $([v_a],\delta)$ and $(\delta,[v_b])$ represent the same path $\tilde{\delta}$ in $\tilde{Q}$, which depending on the presentation is called a *lifting* of δ to $\tilde{Q}$ starting at $[v_a]$, respectively, ending at $[v_b]$.

It is clear that $\Pi_1(Q,I)$ acts on $\tilde{Q}$ by quiver automorphisms defined by composition of appropriate walks, and that the mapping $[v_a]_\sim \mapsto a$, $v_a\in\mathcal{W}_Q(a_0,a)$, yields a quiver morphism $p:\tilde{Q}\to Q$. The map p is a Galois covering of quivers with group $\Pi_1(Q,I)$. In fact $p:(\tilde{Q},\tilde{I})\to(Q,I)$ is a morphism of bounded quivers, where $\tilde{I}$ by definition is the ideal generated by liftings of all minimal and zero relations in I to $\tilde{Q}$. Moreover, the functor $\bar{F}: R(\tilde{Q},\tilde{I})\to R(Q,I)$ induced by p is a Galois covering functor with group $\Pi_1(Q,I)$.

We finish this section with some practical remark, which is useful when one wants to compute the universal covering $(\tilde{Q},\tilde{I})$ for a concrete finite bounded quiver (Q,I). The admissible ideal I in $R(Q)$ is usually given in the form $I=\langle\rho_l : l\in\Lambda\rangle$, where all ρ_l are elements of the spaces $I(b,a)$, $a,b\in Q_0$, and Λ is a finite set, in case Q is finite. Without loss of generality one can always assume that each ρ_l is either zero or a minimal linear relation.

Then it is easily seen that the equivalence relation $\sim$ is generated by all relations of type (i) and only those of type (ii) for which δ and δ' appear in some minimal linear relation ρ_l, $l \in \Lambda$.

1.4. Recently in [8], a certain class of G-coverings was introduced and investigated, more general than the Galois coverings. They are called almost Galois coverings and seem to be important in the context of a better understanding of the behavior of nonstandard algebras. Below we present a little different definition of them, which is equivalent to but more handy than the original one (cf. [8, Definition 2.2.1], [9, 3.1]).

Let $G \subseteq \mathrm{Aut}_{k\text{-cat}}(R)$ be a group of k-automorphisms acting freely on $\mathrm{ob}\, R$, and $(\mathrm{ob}\, R)_0$ a fixed set of representatives of G-orbits in $\mathrm{ob}\, R$. If $p : G \to L$ is a surjective group homomorphism then the set $(\mathrm{ob}\, R)_0$ yields a G-invariant L-grading $|-|$ on the morphisms of R: for $\sigma \in R(g_1\bar{x}, g_2\bar{y})$ with $\bar{x}, \bar{y} \in (\mathrm{ob}\, R)_0$ we set $|\sigma| := p(g_2^{-1}g_1)$. Clearly, $|\tau\sigma| = |\tau| \cdot |\sigma|$ and $|g(\sigma)| = |\sigma|$ for all $\sigma \in R(x, y)$, $\tau \in R(y, z)$ and $g \in G$. Notice that we always have a G-grading of $\bar{R}$ defined by the decompositions $\bar{R}(\bar{x}, \bar{y}) = \bigoplus_{g\in G} \bar{R}(\bar{x}, \bar{y})_g$, for $\bar{x}, \bar{y} \in (\mathrm{ob}\, R)_0$; moreover, this grading is compatible via $\bar{F}$ and p with $|(-)|$.

Note that if $L = (L, \leq)$ is an ordered group (so torsion-free) then the homomorphism p induces on G the canonical structure of an ordered group $(G, \preceq)$, where $g_1 \prec g_2$ if and only if $p(g_1) < p(g_2)$, for $g_1, g_2 \in G$. We usually assume that $L = (L, \leq)$ is a totally ordered abelian (torsion-free) group and then we say that $G = (G, \preceq)$ is *L-totally ordered*. In case $L = (\mathbb{Z}, \leq)$, $\leq$ is the standard order in $\mathbb{Z}$. Notice that each group G is always $\{0\}$-totally ordered, and if G is a free (resp. an abelian free) group then it is $\mathbb{Z}$-totally ordered in a canonical way since G admits a standard homomorphism $p : G \to \mathbb{Z}$ (free generators are mapped to 1).

Let $F : R \to R'$ be a G-covering functor. For any $x \in \mathrm{ob}\, R$, $b \in \mathrm{ob}\, R'$ and $g \in G$, we denote by ${}_b\phi^{(x,g)}$ the k-isomorphism

$${}_b\phi^{(x,g)} := ({}_bf^x)^{-1} \circ {}_bf^{gx} : \bigoplus_{y\in F^{-1}(b)} R(gx, y) \to \bigoplus_{y'\in F^{-1}(b)} R(x, y').$$

F is said to be an *almost Galois covering* of type L (with group G) if G admits a surjective homomorphism $p : G \to L$, where L is a totally ordered group as above, such that for any $\sigma \in R(gx, y)$ the homogeneous coordinate presentation

$$(*) \qquad {}_b\phi^{(x,g)}(\sigma) \ (= ({}_bf^x)^{-1}(F(\sigma))) = \sum_{y'} \tau_{y'}$$

has the following shape: $\tau_{g^{-1}y} = g^{-1}(\sigma)$ and $|\tau_{y'}| > |\sigma|$ if $\tau_{y'} \neq 0$ for all $y' \in F^{-1}(b) \setminus \{g^{-1}y\}$, where $\tau_{y'} \in R(x, y')$ (in particular, $\tau_{hg^{-1}y} = 0$ for each nontrivial $h \in \ker p$). In the above we identify an element in $R(x, z)$ with its image under the canonical inclusion in $\bigoplus_{z\in F^{-1}(b)} R(x, z)$.

Note that we can briefly rephrase the condition above in the form

$$(**)\qquad {}_b\phi^{(x,g)}(\sigma) = g^{-1}(\sigma) + \sum_{l>|\sigma|} \tau_l$$

where $\tau_l = \sum_{|\tau_{y'}|=l} \tau_{y'}$ for $l \in L$. It is easily seen that this can be required only for $x = \bar{x} \in (\operatorname{ob} R)_0$. One can show that the definition does not depend on the choice of the set $(\operatorname{ob} R)_0$.

In case $L = \mathbb{Z}$ (resp. $L = \{0\}$, L is abelian), F is called an almost Galois G-covering functor of *integral* (resp. *trivial*, *abelian*) *type*. It is easily seen that the class of almost Galois G-coverings of trivial type coincides exactly with the class of Galois covering functors with the group G.

Recall that due to [8, Theorem 3.1.1(b)] and [10, Corollary 3.4], the push-down functors for almost Galois coverings of integral type behave similarly to those for Galois coverings. Namely, if $F : R \to R'$ is an almost Galois G-covering functor of abelian type then (i) $F_\bullet F_\lambda(N) \cong \bigoplus_{g\in G} {}^gN$ for any indecomposable N in $\operatorname{mod} R$ satisfying $\operatorname{Ext}^1_R(N, {}^hN) = 0$ for all $h \in G^{\prec e}$, where $\prec$ is defined as above. If F is of integral type then for such N also (ii) $F_\lambda({}^gN) \cong F_\lambda(N)$ for every $g \in G$. Notice that in contrast to the Galois covering case these isomorphisms are not necessarily natural. Observe also that the Ext-vanishing condition holds automatically for N if $\operatorname{supp} {}^hN \cap \widehat{\operatorname{supp} N} = \emptyset$ for all $h \neq e$. In particular, if R is locally support-finite it holds for all N in $\operatorname{ind} R$, provided the group G is small enough in the sense that for every $x \in \operatorname{ob} R$, $hR(x) \cap \widehat{R(x)} = \emptyset$ for all $h \neq e$.

In this paper we will make a deeper analysis of the isomorphism (i) in a more specific situation (see 3.2).

Finally recall that a nice behavior and properties of the class of almost Galois coverings of integral type are partially connected with the notion of degeneration of functors introduced in [10, Definition 2.4]. More precisely, they follow from [10, Theorem 2.6], which says that any such covering functor admits a degeneration of the best possible kind to the canonical Galois covering functor $\bar{F} : R \to \bar{R} = R/G$ associated with an action of G on R.

2. The representation-finite case

2.1. The main aim of this section is to prove the following result.

THEOREM (A). *Let G be a group, Q a connected quiver equipped with the action $\cdot : G \times Q \to Q$ of G, which is free on Q_0, $I \lhd R(Q)$ an admissible G-invariant ideal and $F : R \to R'$ a G-covering functor with respect to the induced action of G on R, where $R := R(Q, I)$. Assume that the following properties are satisfied:*

- *R' is square-free,*

- *I is homogeneous with respect to the path length $\mathbb{Z}$-grading on $R(Q)$,*
- *$\Pi(Q, I) \cong \{1\}$.*

Then there exists an action $\star : G \times R \to R$ which coincides with $\cdot$ on $\operatorname{ob} R$ $(= Q_0)$ *and has the following properties:*

(a) *F is an almost Galois G-covering of integral or trivial type with respect to $\star$.*

(b) *${}^{g\star}N \cong {}^{g}N$ for any N in* $\operatorname{mod} R$ *and $g \in G$, where ${}^{g\star}N := N \circ g^{-1}\star(-)$ and ${}^{g}N := N \circ g^{-1}\cdot(-)$.*

Consequently, for N in $\operatorname{ind} R$ *we have $F_\bullet F_\lambda(N) \cong \bigoplus_{g\in G} {}^{g}N$ and $F_\lambda({}^{g}N) \cong F_\lambda(N)$ for all $g \in G$, provided N satisfies the condition* $\operatorname{Ext}^1_R(N, {}^{h}N) = 0$ *for all $h \in G^{\prec e}$, where $\prec$ is defined as in* 1.4.

The proof of Theorem (A) needs some preparations and will be given in 2.6. It intensively uses the notion of an abstract grading which we introduce in the next subsection. However, first we discuss certain important consequences of our result for the case of representation-finite algebras.

Let A be a representation-finite locally bounded k-category. We denote by $(\operatorname{ind} A)_0$ the full subcategory of $\operatorname{ind} A$ formed by a fixed selection of representatives of isoclasses containing all projectives of the form $P_x := A(-, x)$ for $x \in \operatorname{ob} A$, by $\Pi(\Gamma_A)$ the fundamental group of Γ_A and by $k(\tilde{\Gamma}_A)$ the mesh category of the universal covering $\tilde{\Gamma}_A$ of Γ_A, where Γ_A is regarded as a translation quiver. It is well known that there exists a classical $\Pi(\Gamma_A)$-covering functor $F_A : k(\tilde{\Gamma}_A) \to (\operatorname{ind} A)_0$ (see [3] for the definitions and the construction of F_A). Then the universal cover $\tilde{A}$ of A is the full subcategory of $k(\tilde{\Gamma}_A)$ formed by the union of the sets $F_A^{-1}(P_x)$ for $x \in \operatorname{ob} A$ (see [18]).

Theorem (B). *Let A be a category as above. Assume that A is finite.*

(a) *$F_A : k(\tilde{\Gamma}_A) \to (\operatorname{ind} A)_0$ is an almost Galois $\Pi(\Gamma_A)$-covering of integral or trivial type with respect to some action $\star$ of $\Pi(\Gamma_A)$ on $k(\tilde{\Gamma}_A)$, which coincides on objects with the canonical action $\circ : \Pi(\Gamma_A) \times \tilde{\Gamma}_A \to \tilde{\Gamma}_A$ given by composition of paths.*

(b) *A admits an almost Galois G-covering $F : R \to A$ of integral or trivial type, where G is a free group and R is a simply connected representation-finite locally bounded k-category equipped with an action of G which is free on* $\operatorname{ob} R$. *More precisely, for R we can take the common universal cover $\tilde{A}$ of A and of the standard form $\bar{A}$ of A, for G the fundamental group $\Pi(\Gamma_A)$, and then the functor $F : \tilde{A} \to A$ admits a degeneration in the sense of* [10] *to the canonical Galois covering functor $\bar{F} : \tilde{A} \to \bar{A} := \tilde{A}/\Pi(\Gamma_A)$ given by the projection with respect to restriction of the action $\circ$.*

The proof of Theorem (B) will be given in 2.7.

2.2. Let S be a set and L a group. A function $d : S^2 \to L$ is called an *L-grading* (on S) if

$$(\bullet) \qquad d(y,z)d(x,y) = d(x,z) \quad \text{for all } x,y,z \in S.$$

Assume that $\cdot : G \times S \to S$ is an action of a group G on S. We say that $d : S^2 \to L$ is a *G-invariant L-grading* (on S) if d is an L-grading such that

$$(\bullet\bullet) \qquad d(gx,gy) = d(x,y) \quad \text{for all } x,y \in S \text{ and } g \in G.$$

Clearly, L-gradings on S are a special case of G-invariant L-gradings, for G being the trivial group.

Below we formulate straightforward properties of the notions introduced.

LEMMA. *Let d be a G-invariant L-grading on S, where G, S, L are as above.*

(a) *$d(x,x) = e_L$ for any $x \in S$.*
(b) *For any $x \in S$, the map $p_x = p_x^{(d)} : G \to L$ given by $p_x(g) := d(x, g^{-1}x)$ for $g \in G$ is a group homomorphism. Moreover, for any $x, y \in S$ we have $p_y = d(x,y)\, p_x(-)\, d(x,y)^{-1}$; in particular, $p_x = p_y$ if L is abelian.*
(c) *$d(g_1x, g_2y) = d(x,y)\, p_x(g_2^{-1}g_1) = p_y(g_2^{-1}g_1)\, d(x,y)$ and $d(g_1x, g_2y) = d(z,y)\, p_z(g_2^{-1}g_1)\, d(z,x)^{-1}$ for all $x,y,z \in S$ and $g_1, g_2 \in G$.*

REMARK. (i) If p_{x_0} is not surjective for some $x_0 \in S$ (equivalently, p_x is not surjective for every $x \in S$) then $d(S^2)$ is not necessarily contained in $L_0 := p_{x_0}(G)$ (clearly $d(S^2) = L_0$ in the opposite situation).

(ii) If $G \subseteq \mathrm{Aut}_{k\text{-cat}}(R)$ is a subgroup of the group $\mathrm{Aut}_{k\text{-cat}}(R)$ then any G-invariant L-grading $d : (\operatorname{ob} R)^2 \to L$ of the set $\operatorname{ob} R$ yields a G-invariant homogeneous L-grading of R: $R(x,y) = \bigoplus_{l \in L} R_l(x,y)$ for $x, y \in \operatorname{ob} R$, where $R_l(x,y) = R(x,y)$ if $l = d(x,y)$, and $R_l(x,y) = 0$ otherwise. (Clearly, not conversely, since by definition a homogeneous L-grading of R is a function $d : \{(x,y) \in (\operatorname{ob} R)^2 : R(x,y) \neq 0\} \to L$ satisfying the equality in $(\bullet)$ only for the triples (x,y,z) such that $R(y,z) \cdot R(x,y) \neq 0$).

EXAMPLE. (a) Let $\Delta = (\Delta_0, \Delta_1)$ be a quiver. Then for any walk (i.e. unoriented path) $w = \alpha_1^{\epsilon_1} \ldots \alpha_n^{\epsilon_n}$, where $\alpha_1, \ldots, \alpha_n \in \Delta_1$ and $\epsilon_1, \ldots, \epsilon_n \in \{\pm 1\}$, we denote by $\ell(w)$ its "oriented length"

$$\ell(w) := \epsilon_1 + \cdots + \epsilon_n.$$

Clearly, if w' is a walk such that $s(w') = t(w)$ then $\ell(ww') = \ell(w) + \ell(w')$.

Assume now that Δ is a connected and simply-connected quiver equipped with the action of the group G which is free on Δ_0. Then the map

$$d : (\Delta_0)^2 \to \mathbb{Z}$$

defined by setting

$$d(x,y) = \ell(w)$$

where w is the unique walk with the starting point $s(w) = y$ and the ending point $t(w) = x$, is a G-invariant L-grading on Δ_0, with $L := \mathbb{Z}$.

(b) If $\Delta = \tilde{Q}$ and $G = \Pi(Q)$, where $\tilde{Q}$ is the universal cover of the quiver Q, and $\Pi(Q)$ the fundamental group of Q, then G acts in a standard way on Δ. Denote by $Q^{(0)}$ the Kronecker quiver and by $Q^{(m)}$, for $m \geq 1$, the oriented cycle with m vertices. Then for $Q = Q^{(m)}$ with $m \in \mathbb{N}$ and the G-invariant grading d defined as above we have

$$\operatorname{Im} p_x^{(d)} = m\mathbb{Z} \quad \text{for every } x \in \Delta_0,$$

and clearly $d((\Delta_0)^2) \nsubseteq L_0$.

From now on we assume that the action $\cdot$ is free. Let $\bar{S}$ be a fixed set of representatives of G-orbits in S. For any $x \in S$ we denote by $\bar{x}$ the unique element in $\bar{S}$ such that $x \in G\bar{x}$.

Corollary. *Let G, S, L and $\bar{S}$ be as above. Then for a fixed $\bar{x}_0 \in \bar{S}$, the mapping $d \mapsto d_| := (p_{\bar{x}_0}, \{d(\bar{x}_0, \bar{x})\}_{\bar{x} \in \bar{S}\setminus\{\bar{x}_0\}})$ yields a bijection between the set of all G-invariant L-gradings on S and the set* $\operatorname{Hom}(G, L) \times L^{\bar{S}\setminus\{\bar{x}_0\}}$.

Proof. By (b) of the Lemma, the map is well defined. Given a collection $\partial = (p, \{\partial_{\bar{x}}\}) \in \operatorname{Hom}(G, L) \times L^{\bar{S}\setminus\{\bar{x}_0\}}$, it is easy to show that $\tilde{\partial} : S^2 \to L$ defined by setting

$$\tilde{\partial}(x, y) := \partial_{\bar{y}}\, p(g_2^{-1} g_1)\, \partial_{\bar{x}}^{-1}$$

for $(x, y) \in S^2$, where $x = g_1\bar{x}$, $y = g_2\bar{y}$ and $\partial_{\bar{x}_0} = e$, is a G-invariant L-grading of S. The mapping $\partial \mapsto \tilde{\partial}$ defines an inverse to $d \mapsto d_|$. ∎

2.3. Now we consider the case $L = G$. (We then use the abbreviated name *G-grading* of S.) We say that a G-grading $d : S^2 \to G$ is *trivial* if there exists a collection $\bar{G} = \{g_{\bar{x}}\}_{\bar{x} \in \bar{S}} \subset G$ such that

$$(*) \qquad d(x, y) = g_{\bar{y}} g_2^{-1} g_1 g_{\bar{x}}^{-1}$$

for all $(x, y) \in S^2$, where $x = g_1\bar{x}$, $y = g_2\bar{y}$.

Lemma.

(a) *For a subset $\bar{G} = \{g_{\bar{x}}\}_{\bar{x} \in \bar{S}} \subset G$, the function $d = d^{(\bar{G})} : S^2 \to G$ defined above is a G-grading of S such that $p_x := p_x^{(d)} \in \operatorname{Aut}_0(G)$ for every $x \in S$, where $\operatorname{Aut}_0(G)$ denotes the group of all inner automorphisms of G, and $p_{g_{\bar{x}}\bar{x}} = \mathrm{id}_G$ for every $\bar{x} \in \bar{S}$ (in particular, $p_x = \mathrm{id}_G$ for every $x \in S$ if G is abelian). Moreover, if $\bar{G}' = \{g'_{\bar{x}}\}_{\bar{x} \in \bar{S}} \subset G$ is another collection then $d^{(\bar{G})} = d^{(\bar{G}')}$ if and only if there exists $h \in \mathrm{Z}(G)$ such that $g_{\bar{x}} = hg'_{\bar{x}}$ for all $\bar{x} \in \bar{S}$.*

(b) *The definition does not depend on the choice of $\bar{S}$, and the G-grading $d : S^2 \to G$ is trivial if and only if there exists a subset $\bar{S}' = \{\bar{x}'\}_{x \in S}$ of representatives of G-orbits in S such that $d(x, y) = g_2^{-1} g_1$ for all*

$(x,y) \in S^2$, *where* $g_1, g_2 \in G$ *satisfy* $x = g_1\bar{x}'$, $y = g_2\bar{y}'$ (*then clearly* $p_{\bar{x}'} = \mathrm{id}_G$ *for every* $\bar{x}' \in \bar{S}'$).

(c) *For a fixed* $\bar{x}_0 \in \bar{S}$ *the mapping* $d \mapsto d_|$ *as in Corollary* 2.2 *yields a bijection between the set of all trivial* G*-gradings on* S *and the subset* $\mathrm{Aut}_0(G) \times G^{\bar{S}\setminus\{\bar{x}_0\}} \subseteq \mathrm{Hom}(G,G) \times G^{\bar{S}\setminus\{\bar{x}_0\}}$.

Proof. An easy check on definitions. ∎

REMARK. (i) If $d : S^2 \to G$ is a G-grading and $\pi : G \to L$ is a group homomorphism then $d' := \pi d : S^2 \to L$ is a G-invariant L-grading of S. In case d is a trivial grading, d' can be regarded as an L_0-grading for $L_0 := p'_{\bar{x}_0}(G)$ $(= \pi(G))$, where $x_0 \in S$ is arbitrarily fixed and $p'_x := p_x^{(d')}$; moreover, $p'_x = \pi$ for every $x \in \bar{S}'$, where $\bar{S}' = \{g_{\bar{x}}\bar{x} : \bar{x} \in \bar{S}\}$ (cf. (b)).

(ii) The grading $|-|$ of morphisms in the definition of an almost Galois covering is in fact given by the composition $d' := p\bar{d}$ of a homomorphism $p : G \to L$ and the trivial grading $\bar{d} = d^{(\bar{G})} : (\mathrm{ob}\, R)^2 \to G$ defined by the collection $\bar{G}$ whose members are all equal to the unit $e \in G$, for a fixed set $(\mathrm{ob}\, R)_0$ (cf. Remark 2.2(ii)). More precisely, for $\sigma \in R(x,y)$ we have $|\sigma| = d'(x,y)$ $(= p(g_2^{-1}g_1))$, where $x = g_1\bar{x}$ and $y = g_2\bar{y}$ with $\bar{x}, \bar{y} \in (\mathrm{ob}\, R)_0$.

The following result will play an important role in further considerations.

PROPOSITION. *Let* $L = (L, <)$ *be an ordered group,* $d : S^2 \to L$ *a* G*-invariant* L*-grading of* S, $\bar{d} : S^2 \to G$ *a trivial* G*-grading of* S *and* $d' := p\bar{d} : S^2 \to L_0$ *the induced* G*-invariant* L_0*-grading, where* $p := p_{x_0}^{(d)}$ *for a fixed* $x_0 \in S$ *and* $L_0 := p(G)$ $(= p_x^{(d')}(G)$ *for all* $x \in S)$. *Then for any* $x = g_1\bar{x}$, $x' = g_1'\bar{x} \in G\bar{x}$ *and* $y = g_2\bar{y}$, $y' = g_2'\bar{y} \in G\bar{y}$ *the following conditions are equivalent:*

- $d(x,y) < d(x',y')$,
- $d'(x,y) < d'(x',y')$,
- $p(g_2^{-1}g_1) < p(g_2'^{-1}g_1')$.

Proof. The assertion follows immediately from the formulas

$$d(x,y) = d(x_0,\bar{y})\, p(g_2^{-1}g_1)\, d(x_0,\bar{x})^{-1},$$
$$d'(x,y) = p(g_{\bar{y}})\, p(g_2^{-1}g_1)\, p(g_{\bar{x}}^{-1}),$$

and the analogous one for the pair x', y' (see Lemma 2.2(c), $(*)$ and the definition of d'). ∎

2.3. Now we briefly discuss a functorial description of k^*-gradings, which we use in the proof of Theorem 2.1(A).

LEMMA. *Let* R *be a locally bounded* k*-category.*

(a) *The mapping* $\alpha \mapsto u(x,y) \cdot \alpha$, *for* $\alpha \in R(x,y)$ *and* $x, y \in \mathrm{ob}\, R$, *given by a collection* $u' = (u(x,y))$ *of nonzero scalars indexed by the set*

$\{(x,y)\in(\operatorname{ob}R)^2 : R(x,y)\neq 0\}$ defines an automorphism of R if and only if $u(y,z)u(x,y) = u(x,z)$ whenever $R(y,z)\cdot R(x,y)\neq 0$. In particular, each k^*-grading $u:(\operatorname{ob}R)^2\to k^*$ yields an automorphism $\varphi^u : R\to R$, given by the formula above.

(b) *If $\varphi = \varphi^u : R\to R$ is the automorphism given by a k^*-grading $u:(\operatorname{ob}R)^2\to k^*$ then the autoequivalence ${}^{\varphi}(-) : \operatorname{MOD}R\to\operatorname{MOD}R$, attaching to an R-module N the module ${}^{\varphi}N := N\circ\varphi^{-1}$, is isomorphic to the functor $\operatorname{Id}_{\operatorname{MOD}R}$.*

Proof. (a) An easy check on definitions.

(b) Fix $x\in\operatorname{ob}R$. For N in $\operatorname{MOD}R$ we define a map $\xi_N : {}^{\varphi}N\to N$ by setting

$$\xi_N(y) := u(x,y)^{-1}\operatorname{id}_{N(y)}$$

for $y\in\operatorname{ob}R$. Note that ξ_N is an R-homomorphism, since for any $\alpha\in R(y,z)$ we have

$$\begin{aligned}\xi_N(y)\circ{}^{\varphi}N(\alpha) &= (u(x,y)^{-1}\operatorname{id}_{N(y)})\circ(u(y,z)^{-1}N(\alpha))\\ &= (u(x,y)^{-1}\,u(y,z)^{-1})N(\alpha)\\ &= u(x,z)^{-1}\,N(\alpha) = N(\alpha)\circ(u(x,z)^{-1}\operatorname{id}_{N(z)})\\ &= N(\alpha)\circ\xi_N(z).\end{aligned}$$

It is clear that the homomorphisms ξ_N, for N in $\operatorname{MOD}R$, yield a natural family of isomorphisms, and hence the required isomorphism of functors. ∎

2.5. In the proof of Theorem 2.1(A) we will also use the notion of minimal (linear) relations of some special kind (cf. 1.3).

Let (Q,I) be a bounded quiver, where $Q=(Q_0,Q_1)$ is a quiver and I an admissible ideal in the path k-category $R(Q)$ of Q. A nonzero element $\rho=\sum_{j=1}^m t^{(j)}\delta^{(j)}\in I(x,y)$, where $x,y\in Q_0$, $m\geq 2$ and $t^{(j)}\in k^*$, is called a *strictly minimal* (linear) *relation* in I if $\sum_{j\in\Omega}s^{(j)}\delta^{(j)}\notin I(x,y)$ for any nonempty $\Omega\subsetneq\{1,\dots,m\}$ and $(s^{(j)})\in(k^*)^{\Omega}$ (in particular, $\delta^{(j)}\notin I(x,y)$ for every $j\in[m]$). Clearly, a strictly minimal relation is minimal.

Remark. If $\rho=\sum_{j=1}^m t^{(j)}\delta^{(j)}\in I(x,y)$ is a strictly minimal linear relation and $\rho' := \sum_{j=1}^m s^{(j)}\delta^{(j)}$ belongs to $I(x,y)$, where $(0)\neq(s^{(j)})\in k^{[m]}$, then $(s^{(j)})\in(k^*)^{[m]}$ and $\rho'=c\cdot\rho$ for some $c\in k^*$, i.e. $(t^{(j)})$ and $(s^{(j)})$ define the same point in $\mathbb{P}^{m-1}(k)$. Indeed, we can assume that $s^{(m)}\neq 0$ and then $\rho'' := \rho-(t^{(m)}/s^{(m)})\rho'\in I(x,y)\cap(\sum_{j=1}^{m-1}k\delta^{(j)})$, so $\rho''=0$.

Lemma.

(a) *The ideal I is generated by all zero relations and strictly minimal relations in I.*

(b) *For any* $w, w' \in \mathcal{W}(y,x)$, *we have* $w \sim w'$ *if and only if there exist* $w_0 = w, w_1, \dots, w_r = w' \in \mathcal{W}(y,x)$ *and strongly minimal relations* $\rho_1 = \sum_{j=1}^{m_1} t_1^{(j)} \delta_1^{(j)} \in I(z_1', z_1), \dots, \rho_r = \sum_{j=1}^{m_r} t_r^{(j)} \delta_r^{(j)} \in I(z_r', z_r)$, *with* $t_i^{(j)} \in k^*$, *such that* $w_i = u_i \delta_i^{(1)} u_i'$ *and* $w_{i+1} = u_i \delta_i^{(2)} u_i'$ *for every* $i \in [r]$, *where* $u_i \in \mathcal{W}(y, z_i)$, $u_i' \in \mathcal{W}(z_i', x)$ *are some walks.*

Proof. We start by fixing some notation. For $\sigma := \sum_{\delta \in \mathcal{P}(y,x)} t^{(\delta)} \delta \in R(Q)$, where $x, y \in Q_0$ and $t^{(\delta)} \in k$, we denote by $\Omega(\sigma)$ the *path support* of σ, which is by definition the set $\{\delta \in \mathcal{P}(y,x) : t^{(\delta)} \in k^*\}$. We claim that each minimal relation $\rho := \sum_{j=1}^{m} t^{(j)} \delta^{(j)} \in I(x,y)$ is a linear combination of strongly minimal relations $\sigma_1, \dots, \sigma_p \in I(x,y)$ such that $\Omega(\sigma_1), \dots, \Omega(\sigma_p) \subseteq \Omega(\rho) = \{\delta^{(1)}, \dots, \delta^{(m)}\}$. Notice that I is generated by all zero relations and minimal relations in I, so by the observation after the definition of strongly minimal relation, our claim immediately implies (a).

To prove the claim observe that for any $\rho = \sum_{j=1}^{m} t^{(j)} \delta^{(j)} \in I(x,y)$ with $m \geq 2$ and $t^{(j)} \in k^*$ such that $\delta^{(j)} \notin I(x,y)$ for every j, we have the following: either ρ is a strongly minimal relation, or there exists a nonempty subset $\Omega \subsetneq \{1, \dots, m\}$ and a strongly minimal relation $\sigma = \sum_{j \in \Omega} s^{(j)} \delta^{(j)} \in I(x,y)$ with $\Omega(\sigma) = \{\delta^{(j)} : j \in \Omega\}$. Moreover, in the latter case there exists $c \in k^*$ such that that $\Omega(\rho') \subsetneq \Omega(\rho)$ for $\rho' := \rho - c\sigma \in I(x,y)$. (Set $c := t^{(j)}/s^{(j)}$ for a fixed $j \in \Omega$.) Note that $|\Omega(\rho')| \geq 2$ by our assumption on ρ.

Now starting with the minimal relation ρ and applying inductively the procedure above we immediately obtain a presentation of ρ as a linear combination of strongly minimal relations, and thus assertion (a) is proved.

It is easily seen that to prove (b) it suffices to show that if $\rho := \sum_{j=1}^{m} t^{(j)} \delta^{(j)} \in I(x,y)$, with $m \geq 2$ and $t^{(j)} \in k^*$, is a minimal relation then for any pair $j, j' \in [m]$ there exist $\delta_0 = \delta^{(j)}, \delta_1, \dots, \delta_{r'} = \delta^{(j')} \in \mathcal{P}(y,x)$ such that for every $i \in [r']$ the paths δ_{i-1}, δ_i are distinct and belong to $\Omega(\sigma_{l(i)})$ for some $l(i) \in [p]$, where $\sigma_1, \dots, \sigma_p$ are strongly minimal relations defined for ρ as above.

It is clear that $\Omega(\sigma_1) \cup \dots \cup \Omega(\sigma_p) = \Omega(\rho)$. Now, for any sequence $\underline{i} : 1 \leq i_1 < \dots < i_{p'} \leq p$, where $p' \geq 1$, we set $\Omega'(\underline{i}) := \bigcup_{i \in \{i_1, \dots, i_{p'}\}} \Omega(\sigma_i)$ and $\Omega''(\underline{i}) := \bigcup_{i \in [p] \setminus \{i_1, \dots, i_{p'}\}} \Omega(\sigma_i)$. We will show that if $\Omega'(\underline{i}) \cap \Omega''(\underline{i}) = \emptyset$ then $p' = p$ (i.e. $\{i_1, \dots, i_{p'}\} = [p]$); hence, $\Omega'(\underline{i}) = \Omega(\rho)$ and $\Omega''(\underline{i}) = \emptyset$. Indeed, we have $\rho = \rho' + \rho''$ for $\rho' := \sum_{i \in \{i_1, \dots, i_{p'}\}} c_i \sigma_i$ and $\rho'' := \sum_{i \in [p] \setminus \{i_1, \dots, i_{p'}\}} c_i \sigma_i$, where $\rho = \sum_{i=1}^{p} c_i \sigma_i$ with $c_i \in k$. As $\Omega(\rho') \subseteq \Omega' := \Omega'(\underline{i})$ and $\Omega(\rho'') \subseteq \Omega'' := \Omega''(\underline{i})$, we have $\rho' = \sum_{j:\, \delta^{(j)} \in \Omega'} t^{(j)} \delta^{(j)} \in I(x,y)$, $\rho'' = \sum_{j: \delta^{(j)} \in \Omega''} t^{(j)} \delta^{(j)} \in I(x,y)$ by the disjointness of Ω' and Ω''. On the other hand Ω' is nonempty, so from minimality of ρ it follows that $\Omega' = \Omega(\rho)$, $\Omega'' = \emptyset$ and $p' = p$.

As a consequence, for any $i, i' \in [p]$, $i \neq i'$, there exists a sequence $i_1 = i, \ldots, i_{r''} = i' \in [p]$, where $r'' \geq 1$, such that $\Omega(\sigma_{i_1}) \cap \Omega(\sigma_{i_2}), \ldots, \Omega(\sigma_{i_{r''-1}}) \cap \Omega(\sigma_{i_{r''}}) \neq \emptyset$. Hence, for any $j, j' \in [m]$, $j \neq j'$, there exist $\delta_0 = \delta^{(j)}, \delta_1, \ldots, \delta_{r'} = \delta^{(j')} \in \mathcal{P}(y, x)$, $r' \geq 1$, such that for every $i \in [r']$ the paths δ_{i-1}, δ_i belong $\Omega(\sigma_{l(i)})$ for some $l(i) \in [p]$. If now $\delta_{i-1} = \delta_i$ for some i, then we can always remove one of these two paths from the sequence. Consequently, the sequence of minimal length, among all sequences with fixed j, j', also satisfies the condition $\delta_{i-1} \neq \delta_i$, for every $i \in [r']$. Thus the proof of the second claim and of (b) is complete. ∎

2.6. *Proof of Theorem 2.1(A).* We start by specifying a surjective homomorphism $p : G \to L$ appearing in the definition of an almost Galois covering. To do this we define a G-invariant $\mathbb{Z}$-grading $d : (\operatorname{ob} R)^2 \to \mathbb{Z}$ as in Example 2.2. For any $x, y \in \operatorname{ob} R = Q_0$ we set

$$d(x, y) = \ell(w) := \epsilon_1 + \cdots + \epsilon_n$$

where $w = \alpha_1^{\epsilon_1} \ldots \alpha_n^{\epsilon_n}$ is a fixed walk in Q with starting point $s(w) = y$ and ending point $t(w) = x$. Note that $\mathcal{W}(y, x) \neq \emptyset$ since Q is connected. Moreover, the definition of $d(x, y)$ does not depend on the choice of w. Indeed, any $w, w' \in \mathcal{W}(y, x)$ are equivalent in the sense of the homotopy relation $\sim\, = \,\sim_I$, since $\Pi(Q, I) = \{1\}$ (see [22] for the precise definition). This means that there exist $w_0 = w, w_1, \ldots, w_r = w' \in \mathcal{W}(y, x)$ such that for every i we have $w_i = u_i \delta_i^{(1)} u_i'$ and $w_{i+1} = u_i \delta_i^{(2)} u_i'$, where $u_i \in \mathcal{W}(y, z_i)$, $u_i' \in \mathcal{W}(z_i', x)$ are some walks and $\rho_i = \sum_{j=1}^{m_i} t_i^{(j)} \delta_i^{(j)} \in I(z_i', z_i)$, with $m_i \geq 2$ and $t_i^{(j)} \in k^*$, are minimal (linear) relations. But the ideal $I \lhd R(Q)$ is length homogeneous, so minimal relations are linear combinations of oriented paths of the same length and we have $\ell(w) = \ell(w')$.

It is clear that the function d defined above is a G-invariant $\mathbb{Z}$-grading, since $\ell(w'w) = \ell(w') + \ell(w)$ and $\ell(g(w)) = \ell(w)$ for any $w \in \mathcal{W}(y, x)$, $w' \in \mathcal{W}(z, y)$ and $g \in G$.

Now fix $x \in Q_0$ and denote by $p : G \to L$ the homomorphism $p_x^{(d)} : G \to \mathbb{Z}$ treated as a surjective homomorphism with codomain $L := \operatorname{Im} p_x^{(d)}$, where $p_x^{(d)}$ is defined for d as in 2.2. Clearly, either $L = 0$ or $L \cong \mathbb{Z}$, and the definition of p does not depend on the choice of x (see Lemma 2.2(c)).

Our next aim is to define a modified action $\star$ of G on R. For this we need more detailed information on the behavior of the functor F.

We may assume that $R' = R(Q', I')$, where $Q' = Q_{R'}$ and $I' \lhd R(Q')$ is an admissible ideal. Observe that R is square-free, since so is R'. Consequently, F induces a quiver map $f : Q \to Q'$, which is a covering of quivers (see [3, 3.3]). Moreover, f is G-invariant, since F is a G-covering functor.

By the very definition of f, for any arrow $\alpha \in Q_1(y,x)$ there exists a scalar $a_\alpha \in k^*$ such that

(i) $$F(\overline{\alpha}) + (J')^2 = a_\alpha\, \overline{f(\alpha)} + (J')^2$$

where $\overline{\alpha}$ denotes the morphism in $J(x,y) \subseteq R(x,y)$ defined by the arrow α, $\overline{f(\alpha)}$ the morphism in $J'(Fy,Fx) \subseteq R'(Fy,Fx)$ defined by the arrow $f(\alpha) \in Q'_1(f(y), f(x))$, $J = J(R)$ and $J' = J(R')$. (We use the convention that for $\rho \in R(Q)(x,y)$ we denote by $\overline{\rho}$ the coset $\rho + I \in R(Q,I)(x,y)$, and similarly for $R' = R(Q', I')$.)

For any $g \in G$ and $\alpha \in Q_1$ we set $c_\alpha(g) := a_{g(\alpha)}\, a_\alpha^{-1}$. Note that by (i) we have

(ii) $$F(\overline{g(\alpha)}) + (J')^2 = c_\alpha(g) F(\overline{\alpha}) + (J')^2$$

since $f(g(\alpha)) = f(\alpha)$.

For a fixed $g \in G$, the family $\{c_\alpha(g)\}_{\alpha\in Q_1}$ yields an automorphism

$$\theta_g : R(Q) \to R(Q)$$

of the path category $R(Q)$, which is determined by the formulas $\theta_g(x) = gx$ for $x \in Q_0$ and $\theta_g(\alpha) = c_\alpha(g)\, g(\alpha)$ for $\alpha \in Q_1$. (For any oriented path $\delta = \alpha_1 \dots \alpha_n$ in Q we set $\theta_g(\delta) = c_\delta(g)\, g(\delta)$, where $c_\delta(g) := c_{\alpha_1}(g) \dots c_{\alpha_n}(g)$.) We show that

(iii) $$\theta_{g_2} \circ \theta_{g_1} = \theta_{g_2 g_1}$$

for $g_1, g_2 \in G$. Observe that for any arrow $\alpha \in Q_1(y,x)$ we have

$$c_\alpha(g_2 g_1) = c_\alpha(g_1) c_{g_1(\alpha)}(g_2)$$

since

$$\begin{aligned} c_\alpha(g_2g_1)F(\overline{\alpha}) + (J')^2 &= F(\overline{g_2g_1(\alpha)}) + (J')^2 \\ &= c_{g_1(\alpha)}(g_2)F(\overline{g_1(\alpha)}) + (J')^2 = c_\alpha(g_1)c_{g_1(\alpha)}(g_2)F(\overline{\alpha}) + (J')^2 \end{aligned}$$

and $F(\overline{\alpha}) + (J')^2$ is a nonzero coset in $J'/(J')^2(Fx,Fy)$ (see (ii) and (i)). Now, by an easy check we obtain the required equalities $\theta_{g_2g_1}(\alpha) = \theta_{g_2}(\theta_{g_1}(\alpha))$.

Next we show that each θ_g induces an automorphism

$$\bar{\theta}_g : R \to R,$$

equivalently, the inclusion

(∗) $$\theta_g(I) \subseteq I$$

holds. For this we need further properties of the covering functor F.

First observe that for any path $\delta = \alpha_1 \dots \alpha_n \in \mathcal{P}(y,x)$ we have

(iv) $$F(\overline{\delta}) + (J')^{n+1} = a_\delta\, \overline{f(\delta)} + (J')^{n+1}$$

where $a_\delta := a_{\alpha_1} \dots a_{\alpha_n}$, $f(\delta) := f(\alpha_1) \dots f(\alpha_n) \in \mathcal{P}_{Q'}(f(y), f(x))$ and $\overline{f(\delta)}$ is a morphism in $(J')^n(Fx,Fy) \subseteq R'(Fx,Fy)$ defined by $f(\delta)$. Formula (iv) follows easily by induction on $n \geq 1$, upon applying (i) and the equality

$$F(\overline{\delta}) - a_\delta\, \overline{f(\delta)} = (F((\overline{\alpha_1 \ldots \alpha_{n-1}})\,\overline{\alpha_n}) - a_{\alpha_n} F(\overline{\alpha_1 \ldots \alpha_{n-1}})\,\overline{f(\alpha_n)}) \\ + (a_{\alpha_n} F(\overline{\alpha_1 \ldots \alpha_{n-1}})\,\overline{f(\alpha_n)} - a_{\alpha_n} a_{\alpha_1 \ldots \alpha_{n-1}}\, \overline{f(\alpha_1 \ldots \alpha_{n-1})}\ \ \overline{f(\alpha_n)}).$$

An immediate consequence of (iv) is the formula

$$F(\overline{\delta}) + (J')^{n+1} = c_\delta(g) F(\overline{g(\delta)}) + (J')^{n+1}$$

for any $g \in G$. In particular,

$$F(\overline{\delta}) - c_\delta(g) F(\overline{g(\delta)}) - \zeta \in (J')^{n+2}$$

where the element ζ is a linear combination of morphisms defined by paths $\omega \in \mathcal{P}_{Q'}(f(y), f(x))$ such that $\ell_{Q'}(\omega) = n+1$. Then lifting all paths ω along f we obtain by (iv) the equality

$$F(\overline{\delta}) + (J')^{n+2} = c_\delta(g) F(\overline{g(\delta)}) + \sum_{\delta' \in \tilde{\mathcal{P}}_{n+1}} c'_{\delta'} F(\overline{g(\delta')}) + (J')^{n+2}$$

where $c'_{\delta'} \in k$ and $\tilde{\mathcal{P}}_l$, for a fixed $l \in \mathbb{N}$, denotes the set of all $\delta' \in \tilde{\mathcal{P}} = \tilde{\mathcal{P}}(Fy, x) := \bigcup_{y' \in F^{-1}(Fy)} \mathcal{P}(y', x)$ such that $\ell(\delta') = l$. Proceeding in an analogous way, we can prove by induction on $l > \ell(\delta) = n$ that

$$F(\overline{\delta}) + (J')^{l+1} = c_\delta(g) F(\overline{g(\delta)}) + \sum_{\delta' \in \tilde{\mathcal{P}}_l} c'_{\delta'} F(\overline{g(\delta')}) + (J')^{l+1}$$

for some $c'_{\delta'} \in k$. Consequently,

$$\text{(v)} \qquad F(\overline{\delta}) = c_\delta(g) F(\overline{g(\delta)}) + \sum_{\delta' \in \tilde{\mathcal{P}}:\, \ell(\delta') > n} c'_{\delta'} F(\overline{g(\delta')})$$

since R' is locally bounded.

Now we prove $(*)$. Let $\rho = \sum_{j=1}^m t^{(j)} \delta^{(j)} \in I(x,y)$ be a generator of I. Since I is a homogeneous ideal, we can assume that ρ is a homogeneous element, so all $\ell(\delta^{(j)})$ are equal to some $n \geq 2$. Then by (v) we have

$$\begin{aligned} 0 = F(\overline{\rho}) &= \sum_{j=1}^m t^{(j)} F(\overline{\delta^{(j)}}) \\ &= \sum_{j=1}^m t^{(j)} c_{\delta^{(j)}}(g) F(\overline{g(\delta^{(j)})}) + \sum_{\delta' \in \tilde{\mathcal{P}}:\, \ell(\delta') > n} c''_{\delta'} F(\overline{g(\delta')}) \\ &= F\Big(\sum_{j=1}^m t^{(j)} c_{\delta^{(j)}}(g)\, \overline{g(\delta^{(j)})}\Big) + \sum_{\delta' \in \tilde{\mathcal{P}}:\, \ell(\delta') > n} c''_{\delta'} F(\overline{g(\delta')}) \end{aligned}$$

for some $c''_{\delta'} \in k$. Note that $\sum_{j=1}^m t^{(j)} c_{\delta^{(j)}}(g)\, \overline{g(\delta^{(j)})}$ belongs to $R(gx, gy)$ and $d(gx, gy) = d(x,y) = n$, whereas the element $\sum_{\delta' \in \tilde{\mathcal{P}}:\, \ell(\delta') > n} c''_{\delta'}\, \overline{g(\delta')}$ belongs to the direct sum of all morphism spaces $R(gx, gy')$, where $y' \in Gy$ satisfies $d(gx, gy') = d(x, y') > n$. Consequently, $\sum_{j=1}^{m} t^{(j)} c_{\delta^{(j)}}(g)\, \overline{g(\delta^{(j)})} = 0$, by a property of covering functors, and $\sum_{j=1}^{m} t^{(j)} \theta_g(\delta^{(j)}) = \sum_{j=1}^{m} t^{(j)} c_{\delta^{(j)}}(g) g(\delta^{(j)}) \in I(gx, gy)$.

As a result, each θ_g induces an automorphism $\bar{\theta}_g$ of R and by (iii), the family $\{\bar{\theta}_g\}_{g\in G}$ yields an action $\star : G \times R \to R$ of the group G on the category R. Now we can show that F is an almost Galois G-covering functor of integral or trivial type with respect to this action (see (a)).

Observe that by the definition of $\star$, replacing in (v) g by g^{-1} and x by gx, for any $x \in \operatorname{ob} R$, $b \in \operatorname{ob} R'$, $g \in G$ and any path $\delta \in \mathcal{P}(y, gx)$ with $y \in F^{-1}(b)$ we get

$$(\mathrm{v}') \qquad F(\overline{\delta}) = F(g^{-1} \star \overline{\delta}) + \sum_{\delta' \in \tilde{\mathcal{P}}:\, \ell(\delta') > \ell(\delta)} c'_{\delta'} F(\overline{g(\delta')})$$

for some $c'_{\delta'} \in k$, where $\tilde{\mathcal{P}} = \tilde{\mathcal{P}}(b, gx)$, and consequently

$$(\mathrm{vi}) \qquad {}_b\phi^{(x,g)}(\overline{\delta}) = g^{-1} \star \overline{\delta} + \sum_{\delta' \in \tilde{\mathcal{P}}:\, \ell(\delta') > \ell(\delta)} c'_{\delta'}\, \overline{g(\delta')}.$$

On the other hand, by Proposition 2.3, we have the equivalence

$$(\mathrm{vii}) \qquad |\,\overline{\delta'}| > |\,\overline{\delta}| \;\Leftrightarrow\; \ell(\delta') > \ell(\delta)$$

where $|-|$ is determined as in 1.4 by a fixed selection of $(\operatorname{ob} R)_0$ and the surjective homomorphism p defined in the first part of the proof (we apply Proposition 2.3 for $L = \mathbb{Z}$ and $L_0 = \operatorname{Im} p_x^{(d)} =: L$, see also Remark 2.3(ii) and the definition of d). Consider two cases:

- $L = \{0\}$. Then by (vii), for δ, there exists no $\delta' \in \tilde{\mathcal{P}}$ such that $\ell(\delta') > \ell(\delta)$. Consequently, ${}_b\phi^{(x,g)}(\overline{\delta}) = g^{-1} \star \overline{\delta}$ for all collections (x, b, g, δ), hence $F(\sigma) = F(g^{-1} \star \sigma)$ for every $\sigma \in R(gx, y)$ and F is a Galois G-covering functor with respect to the action $\star$.
- $L \neq \{0\}$. Then we can assume that $L = \mathbb{Z}$ as ordered groups and by (vii), equality (vi) can be be written in the form

$$(\mathrm{vi}') \qquad {}_b\phi^{(x,g)}(\overline{\delta}) = g^{-1} \star \overline{\delta} + \sum_{\delta' \in \tilde{\mathcal{P}}:\, |\,\overline{\delta'}| > |\,\overline{\delta}|} \overline{g(\delta')}.$$

Consequently, for any $\sigma \in R(gx, y)$ the formula 1.4(∗∗) holds and F is an almost Galois G-covering functor of integral type with respect to the action $\star$.

To prove that (b) of Theorem 2.1(A) holds for $\star$, fix N in $\operatorname{mod} R$ and $g \in G$. Since ${}^{g\star}N \cong {}^{g}N$ if and only if ${}^{g^{-1}}({}^{g\star}N) \cong N$, and ${}^{g^{-1}}({}^{g\star}N) = {}^{\varphi}N$ for $\varphi := g^{-1} \cdot (-) \circ g \star (-) : R \to R$, it suffices to show that the automorphism φ is defined by some k^*-grading u of $\operatorname{ob} R$ (i.e. $\varphi = \varphi^u$, see Lemma 2.4).

To see this, note first that φ is given by the mapping $\overline{\delta} \mapsto c_\delta(g)\overline{\delta}$ for $\delta \in \mathcal{P}(y,x)$, where $x, y \in \operatorname{ob} R$. Next for any $w = \beta_1 \dots \beta_n \in \mathcal{W}(y,x)$ with $\beta_1, \dots, \beta_n \in Q_1 \cup Q_1^{-1}$, we set

$$\text{(viii)} \qquad c_w(g) := c_{\beta_1}(g) \dots c_{\beta_n}(g)$$

where $c_{\beta_i}(g) := c_{\beta_i^{-1}}(g)^{-1}$ if $\beta_i^{-1} \in Q_1$. We show that $c_w(g) = c_{w'}(g)$ for any $w, w' \in \mathcal{W}(y,x)$.

Notice that $w \sim w'$, since $\Pi(Q,I) = \{1\}$. Consequently, by Lemma 2.5(b) and formula (viii) it suffices to show that $c_{\delta^{(1)}} = \dots = c_{\delta^{(m)}}$, where $\rho := \sum_{j=1}^m t^{(j)}\delta^{(j)} \in I(x,y)$ is a strongly minimal (linear) relation. Indeed,

$$g^{-1}\theta_g(\rho) = g^{-1}\Big(\sum_{j=1}^m t^{(j)} c_{\delta^{(j)}}(g) g(\delta^{(j)})\Big) = \sum_{j=1}^m t^{(j)} c_{\delta^{(j)}}(g)\delta^{(j)} \in I(x,y),$$

since I is invariant with respect to g^{-1} and θ_g. Hence, by Remark 2.5, the coefficients $c_{\delta^{(j)}}(g)$ for $j \in [m]$ are all equal, so $c_w(g) = c_{w'}(g)$.

Now for any $x, y \in \operatorname{ob} R$, we set

$$u(x,y) := c_w(g)$$

where $w \in \mathcal{W}(y,x)$ is a fixed unoriented path in Q (R is connected). It is easily seen that this defines a k^*-grading $u : (\operatorname{ob} R)^2 \to k^*$ of $\operatorname{ob} R$; moreover, $\varphi = \varphi^u$ (see the definition of φ). As a result, by Lemma 2.5(b) we have ${}^{g^{-1}}({}^{g\star}N)) = {}^{\varphi}N \cong N$, and hence ${}^{g\star}N \cong {}^{g}N$.

Finally observe that once we have proved that (a) and (b) hold for $\star$, the last assertion of Theorem 2.1(A) follows almost immediately. Indeed, by [8, Theorem 3.1.1] and [10, Corollary 3.4], for any N satisfying the Ext-vanishing condition we have

$$F_\bullet F_\lambda(N) \cong \bigoplus_{g\in G} {}^{g\star}N \quad \text{and} \quad F_\lambda({}^{g\star}N) \cong F_\lambda(N) \quad \text{for all } g \in G.$$

(F is an almost Galois G-covering of integral type with respect to $\star$; in case F is a Galois covering, the formula holds for all N in $\operatorname{mod} R$.) Since ${}^{g\star}N \cong {}^{g}N$ from (b), we hence obtain

$$F_\bullet F_\lambda(N) \cong \bigoplus_{g\in G} {}^{g}N \quad \text{and} \quad F_\lambda({}^{g}N) \cong F_\lambda(N) \quad \text{for all } g \in G,$$

and the proof is complete. ∎

Remark. (a) The functor $F : R \to R'$ admits a degeneration in the sense of [10] to the canonical Galois covering functor $\bar{F}^{(\star)} : R \to \bar{R}^{(\star)} := R/(G,\star)$, defined by the action $\star$ of G on R (see [10, Theorem 2.6]).

(b) If Q is a tree then there exists an automorphism $\varphi : R \to R$, identifying the G-actions $\star$ and $\cdot$ on R, i.e. satisfying the equality $\varphi \circ g\star(-) = g(-) \circ \varphi$ for every $g \in G$. We set $\varphi(\overline{\delta}) = a_\delta^{-1}\,\overline{\delta}$ for $\delta \in \mathcal{P}(y,x)$ with $x, y \in Q_0$. This yields a well defined functor, since Q is a tree, and the G-equivariance follows from the equalities $a_\alpha^{-1} = a_{g(\alpha)}^{-1} c_\alpha(g)$ for $\alpha \in Q_1$ and $g \in G$. Consequently, we can identify $\bar{F}^{(\star)}$ with the canonical Galois covering functor $\bar{F} : R \to \bar{R} := R/(G, \cdot)$, defined by the original action $\cdot : G \times R \to R$. More precisely, φ induces an isomorphism $\bar{\varphi} : \bar{R}^{(\star)} \to \bar{R}$ such that $\bar{F} \circ \varphi = \bar{\varphi} \circ \bar{F}^{(\star)}$; hence, $\bar{F}$ and $\bar{F}^{(\star)}$ are isomorphic in the sense of [10, 2.4]. In this way the functor F also admits a degeneration to $\bar{F}$.

We end this subsection with an interesting corollary from Theorem 2.1(A), concerning algebras which admit a cover that is a string special biserial category (see e.g. [25] for the definition). Recall that if R is a string tree category then any indecomposable R-module is a string module, i.e. it is isomorphic to $N(w)$ for some V-sequence w (see 1.1).

Let Q be a tree and $h : Q \to Q$ a nontrivial quiver automorphism acting freely on Q_0. We say that a line $\mathcal{L}$ in Q is *h-orientable* if $\mathcal{L}$ has a common arrow with some line L' such that $hL'_0 \cap L' \neq \emptyset$, i.e. containing x and hx for some $x \in Q_0$. ($\mathcal{L}$ and $\mathcal{L}'$ have a common arrow if and only if $|\mathcal{L}_0 \cap \mathcal{L}'_0| \geq 2$.) Notice that clearly there also exist examples of lines which are not h-orientable. Observe that if $\mathcal{L}$ is h-orientable then one of the two total orderings of "consecutive vertices" of $\mathcal{L}$ is distinguished, by the condition that it coincides on $\mathcal{L}_0 \cap \mathcal{L}'_0$ with that of the two orderings for $\mathcal{L}'$ in which x is smaller than hx, where x is as above. (Note that the choice of the orderings does not depend on the choice of x and $\mathcal{L}'$!) It is now clear that for $\mathcal{L}$ as above we have the notions of predecessor y^- and successor y^+ for $y \in \mathcal{L}_0$. Moreover, if $\mathcal{L}$ is finite and nontrivial then in the two-element set $B(\mathcal{L})$ consisting of the border points of $\mathcal{L}$, the left border $l(\mathcal{L})$ and the right border $r(\mathcal{L})$ are well defined.

For a reduced walk w we say that w is h-orientable if so is the line $Q(w)$ (see 1.1).

Corollary. *Let G be a group, Q a connected tree equipped with an action $\cdot : G \times Q \to Q$ of G, which is free on Q_0, $I \lhd R(Q)$ an admissible G-invariant ideal such that $R(Q, I)$ is a special biserial category, and $F : R \to R'$ a G-covering functor with respect to the induced action of G on $R := R(Q, I)$, where R' is square-free. Then F admits a degeneration to the canonical Galois covering G-functor $\bar{F} : R \to \bar{R} := R/(G, \cdot)$ in the sense of* [10]. *Moreover, for an indecomposable R-module N in* $\operatorname{mod} R$ *we have $F_\bullet F_\lambda(N) \cong \bigoplus_{g\in G} {}^gN$ and $F_\lambda({}^gN) \cong F_\lambda(N)$ for all $g \in G$, provided the V-sequence w such that $N \cong N(w)$ satisfies the negations of both conditions $1^\circ_{(h)}$ and $2^\circ_{(h)}$ in the Lemma below, for all $h \in G^{\prec e}$ such that w is h-orientable.*

Proof. Note that all the assumptions of Theorem 2.1(A) are satisfied. Thus, there exists an action $\star : G \times R \to R$ which coincides with $\cdot$ on ob R $(= Q_0)$ and has the properties as in the statement of the theorem. Now our first assertion follows immediately from Remark 2.6.

Fix a module N in ind R. By [25] there exists a unique, up to the inverse operation, V-sequence w in $R(Q,I)$ such that $N \cong N(w)$ (see 1.1). Note that if N is a simple module, i.e. w is a trivial walk, then the required isomorphism always holds. Now if w is a nontrivial V-sequence as in the theorem then by the lemma below, we have

$$(*) \qquad \mathrm{Hom}_R({}^hN, \tau N) = 0$$

for all $h \in G^{\prec e}$, where τ is the Auslander–Reiten translate in mod R. By the Auslander–Reiten formula [1, Theorem 2.13], $(*)$ immediately implies the required equality $\mathrm{Ext}^1_R(N, {}^hN) = 0$ for all $h \in G^{\prec e}$. Hence, the proof of the second assertion is finished (see 1.4). ∎

Let Q and G be as above. Then for any $x \in Q_0$ and $h \in G\setminus\{e\}$ we denote by $\mathcal{L}(x,h)$ the full subquiver of Q formed by the set $\bigcup_{m\in\mathbb{Z}} h^m Q[x,hx]_0$, where $Q[x,hx]$ is the unique (finite) line in Q connecting x and hx. Note that $\mathcal{L}(x,h)$ itself is not necessarily a line.

LEMMA. *Let G, Q, I, R be as above and $N = N(w)$ be an R-module, where w is a nontrivial V-sequence in (Q,I). Let $h \in G$. If* $\mathrm{Hom}_R({}^hN, \tau N) \neq 0$ *then either*

$1^{\circ}_{(h)}$: *$Q(w)_0 \cap hQ(w)_0 \neq \emptyset$ and then for $x := l(Q(w) \cap hQ(w))$ and $\mathcal{L} := \mathcal{L}(x,h)$ the following conditions are satisfied:*

- *$y := r(Q(w) \cap hQ(w)) \notin \mathcal{L}_0$,*
- *L is a line having the shape*
$$\cdots\cdots\ h^{-1}x\ \cdots\cdots\ x^- \to x \cdots\cdots\ y \to y^+ \cdots\cdots,$$
- *if $l(Q(w)) \notin \mathcal{L}_0$ then there exists $\beta \in Q(w)_1$ such that $s(\beta) = h^{-1}x$ and $t(\beta) \notin \mathcal{L}_0$,*
- *if $r(w) \notin \mathcal{L}_0$ then there exists $\gamma \in Q(w)_1$ such that $t(\beta) = y$ and $s(\gamma) \notin \mathcal{L}_0$;*

or

$2^{\circ}_{(h)}$: *$Q(w)_0 \cap hQ(w)_0 = \emptyset$ and then there exists $\alpha \in Q_1$ with $s(\alpha) \in B(Q(w))$ and $t(\alpha) \notin Q(w)_0$ such that for $x := t(\alpha)$ and $\mathcal{L} := \mathcal{L}(x,h)$ the following conditions are satisfied:*

- *L is a line having the shape*
$$\cdots\cdots\ h^{-1}x\ \cdots\cdots\ x^- \xrightarrow{\alpha} x \to x^+ \cdots\cdots,$$
- *$h^{-1}x \in Q(w)_0$,*
- *if $B(w) \nsubseteq \mathcal{L}_0$ then there exists $\beta \in Q(w)_1$ such that $s(\beta) = h^{-1}x$ and $t(\beta) \notin \mathcal{L}_0$;*

in particular, in each of the two cases, w is h-orientable.

Proof. We apply the precise description of the V-sequence τw such that $\tau N(w) \cong N(\tau w)$ [25, 13] and the criterion [12, 3.2] for deciding when for two V-sequences w, w' in $R(Q, I)$ the space $\mathrm{Hom}_R(N(w), N(w'))$ is nonzero. The proof is not difficult, but very technical. It relies on a case-by-case combinatorial analysis of the conditions from the criterion in the situation $w' = \tau w$; therefore, we leave it to the reader. ∎

2.7. Theorem 2.1(B) follows almost immediately from Theorem 2.1(A), once we know the two results presented below. The first of them is a straightforward property of almost Galois covering functors.

LEMMA. *Let $F : R \to R'$ be an almost Galois G-covering functor of type L. If R_1 is a full connected G-invariant subcategory of R then so is the restriction $F_1 : R_1 \to R'_1$ of F to R_1, where R'_1 is a full subcategory of R' formed by $F(\mathrm{ob}\, R_1)$.*

To formulate the second one, we recall [4] that a locally bounded k-category R is *schurian* if $\dim_k R(x, y) \leq 1$ for all $x, y \in \mathrm{ob}\, R$. A schurian k-category R admits a multiplicative basis, if we can choose a basic vector ${}_y\beta_x$ in each non-zero morphism space $R(x, y)$ in such a way that ${}_z\beta_y \cdot {}_y\beta_x = {}_z\beta_x$ whenever $R(y, z) \cdot R(x, y) \neq 0$. Any collection $\{{}_y\beta_x\}$ as above is called a *multiplicative basis* of R.

PROPOSITION. *Let R be a connected locally representation-finite locally bounded k-category such that that $\Pi(\Gamma_R) = \{1\}$ (hence, R is schurian!). Assume that $\diamond : G \times R \to R$ is an action of a group G that is free on* $\mathrm{ob}\, R$. *If G is a free group then there exists a multiplicative basis of R which is G-invariant with respect to $\diamond$.*

Proof. Repeat the arguments from [4, 3.2]. ∎

Proof of Theorem 2.1(B). (a) Since all the assumptions of Theorem 2.1(A) are satisfied, (a) automatically holds (see [3] for all the necessary details).

(b) Let $R := \tilde{A}$, where $\tilde{A}$ is the full subcategory of $k(\tilde{\Gamma}_A)$ formed by all projective vertices (see 2.1). Then on identifying A with the full subcategory of $(\mathrm{ind}\, A)_0$ formed by all projective objects, the restriction of F_A to R can be regarded as a G-functor $F : R \to A$, where $G := \Pi(\Gamma_A)$ (see [3, 18] for all the necessary details). Now by (a) and the lemma above, F is an almost Galois G-covering functor $F : R \to A$ of the respective type. Note that $\Pi(\Gamma_A)$ is a free group [3, 4.2].

Let now $\bar{F} : R \to \bar{A} := R/(G, \cdot)$ and $\bar{F}^{(\star)} : R \to \bar{A}^{(\star)} := R/(G, \star)$ be the canonical Galois covering functors defined by the restriction of the actions $\circ$ and $\star$ of G to R. Then the functor $F : R \to A$ admits a degeneration to $\bar{F}^{(\star)}$ (see [10, Theorem 2.6]). On the other hand G is a free group so by the Proposition there exist multiplicative bases $\mathcal{B}$ and $\mathcal{B}^{(\star)}$ of the schurian category R which are G-invariant with respect to $\circ$ and $\star$, respectively. The obvious bijection between $\mathcal{B}$ and $\mathcal{B}^{(\star)}$ yields an automorphism $\varphi : R \to R$ such that $\varphi \circ g \star (-) = g \cdot (-) \circ \varphi$ for every $g \in G$. Consequently, as in Remark 2.6, the functors $\bar{F}^{(\star)}$ and $\bar{F}$ are isomorphic in the sense of [10, 2.4], since φ induces an isomorphism $\bar{\varphi} : \bar{R}^{(\star)} \to \bar{R}$ such that $\bar{F} \circ \varphi = \bar{\varphi} \circ \bar{F}^{(\star)}$. In this way the functor F also admits a degeneration to $\bar{F}$. ∎

3. An interesting class of coverings. In this section we discuss a certain special class of almost Galois G-covering functors of integral type between bounded quiver categories (with G being an infinite cyclic group), containing the series of important natural examples presented in [8, Theorem 4.1.1]. Functors from this class behave more regularly than usual with respect to nice properties of the associated push-down and pull-up functors (see 1.4). We start by establishing some notation.

3.1. Let (Q, I) be a connected bounded quiver with fundamental group $G = \Pi(Q, I)$ and universal covering $(\tilde{Q}, \tilde{I})$, where $\Pi(Q, I) = \Pi((Q, I), a_0)$ and $\tilde{Q} = \tilde{Q}(a_0)$ for some fixed $a_0 \in Q_0$. Suppose that $\{w_a\}$, $a \in Q_0$, is a fixed collection of paths such that $w_a \in \mathcal{W}(a_0, a)$. Then $[w_a]$, $a \in Q_0$, forms a set of representatives of fibers of the canonical Galois functor

$$\bar{F} : \tilde{R} \to \bar{R}$$

and we have $\bar{F}^{-1}(a) = G[w_a]$ for any $a \in Q_0$, where $\bar{R} = R(Q, I)$ and $\tilde{R} = R(\tilde{Q}, \tilde{I})$. In particular, setting $(\operatorname{ob} \tilde{R})_0 := \{[w_a] : a \in Q_0\}$ we obtain the trivial G-invariant G-grading $d : (\operatorname{ob} R)^2 \to G$ on $\operatorname{ob} R = \tilde{Q}_0$, given by the formula $d(y, x) := [w_b v_b^{-1} v_a w_a^{-1}]$ for $x = [v_a]$, $y = [v_b] \in \tilde{Q}_0$, where $v_a \in \mathcal{W}(a_0, a)$ and $v_b \in \mathcal{W}(a_0, b)$ (cf. 2.3 and [8, 2.3]). Recall that the collection above also yields the *degree function*

$$\deg : \mathcal{W} \to G$$

given by the formula $\deg(v) := [w_b v w_a^{-1}]$ for $v \in \mathcal{W}_Q(b, a)$ (see [8, Corollary 2.3.2]).

For any $g = [u] \in G$ and $a \in Q_0$ we set

$$a_g = g[w_a] = [u w_a] \in Q_0.$$

If $\delta \in \mathcal{P}(b, a)$ is a path in Q then the lifting

$$\tilde{\delta} = ([u w_b], \delta) : [u w_b] \to [u w_b \delta]$$

of δ to $\tilde{Q}$ is denoted by $\tilde{\delta}_g$. Note that

$$\tilde{\delta}_g \in \tilde{\mathcal{P}}(b_g, a_{g \deg(\delta)}).$$

Assume now that G is an infinite cyclic group with a fixed generator $g = [u]$ for some $u \in \mathcal{W}(a_0, a_0)$. Then we have the identification $\mathbb{Z} = G$ given by $i \mapsto g^i$ for $i \in \mathbb{Z}$, and we write a_i (resp. $\tilde{\delta}_i$) instead of a_{g^i} (resp. $\tilde{\delta}_{g^i}$); moreover,

$$\tilde{\delta}_i \in \tilde{\mathcal{P}}(s(\delta)_i, t(\delta)_{i+\deg(\delta)}).$$

Observe that if $\delta = \delta^{(1)} \dots \delta^{(m)}$ then

$$(*) \qquad \tilde{\delta}_i = \widetilde{\delta^{(1)}}_i \, \widetilde{\delta^{(2)}}_{i+i_1} \cdots \widetilde{\delta^{(m)}}_{i+i_1+\cdots+i_{m-1}}$$

where $i_l = \deg \delta^{(l)}$ for $l = 1, \dots, m$ (see [8, Proposition 2.3.5]).

Note that clearly in this case G is $\mathbb{Z}$-totally ordered in a natural way and then for any $v_1, v_2 \in \mathcal{W}(b, a)$ we have $v_1 \prec' v_2$ if and only if $\deg(v_1) < \deg(v_2)$, where by definition $v_1 \prec' v_2$ if and only if $0 = [\varepsilon_{a_0}] < [w_a v_1^{-1} v_2 w_a^{-1}]$ in $G = \mathbb{Z}$ (see [8, Lemma 2.3.1]).

The following specialization of [8, Theorem 2.3.3] plays a crucial role in our further considerations.

Theorem. *Let (Q, I), a_0, G, R, $\tilde{R}$ be as above and I' be an admissible ideal in the path category $R(Q)$ such that $\underline{\dim}\, R' = \underline{\dim}\, \bar{R}$, where $R' = R(Q, I')$. Assume that $F : R(\tilde{Q}) \to R'$ is a k-functor satisfying the conditions:*

- (a) *$F_{\mathrm{ob}} : \mathrm{ob}\, R(\tilde{Q}) \to \mathrm{ob}\, R'$ is given by $p_0 : \tilde{Q}_0 \to Q_0$,*
- (b) *$F(\tilde{\alpha}) = (\alpha + \sum_{\alpha \prec' \delta} a_{\delta,\tilde{\alpha}} \delta) + I'$ for any lifting $\tilde{\alpha} \in \tilde{Q}_1$ of $\alpha \in Q_1$, where δ are oriented paths in Q (not belonging to I') and $a_{\delta,\tilde{\alpha}} \in k$,*
- (c) *$F(\tilde{I}) = 0$.*

Then the functor $F' : \tilde{R} \to R'$ induced by F is an almost Galois G-covering functor of integral type.

(Notice that deg determines the grading $|-|$, which is a basic tool for verification that F' is an almost Galois G-covering functor; cf. 1.4($**$), Remark 2.3(ii) and [8, proof of Theorem 2.3.3].)

Finally recall that we also have at our disposal the natural filtration on the category R', which for any pair $a, b \in Q_0$ is given by the formula

$$(**) \qquad R'(a, b)_{(g)} := \sum_{\delta \in \mathcal{P}(b,a):\, g \le \deg \delta} k(\delta + I')$$

where $g \in G$ (see [8, 2.3]).

3.2. Let $F' : \tilde{R} \to R'$ be an almost Galois covering as in Theorem 3.1, where G is an infinite cyclic group (with a fixed generator) and $L = \mathbb{Z}$. Assume that F' satisfies the following condition:

(✠): $F(\tilde{\beta}_i) = (\beta + b_{\beta,i} v_\beta) + I'$ for every $(\beta, i) \in Q_1 \times \mathbb{Z}$, where $v_\beta \in \mathcal{P}(s(\beta), t(\beta))$ depends only on β, and $\deg v_\beta > \deg \beta$ if $(b_{\beta,i'})_{i'} \neq 0$ in $k^{\mathbb{Z}}$.

Observe now that if $\delta = \beta^{(1)} \dots \beta^{(m)}$, where all $\beta^{(j)}$ are arrows, then

$$(*) \qquad F(\widetilde{\delta}_i) = \Big(\delta + \sum_{\underline{e}\in \mathrm{Sub}(m)} b_{\delta,i,\underline{e}}\, \delta_{(\underline{e})}\Big) + I'$$

for any $i \in \mathbb{Z}$, where $b_{\delta,i,\underline{e}} \in k$ is a product $b_1 \dots b_m$ with

$$b_j = \begin{cases} 1 & \text{if } j \neq e_1, \dots, e_p, \\ b_{\beta^{(j)}, i+i_1+\dots+i_{l-1}} & \text{if } j = e_l, \end{cases}$$

and the path $\delta_{(\underline{e})}$ is a composition $\upsilon^{(1)} \dots \upsilon^{(m)}$ with

$$\upsilon^{(j)} = \begin{cases} \beta^{(j)} & \text{if } j \neq e_1, \dots, e_p, \\ \upsilon_{\beta^{(j)}} & \text{if } j = e_l, \end{cases}$$

for any $\underline{e} \in \mathrm{Sub}(m) := \{\underline{e}' = (e'_1, \dots, e'_p) \in \mathbb{N}^p : 1 \leq e'_1 < \dots < e'_p \leq m,$ $p \geq 1\}$. In particular, we have

$$(**) \qquad \deg \delta_{(\underline{e})} > \deg \delta$$

if $b_{\delta,i,\underline{e}} \neq 0$ (see [8, Corollary 2.3.2]).

Definition. Let $K = (\omega; \alpha^{(1)}, \dots, \alpha^{(2s)}; \kappa^{(1)}, \dots, \kappa^{(2s)})$ be a collection formed by an oriented cycle ω, arrows $\alpha^{(1)}, \dots, \alpha^{(2s)}$ and oriented paths $\kappa^{(1)}, \dots, \kappa^{(2s)}$ in Q, where $s \in \mathbb{N}$. Denote by $Q_0(\omega)$ the set of all vertices of Q visited by ω, and by $Q_1(\omega)$ the set of all arrows of Q forming ω. Then K is called an *s-flower in Q* if it satisfies the following conditions:

- for each $x \in Q_0(\omega)$ there exists exactly one arrow $\gamma \in Q_1(\omega)$ such that $s(\gamma) = x$ (respectively, $\gamma' \in Q_1(\omega)$ such that $t(\gamma') = x$),
- $s(\alpha^{(1)}), t(\alpha^{(2)}), s(\alpha^{(3)}), \dots, s(\alpha^{(2s-1)}), t(\alpha^{(2s)}) \in Q_0(\omega)$, respectively, $t(\alpha^{(1)}), s(\alpha^{(2)}), t(\alpha^{(3)}), \dots, t(\alpha^{(2s-1)}), s(\alpha^{(2s)}) \notin Q_0(\omega)$,
- $t(\alpha^{(1)}) = s(\kappa^{(1)})$, $t(\kappa^{(1)}) = s(\alpha^{(2)})$, $t(\alpha^{(2)}) = s(\kappa^{(2)}), \dots, t(\alpha^{(2s)}) = s(\kappa^{(2s)})$, $t(\kappa^{(2s)}) = s(\alpha^{(1)})$,
- $\kappa^{(l)}$ contains no arrow β such that $s(\beta) \in Q_0(\omega)$ and $t(\beta) \notin Q_0(\omega)$ (respectively, $t(\beta) \in Q_0(\omega)$ and $s(\beta) \notin Q_0(\omega)$), for any $l = 1, \dots, 2s$.

Let K be an s-flower as above. We fix some extra notation.

We denote by ω_K the oriented cycle $\alpha^{(1)}\kappa^{(1)}\alpha^{(2)} \dots \alpha^{(2s)}\kappa^{(2s)}$ in Q. If the group G is cyclic and is generated by $[\omega]$ then we denote by $r(K)$ the integer $r \in \mathbb{Z}$ such that $[\omega_K] = [\omega]^r$ (we assume that $a_0 = s(\omega)$).

For any $x \in Q_0(\omega)$, let $\omega(x)$ denote the oriented cycle $\gamma^{(j)} \dots \gamma^{(m)} \dots \gamma^{(j-1)}$, where $x = s(\gamma^{(j)})$ and $\omega = \gamma^{(1)} \dots \gamma^{(m)}$. Additionally, for $l = 1, \dots, 2s$, we set $\omega(l) := \omega(s(\alpha^{(l)}))$ if l is odd, and $\omega(l) = \omega(t(\alpha^{(l)}))$ if l is even.

For any $\beta \in Q_1$ we set for simplicity $d_\beta = \deg \beta$; moreover, for $l = 1, \dots, 2s$ we set $d_l = \deg \alpha^{(l)}$ and $d'_l = \deg \kappa^{(l)}$.

If $\beta = \alpha^{(l)}$, then for any $i \in \mathbb{Z}$ we set $a_i^{(l)} = b_{\beta,i}$; if all $b_{\beta,i}$ for $i \in \mathbb{Z}$ are equal for some β, then their common value is denoted by b_β.

Assume that $\deg \upsilon_{\alpha^{(l)}} = d_l + 1$ for every l. Then for a fixed $r \in \mathbb{N}$, to any N in $\operatorname{mod} \tilde{R}$ we associate the $R(\tilde{Q})$-module $\dot{N} = \dot{N}_{(0,\dots,r-1)}$ in $\operatorname{mod} R(\tilde{Q})$ defined as follows:

For any object $x_j \in \tilde{Q}_0$, $(x,j) \in Q_0 \times \mathbb{Z}$, we set

$$\dot{N}(x_j) = N(x_{j-r+1}) \oplus \dots \oplus N(x_j);$$

for any arrow $\beta_j : x_j \to y_{j+d_\beta}$, $(\beta, j) \in Q_1 \times \mathbb{Z}$, the map $\dot{N}(\beta_j) : \dot{N}(x_j) \to \dot{N}(y_{j+d_\beta})$ is given in matrix form

$$[\dot{N}(\beta_j)^{(i',i)}] : \bigoplus_{i=j-r+1}^{j} N(x_i) \to \bigoplus_{i'=j+d_\beta-r+1}^{j+d_\beta} N(x_{i'})$$

where

$$\dot{N}(\beta_j)^{(i',i)} = \begin{cases} N(\widetilde{\alpha^{(l)}}_i) & \text{if } i' = i + d_l,\ j-r+1 \le i \le j, \\ (a_j^{(l)} - a_{i+r}^{(l)})\, N((\widetilde{\upsilon_{\alpha^{(l)}}})_i) & \text{if } i' = i + d_l + 1,\ j-r+1 \le i < j, \\ 0 & \text{otherwise,} \end{cases}$$

whenever $\beta = \alpha^{(l)}$ for some $l = 1, \dots, 2s$, and

$$\dot{N}(\beta_j)^{(i',i)} = \begin{cases} N(\beta_i) & \text{if } i' = i + d_\beta,\ j-r+1 \le i \le j, \\ 0 & \text{otherwise,} \end{cases}$$

in the remaining case. It is clear that the mapping $N \mapsto \dot{N}$ extends naturally to a k-linear functor

$$\Phi = \Phi_r : \operatorname{mod} \tilde{R} \to \operatorname{mod} R(\tilde{Q}).$$

For any $m = 1, \dots, r$ we denote by $\dot{N}^{(m)}$ the family $\{\dot{N}^{(m)}(x_j)\}_{x_j \in \tilde{Q}_0}$ of k-spaces, where

$$\dot{N}^{(m)}(x_j) = N(x_{j-m+1}) \oplus \dots \oplus N(x_j).$$

Lemma. *$\dot{N}^{(0)} = \{0\}, \dot{N}^{(1)}, \dots, \dot{N}^{(r)}$ form an ascending chain*

$$\dot{N}^{(0)} \subseteq \dot{N}^{(1)} \subseteq \dots \subseteq \dot{N}^{(r)} = \dot{N}$$

of $\tilde{R}$-submodules of $\dot{N}$ such that $\dot{N}^{(m)}/\dot{N}^{(m-1)} \cong {}^{m-1}N$ for every $m \in \mathbb{Z}$ $(= G)$, where the identification is given by $m \mapsto [\omega]^m$.

Proof. An easy check on definitions. ∎

The main aim of this section is to prove the general result announced in [8, 4.3], which in an abstract situation described by certain combinatorial conditions guarantees nice, more regular than usual, properties of the push-down and pull-up functors associated with an almost Galois covering of integral type. This result will play a crucial role in the proof of [8, Theorem 4.3.1] (only formulated there).

THEOREM. *Let $F' : \tilde{R} \to R'$ be an almost Galois covering as in* 3.2(✠). *Assume that Q contains an s-flower $K = (\omega; \alpha^{(1)}, \ldots, \alpha^{(2s)}; \kappa^{(1)}, \ldots, \kappa^{(2s)})$ satisfying the conditions below:*

(i) $\langle[\omega]\rangle = G$ *(we assume that $a_0 = s(\omega)$);*

(ii) *for any $l = 1, \ldots, 2s$, there exist $m_l \in \mathbb{N}$, paths $\nu^{(1,l)}, \ldots, \nu^{(m_l,l)} \in \mathcal{P}(s(\alpha^{(l)}), t(\alpha^{(l+1)}))$ and scalars $f_{1,l}, \ldots, f_{m_l,l} \in k$ such that:*

- *for any $j=1, \ldots, m_l$, $\deg \nu^{(j,l)} \geq \deg(\alpha^{(l)}\kappa^{(l)}\alpha^{(l+1)})$, and $\deg \nu^{(j,l)} > \deg(\alpha^{(l)}\kappa^{(l)}\alpha^{(l+1)})$ if $\alpha^{(l')} \in Q_1(\nu^{(j,l)})$ for some l',*
- $\alpha^{(l)}\kappa^{(l)}\alpha^{(l+1)} - \sum_{j=1}^{m_l} f_{j,l}\nu^{(j,l)} \in I'$;

(iii) *for any $l = 1, \ldots, 2s$:*

- *if $l \in 2\mathbb{N}+1$ then $\omega(l)\alpha^{(l)}\kappa^{(l)}\alpha^{(l+1)} - \alpha^{(l)}\kappa^{(l)}\alpha^{(l+1)}\omega(l+1) \in I$ and $\omega(l)\alpha^{(l)}\kappa^{(l)}\alpha^{(l+1)} \notin I$,*
- *if $l \in 2\mathbb{N}$ then $\alpha^{(l)}\omega(l)\kappa^{(l)}\alpha^{(l+1)} - \alpha^{(l)}\kappa^{(l)}\omega(l+1)\alpha^{(l+1)} \in I$ and $\alpha^{(l)}\omega(l)\kappa^{(l)}\alpha^{(l+1)} \notin I$;*

(iii)′ *for any $\beta \in Q_1 \setminus \{\alpha^{(1)}, \ldots, \alpha^{(2s)}\}$:*

- *$\omega(s(\beta))\beta \in I'$ if $s(\beta) \in Q_0(\omega)$ and $t(\beta) \notin Q_0(\omega)$ (respectively, $\beta\omega(t(\beta)) \in I'$ if $t(\beta) \in Q_0(\omega)$ and $s(\beta) \notin Q_0(\omega)$),*
- *$\beta\omega(t(\beta)) - \omega(s(\beta))\beta \in I'$ if $s(\beta), t(\beta) \in Q_0(\omega)$;*

(iv) *for $\beta \in Q_1$:*

- *all $b_{\beta,i} = b_\beta$ for $i \in \mathbb{Z}$ are equal if $\beta \in Q_1 \setminus \{\alpha^{(1)}, \ldots, \alpha^{(2s)}\}$,*
- *$\upsilon_{\alpha^{(l)}} = \omega(l)\alpha^{(l)}$ if $l = 1, \ldots, 2s-1$, while $\upsilon_{\alpha^{(l)}} = \alpha^{(l)}\omega(l)$ if $l = 2, \ldots, 2s$,*
- *$R'(t(\beta), s(\beta))_{(\deg \upsilon_\beta + 1)} = 0$ if $(b_{\beta,i})_{i\in\mathbb{Z}} \neq 0$ in $k^{\mathbb{Z}}$ (see* 3.1(∗∗)).

Then the functor F' has the following properties:

(a) *Φ_r is a functor from* $\operatorname{mod} \tilde{R}$ *to* $\operatorname{mod} \tilde{R}$ *and $F'_\bullet F'_\lambda \cong \bigoplus_{i\in\mathbb{Z}} {}^{ir}\Phi_r$, where $r = r(K)$. In particular, if N in* $\operatorname{mod} \tilde{R}$ *is an indecomposable module such that $\operatorname{Ext}^1_{\tilde{R}}(N, {}^iN) = 0$ for all $i = 1, \ldots, r-1$, then $F'_\bullet F'_\lambda(N) \cong \bigoplus_{i\in\mathbb{Z}} {}^iN$.*

(b) *$F'_\bullet \cong {}^rF'_\bullet$.*

(c) *For any $x, y \in \operatorname{ob} R'$, $\alpha \in R'(y,x)$, and $x_i \in F'^{-1}(x)$, $y_j \in F'^{-1}(y)$ we have $g({}_{x_i}\dot{\alpha}_{y_j}) = {}_{gx_i}\dot{\alpha}_{gy_j}$, where $g = -1$, and hence $F'_\rho \circ {}^1(-) \cong F'_\lambda$.*

First we formulate several simple observations directly connected with our assumptions:

REMARK. To (ii): If $l \in 2\mathbb{N}+1$ then $m_l \geq 1$, since by (ii) we also have $\alpha^{(l)}\kappa^{(l)}\alpha^{(l+1)} \notin I$ (see below).

To (iii): We always have $\omega(l)\alpha^{(l)}\kappa^{(l)}\alpha^{(l+1)} \sim \alpha^{(l)}\kappa^{(l)}\alpha^{(l+1)}\omega(l+1)$, since G is abelian; from (iii) it follows $\omega(l)\alpha^{(l)}\kappa^{(l)}\alpha^{(l+1)}, \alpha^{(l)}\kappa^{(l)}\alpha^{(l+1)}\omega(l+1) \notin$

I' if $l \in 2\mathbb{N}+1$ (respectively, $\alpha^{(l)}\omega(l)\kappa^{(l)}\alpha^{(l+1)} \sim \alpha^{(l)}\kappa^{(l)}\omega(l+1)\alpha^{(l+1)}$; $\alpha^{(l)}\omega(l)\kappa^{(l)}\alpha^{(l+1)}, \alpha^{(l)}\kappa^{(l)}\omega(l+1)\alpha^{(l+1)} \notin I$ if $l \in 2\mathbb{N}$).

To (iii$'$): From (iii$'$) we have $\omega(s(\beta))\beta \in I$ (resp. $\beta\omega(t(\beta)) \in I$) for β as in the first case; always $\beta\omega(t(\beta)) \sim \omega(s(\beta))\beta$ if $s(\beta), t(\beta) \in Q_0(\omega)$.

To (iv): $R'(t(\beta), s(\beta))_{(\deg \upsilon_\beta + 1)} = 0$ if and only if $\delta \in I$ for every $\delta \in \mathcal{P}(s(\beta), t(\beta))$ such that $\deg \delta \geq \deg \upsilon_\beta + 1$.

The proof of the Theorem needs some preparation (it will be given in 3.7). We start by computing a precise formula for the functor $F'_\bullet F'_\lambda$.

3.3. Let N be an $\tilde{R}$-module. Then $F'_\lambda(N)(x) = \bigoplus_{i\in\mathbb{Z}} N(x_i)$ for any $x \in Q_0$.

Lemma. *For any $\beta \in Q_1(x,y)$ the structure map*

$$F'_\lambda(N)(\beta) = [\bar{N}(\beta)^{(i',i)}] : \bigoplus_{i\in\mathbb{Z}} N(x_i) \to \bigoplus_{i'\in\mathbb{Z}} N(x_{i'})$$

is as follows:

$$\bar{N}(\beta)^{(i',i)} = \begin{cases} N(\tilde{\beta}_i) & \text{if } i' = i + d_\beta, \\ -b_{\beta,i} N((\widetilde{\upsilon_\beta})_i) & \text{if } i' = i + \deg \upsilon_\beta, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Fix β. We show that

$$(*) \qquad \beta + I' = (F(\tilde{\beta}_i) - b_{\beta,i} F((\widetilde{\upsilon_\beta})_i)) + I'$$

for any $i \in \mathbb{Z}$. If $b_{\beta,i} = 0$, the equality $(*)$ is trivially satisfied, since $F(\tilde{\beta})_i = (\beta + b_{\beta,i}\upsilon_\beta) + I'$. Assume that $b_{\beta,i} \neq 0$. Then, by 3.2$(*)$, we have

$$\upsilon_\beta + I' = \Big(F((\widetilde{\upsilon_\beta})_i) + \sum_{\underline{e}\in \mathrm{Sub}(l_\beta)} b_{\upsilon_\beta,i,\underline{e}}\,(\upsilon_\beta)_{(\underline{e})}\Big) + I'.$$

Note that for any $\underline{e} \in \mathrm{Sub}(l_\beta)$, either $b_{\upsilon_\beta,i,\underline{e}} = 0$ or, by [8, Corollary 2.3.2], $\deg(\upsilon_\beta)_{(\underline{e})} > \deg \upsilon_\beta$, so $(\upsilon_\beta)_{(\underline{e})}) \in I'$, from assumption (iv).

In this way $(*)$ is proved. Now the assertion follows immediately from the definition of $F'_\lambda(N)$. ∎

Recall that the $\tilde{R}$-module $\tilde{N} = F'_\bullet F'_\lambda(N)$ takes the value

$$\tilde{N}(x_j) = F'_\lambda(N)(F'(x_j)) = \bigoplus_{i\in\mathbb{Z}} N(x_i)$$

at any $x_j \in \tilde{Q}_0$, where $(x,j) \in Q_0 \times \mathbb{Z}$.

Proposition. *For any $\beta \in Q_1(x,y)$ and $j \in \mathbb{Z}$ the structure map*

$$F'_\bullet F'_\lambda(N)(\tilde{\beta}_j) = [\tilde{N}(\tilde{\beta}_j)^{(i',i)}] : \bigoplus_{i\in\mathbb{Z}} N(x_i) \to \bigoplus_{i'\in\mathbb{Z}} N(x_{i'})$$

is given by

$$\tilde{N}(\tilde{\beta}_j)^{(i',i)} = \begin{cases} N(\tilde{\beta}_i) & \text{if } i' = i + d_\beta, \\ (b_{\beta,j} - b_{\beta,i})N((\widetilde{\upsilon_\beta})_i) & \text{if } i' = i + \deg \upsilon_\beta, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Observe first that

$$F'_\bullet F'_\lambda(N)(\tilde{\beta}_j) = F'_\lambda(N)(\beta) + b_{\beta,j} F'_\lambda(N)(\upsilon_\beta),$$

since by definition $F'_\bullet F'_\lambda(N)(\tilde{\beta}_j) = F'_\lambda(N)(F'(\tilde{\beta}_j))$ and $F'(\tilde{\beta}_j) = (\beta + b_{\beta,j}\upsilon_\beta) + I'$. We showed that $F((\widetilde{\upsilon_\beta})_i) = \upsilon_\beta + I'$, so $F'_\lambda(N)(\upsilon_\beta)$ has the form

$$N(\upsilon_\beta) : \bigoplus_{i\in\mathbb{Z}} N(x_i) \to \bigoplus_{i\in\mathbb{Z}} N(y_{i+\deg \upsilon_\beta}).$$

Now the required formula follows immediately from the lemma and the considerations above. ■

3.4. In the proof of the theorem we use some abstract construction that generalizes the description of the $\tilde{R}$-module $F'_\bullet F'_\lambda(N)$ from the proposition above.

Let N be a module in $\operatorname{MOD} \tilde{R}$. Then to any collection $\underline{n} = (n_i) \in \mathbb{Z}^{\mathbb{Z}}$ we associate an $R(\tilde{Q})$-module $\tilde{N}_{\underline{n}}$. For any object $x_j \in \tilde{Q}_0$, $(x, j) \in Q_0 \times \mathbb{Z}$, we set

$$\tilde{N}_{\underline{n}}(x_j) = \bigoplus_{i\in\mathbb{Z}} N(x_i).$$

For any arrow $\beta_j : x_j \to y_{j+d_\beta}$ in $\tilde{Q}$, $(\beta, j) \in Q_1 \times \mathbb{Z}$, the map $\tilde{N}_{\underline{n}}(\tilde{\beta}_j) : \tilde{N}_{\underline{n}}(x_j) \to \tilde{N}_{\underline{n}}(y_{j+d_\beta})$ is defined as a matrix

$$[\tilde{N}_{\underline{n}}(\tilde{\beta}_j)^{(i',i)}] : \bigoplus_{i\in\mathbb{Z}} N(x_i) \;\to\; \bigoplus_{i'\in\mathbb{Z}} N(x_{i'}),$$

where

$$\tilde{N}_{\underline{n}}(\tilde{\beta}_j)^{(i',i)} = \begin{cases} N(\alpha_i^{(l)}) & \text{if } i' = i + d_\beta, \\ (a_j^{(l)} - a_{j+n_{i-j}}^{(l)})N((\widetilde{\upsilon_{\alpha^{(l)}}})_i) & \text{if } i' = i + d_\beta + 1, \\ 0 & \text{otherwise,} \end{cases}$$

whenever $\beta = \alpha^{(l)}$ for some $l = 1, \ldots, 2s$, and

$$\tilde{N}_{\underline{n}}(\tilde{\beta}_j)^{(i',i)} = \begin{cases} N(\beta_i) & \text{if } i' = i + d_\beta, \\ 0 & \text{otherwise.} \end{cases}$$

in the remaining case.

It is easily seen that for a fixed $\underline{n} \in \mathbb{Z}^{\mathbb{Z}}$, the mapping $N \mapsto \tilde{N}_{\underline{n}}$ extends naturally to a k-linear functor

$$\Phi_{\underline{n}} : \operatorname{MOD} \tilde{R} \to \operatorname{MOD} R(\tilde{Q}).$$

Note that $\Phi_{\mathrm{id}_{\mathbb{Z}}} = F'_{\bullet}F'_{\lambda}$, since by Proposition 3.3 we have $\tilde{N}_{\mathrm{id}_{\mathbb{Z}}} = F'_{\bullet}F'_{\lambda}(N)$ for N in MOD $\tilde{R}$.

Let $\underline{n} = (n_i) \in \mathbb{Z}^{\mathbb{Z}}$ be fixed. Then for any $m \in \mathbb{Z}$, we denote by $\underline{n}_m^+$ (resp. $\underline{n}_m^-$) the collection $(n'_i) \in \mathbb{Z}^{\mathbb{Z}}$ such that $n'_m = n_m + r$ (resp. $n'_m = n_m - r$) and $n'_i = n_i$ for all $i \neq m$.

Proposition. *The functors $\Phi_{\underline{n}_m^+}$ and $\Phi_{\underline{n}}$ (resp. $\Phi_{\underline{n}_m^-}$ and $\Phi_{\underline{n}}$) are isomorphic for every $m \in \mathbb{Z}$.*

The following fact plays a crucial role in the proof.

Lemma. *There exist scalars $c_1, \ldots, c_{2s} \in k$ and integers $r_1, \ldots, r_{2s} \in \mathbb{Z}$ such that:*

- $r_1 + \cdots + r_{2s} = r$,
- *for any $i \in \mathbb{Z}$ the following system of equalities holds:*

$$\begin{cases} a_i^{(1)} + a_{i+r_1}^{(2)} = c_1, \\ a_i^{(2)} + a_{i+r_2}^{(3)} = c_2, \\ \qquad \vdots \\ a_i^{(2s)} + a_{i+r_{2s}}^{(1)} = c_{2s}. \end{cases}$$

Proof. To construct the pairs $(c_l, r_l) \in k \times \mathbb{Z}$, $l = 1, \ldots, 2s$, satisfying the required conditions, we analyze the $(d+1)$th component $z_{d+1}(i)$ of the element

$$z := \Big(\alpha^{(l)}\kappa^{(l)}\alpha^{(l+1)} - \sum_{m=1}^{m_l} f_{j,m}\nu^{(m,l)}\Big) + I' \in R'(x,y)$$

via the isomorphisms

$$R'(x,y) \cong \bigoplus_{j\in\mathbb{Z}} \tilde{R}(x_j, y_i) = \bigoplus_{j'\in\mathbb{Z}} \bar{R}(x,y)_{j'}$$

given by F', where $x = t(\alpha^{(l+1)})$, $y = s(\alpha^{(l)})$, $d = d_l + d'_l + d_{l+1}$, and $j' = j - i$ for $i \in \mathbb{Z}$. Note that by [8, Proposition 2.3.5], z belongs to $R'(x,y)_{(d)}$, since $\deg(\alpha^{(l)}\kappa^{(l)}\alpha^{(l+1)}) = d$ and $\deg \nu^{(m,l)} \geq d$ for all m; moreover, for any $i \in \mathbb{Z}$ the dth component $z_d = z_d(i)$ of z is equal to

$$(\alpha^{(l)}\kappa^{(l)}\alpha^{(l+1)})^{\sim}{}_i - \sum_{m\in[m_l]_d} f_{j,m}\widetilde{\nu^{(m,l)}}_i + \tilde{I} \in \tilde{R}(x_{i+d}, y_i),$$

where $[m_l]_{d'} = \{m \in [m_l] : \deg \nu^{(m,l)} = d'\}$ for $d' \in \mathbb{N}$. (When A is a very long expression, for typographical reasons we will write $(A)^{\sim}$ instead of $\widetilde{A}$.)

To compute $z_{d+1}(i)$ we apply formulas 3.1(∗) and 3.2(∗). We assume first that l is odd. Then

$$\begin{aligned} F((\alpha^{(l)}\kappa^{(l)}\alpha^{(l+1)})^{\sim}{}_i) &= F(\widetilde{\alpha^{(l)}}_i)F(\widetilde{\kappa^{(l)}}_{i+d_l})F(\widetilde{\alpha^{(l+1)}}_{i+d_l+d'_l}) \\ &= (\alpha^{(l)} + a_i^{(l)}\omega(l)\alpha^{(l)}) \cdot \Big(\kappa^{(l)} + \sum_{\underline{e}\in\mathrm{Sub}(p_l)} b_{\underline{e}}(\kappa^{(l)})_{(\underline{e})}\Big) \\ &\quad \cdot (\alpha^{(l+1)} + a_{i+d_l+d'_l}^{(l+1)}\alpha^{(l+1)}\omega(l+1)), \end{aligned}$$

where $p_l = \ell(\kappa^{(l)})$ and $b_{\underline{e}} = b_{\kappa^{(l)},i+d_l,\underline{e}}$. Expanding the product on the right hand side, we obtain $\alpha^{(l)}\kappa^{(l)}\alpha^{(l+1)}$ and seven extra summands. Observe that four of them belong to $R'(x,y)_{(d+2)}$, since for any $\underline{e}$ such that $b_{\underline{e}} \neq 0$, we have $\deg(\alpha^{(l)}(\kappa^{(l)})_{(\underline{e})}\alpha^{(l+1)}\omega(l+1))$, $\deg(\omega(l)\alpha^{(l)}(\kappa^{(l)})_{(\underline{e})}\alpha^{(l+1)}) > d+1$, $\deg(\omega(l)\alpha^{(l)}(\kappa^{(l)})_{(\underline{e})}\alpha^{(l+1)}\omega(l+1)) > d+2$, and $\deg(\omega(l)\alpha^{(l)}\kappa^{(l)}\alpha^{(l+1)}\omega(l+1)) = d+2$ (see [8, Proposition 2.3.5] and 3.2(∗∗)). The next two summands, $a_i^{(l)}\omega(l)\alpha^{(l)}\kappa^{(l)}\alpha^{(l+1)} + I'$ and $a_{i+d_l+d'_l}^{(l+1)}\alpha^{(l)}\kappa^{(l)}\alpha^{(l+1)}\omega(l+1) + I'$, belong to $R'(x,y)_{(d+1)}$, as $\deg(\omega(l)\alpha^{(l)}\kappa^{(l)}\alpha^{(l+1)}) = d+1 = \deg(\alpha^{(l)}\kappa^{(l)}\alpha^{(l+1)}\omega(l+1))$. Then by [8, Proposition 2.3.5] and assumption (iii),

$$z'_{d+1}(i) := (a_i^{(l)} + a_{i+d_l+d'_l}^{(l+1)})(\omega(l)\alpha^{(l)}\kappa^{(l)}\alpha^{(l+1)})^{\sim}{}_i + \tilde{I}$$

is the $(d+1)$th component of

$$(a_i^{(l)}\omega(l)\alpha^{(l)}\kappa^{(l)}\alpha^{(l+1)} + a_{i+d_l+d'_l}^{(l+1)}\alpha^{(l)}\kappa^{(l)}\alpha^{(l+1)}\omega(l+1)) + I'.$$

The summand $\sum_{\underline{e}\in\mathrm{Sub}(p_l)} b_{\underline{e}}\alpha^{(l)}(\kappa^{(l)})_{(\underline{e})}\alpha^{(l+1)} + I'$ belongs to $R'(x,y)_{(d+1)}$, since $\deg(\alpha^{(l)}(\kappa^{(l)})_{(\underline{e})}\alpha^{(l+1)}) > d$ if $b_{\underline{e}} \neq 0$ (see 3.2(∗∗)). Consequently, its $(d+1)$th component has the form

$$z''_{d+1}(i) := \sum_{\underline{e}\in\mathrm{Sub}(p_l)_{d+1}} b_{\underline{e}}(\alpha^{(l)}(\kappa^{(l)})_{(\underline{e})}\alpha^{(l+1)})^{\sim}{}_i + \tilde{I},$$

where $\mathrm{Sub}(p_l)_{d+1} = \{\underline{e} \in \mathrm{Sub}(p_l) : \deg(\alpha^{(l)}(\kappa^{(l)})_{(\underline{e})}\alpha^{(l+1)}) = d+1\}$. Note that the coefficients $b_{\underline{e}}$ do not depend on $i \in \mathbb{Z}$, since $\kappa^{(l)}$ does not contain arrows $\alpha^{(l')}$, so the coefficients $b_{\kappa^{(l)},i,\underline{e}}$ for different i are equal (see Definition 3.2 and (iv)).

Now we compute the $(d+1)$th component $z'''_{d+1}(i)$ of $(\sum_{m=1}^{m_l} f_{j,m}\nu^{(m,l)}) + I'$. We apply again the same technique. For any $m \in [m_l]$ we have

$$F(\widetilde{\nu^{(m,l)}}_i) = \Big(\nu^{(m,l)} + \sum_{\underline{e}'\in\mathrm{Sub}(p'_m)} b'_{\underline{e}'}(\nu^{(m,l)})_{(\underline{e}')}\Big) + I',$$

where $p'_m = \ell(\nu^{(m,l)})$ and $b'_{\underline{e}'} = b_{\nu^{(m,l)},i,\underline{e}'}$ for $\underline{e}' \in \mathrm{Sub}(p'_m)$. It is easily seen that

$$z'''_{d+1}(i)$$
$$= \Big(\sum_{m\in[m_l]_{d+1}} f_{j,m}\widetilde{\nu^{(m,l)}}_i + \sum_{m\in[m_l]_d}\sum_{\underline{e}'\in\mathrm{Sub}(p'_m)_{d+1}} f_{j,m}b'_{\underline{e}'}((\nu^{(m,l)})_{(\underline{e}')})^{\sim}{}_i\Big) + \tilde{I},$$

where $\mathrm{Sub}(p'_m)_{d+1} = \{\underline{e}' \in \mathrm{Sub}(p'_m) : \deg((\nu^{(m,l)})_{(\underline{e}')}) = d+1\}$. Note that as above, the coefficient $b'_{\underline{e}'}$ for $\underline{e}' \in \mathrm{Sub}(p'_m)_{d+1}$ does not depend on $i \in \mathbb{Z}$, since $\nu^{(m,l)}$ does not contain arrows $\alpha^{(l')}$ (see (i) and (iv)).

Consequently, we obtain the equality

$$z_{d+1}(i) = -z'_{d+1}(i) - z''_{d+1}(i) - z'''_{d+1}(i).$$

Since $z = 0$ in $R'(x,y)$, it follows that $z_{d+1}(i) = 0$, and we have

$$z'_{d+1}(i) = -z''_{d+1}(i) - z'''_{d+1}(i)$$

in $\tilde{R}(x_{i+d+1}, y_i)$. If we pass to $\bar{R}(x,y)_{d+1}$ then all vectors $z''_{d+1}(i)$ (resp. $z'''_{d+1}(i)$) for $i \in \mathbb{Z}$ collapse to one vector z''_{d+1} (resp. z'''_{d+1}), and we get the equality

$$(a_i^{(l)} + a_{i+d_l+d'_l}^{(l+1)})z' = -z''_{d+1} - z'''_{d+1}$$

in $\bar{R}(x,y)_{d+1}$, where $z' = \omega(l)\alpha^{(l)}\kappa^{(l)}\alpha^{(l+1)} + I$.

Observe that since $z' \neq 0$, all coefficients $a_i^{(l)} + a_{i+d_l+d'_l}^{(l+1)}$, $i \in \mathbb{Z}$, are equal. Consequently, setting $r_l = d_l + d'_l$ and $c_l = a_0^{(l)} + a_{d_l+d'_l}^{(l+1)}$ we obtain the equality $a_i^{(l)} + a_{i+r_l}^{(l+1)} = c_l$ for all $i \in \mathbb{Z}$.

In case l is even we construct the pair (c_l, r_l) in an analogous way. We set again $r_l = d_l + d'_l$. (We assume $\alpha^{(2s+1)} = \alpha^{(1)}$, $a_i^{(2s+1)} = a_i^{(1)}$, and so on.) A slight difference appears in the formula for $F((\alpha^{(l)}\kappa^{(l)}\alpha^{(l+1)})^{\sim}{}_i)$, and for example, in computations of degrees, where we apply a dual set of assumptions. As a result, in a final step of the discussion we consider the vectors $(a_i^{(l)} + a_{i+d_l+d'_l}^{(l+1)})(\alpha^{(l)}\omega(l)\kappa^{(l)}\alpha^{(l+1)} + I) \in \bar{R}(x,y)_{d+1}$. Nevertheless, all arguments used in the consecutive steps are exactly the same.

Finally notice that in fact the proof is already complete, since by the properties of the function deg, the equality $r_1 + \cdots + r_{2s} = r$ follows immediately from the definition of the integers r and r_l for $l = 1, \ldots, 2s$. ∎

Corollary. *There exists a scalar $c \in k$ such that for any $i \in \mathbb{Z}$,*

$$a_i^{(l)} - a_{i+r}^{(l)} = \begin{cases} c & \text{if } l \in 2\mathbb{N}, \\ -c & \text{if } l \in 2\mathbb{N}+1, \end{cases}$$

where $l = 1, \ldots, 2s$.

Proof. Let $l = 1$. Then from the Lemma we have the following $2s$ equalities:

$$\begin{aligned} a_i^{(1)} + \ a_{i+r_1}^{(2)} &= c_1, \\ a_{i+r_1}^{(2)} + a_{i+r_1+r_2}^{(3)} &= c_2, \\ &\vdots \\ a_{i+r_1+\cdots+r_{2s-1}}^{(2s)} + a_{i+r}^{(1)} &= c_{2s}. \end{aligned}$$

If we multiply the lth equality by $(-1)^{l+1}$, for $l = 1, \ldots, 2s$, and next sum up all of them, we obtain

$$a_i^{(1)} - a_{i+r}^{(1)} = c_1 - c_2 + c_3 + \cdots - c_{2s}.$$

Now, modifying slightly the indices in the arguments above, one easily checks that $c = c_1 - c_2 + c_3 + \cdots - c_{2s}$ satisfies the required conditions. ∎

3.5. *Proof of Proposition 3.4.* Fix $m \in \mathbb{Z}$. We show first that the functors $\Phi_{\underline{n}}$ and $\Phi_{\underline{n}_m^+}$ are isomorphic.

For any N in $\operatorname{MOD} R(\tilde{Q})$ we construct the $R(\tilde{Q})$-homomorphism

$$\varphi = \varphi^+_{\underline{n},m}(N) : \tilde{N} \to \tilde{N}',$$

where $\tilde{N} = \tilde{N}_{\underline{n}}$ and $\tilde{N}' = \tilde{N}_{\underline{n}_m^+}$. Note that formally φ is a collection

$$\{\varphi(x_j) : \tilde{N}(x_j) \to \tilde{N}'(x_j)\}_{(x,j)\in Q_0\times\mathbb{Z}}$$

of k-linear maps, where each $\varphi(x_j)$ has matrix form

$$\varphi(x_j) = [\varphi(x_j)^{(i',i)}] : \bigoplus_{i\in\mathbb{Z}} N(x_i) \to \bigoplus_{i'\in\mathbb{Z}} N(x_{i'}).$$

Now we define φ. For any $j \in \mathbb{Z}$ we set

$$\varphi(x_j)^{(i',i)} = \begin{cases} \mathrm{id}_{N(i)} & \text{if } i' = i, \\ cN(\widetilde{\omega(x)}_i) & \text{if } i = j+m,\ i' = j+m+1, \\ 0 & \text{otherwise,} \end{cases}$$

whenever $x \in Q_0(\omega)$, and

$$\varphi(x_j) = \bigoplus_{i\in\mathbb{Z}} \mathrm{id}_{N(i)}$$

in the opposite case.

It is clear that all the maps $\varphi(x_j)$, $(x,j) \in Q_0 \times \mathbb{Z}$, defined above are k-isomorphisms. We have to show that φ is an $R(\tilde{Q})$-homomorphism. For this we have to verify the commutativity of all diagrams

$$(*)_{\beta,j} \qquad \begin{array}{ccc} \tilde{N}(x_j) & \xrightarrow{\varphi(x_j)} & \tilde{N}'(x_j) \\ {\scriptstyle \tilde{N}(\tilde{\beta}_j)}\downarrow & & \downarrow{\scriptstyle \tilde{N}'(\tilde{\beta}_j)} \\ \tilde{N}(y_{j+d_\beta}) & \xrightarrow{\varphi(y_{j+d_\beta})} & \tilde{N}'(y_{j+d_\beta}) \end{array}$$

where $\beta \in Q_1(x,y)$ and $j \in \mathbb{Z}$.

Clearly, for the commutativity of $(*)_{\beta,j}$ it suffices to show the equalities

$$(*)_{\beta,j;i} \qquad \tilde{N}'(\tilde{\beta}_j)\varphi(x_j)_{|N(x_i)} = \varphi(y_{j+d\beta})\tilde{N}(\tilde{\beta}_j)_{|N(x_i)}$$

for all $i \in \mathbb{Z}$, and each equality $(*)_{\beta,j;i}$ is equivalent to the collection of the equalities

$$(*)_{\beta,j;i,i'} \qquad \pi_{i'}\tilde{N}'\tilde{\beta}_j\varphi(x_j)_{|N(x_i)} = \pi_{i'}\varphi(y_{j+d\beta})\tilde{N}(\tilde{\beta}_j)_{|N(x_i)}$$

for all $i' \in \mathbb{Z}$, where $\pi_{i'} : \bigoplus_{i''\in\mathbb{Z}} N(y_{i''}) \to N(y_{i'})$ is the canonical projection.

In fact, for a fixed $j \in \mathbb{Z}$, we have to consider the equalities $(*)_{\beta,j;i}$ only in the following cases:

1°. $i = j+m$ if $\beta \neq \alpha^{(1)}, \dots, \alpha^{(2s)}$,
2°. $i = j+m, j+m-1$ if $\beta = \alpha^{(l)}$ for $l = 1,3,\dots,2s-1$,
3°. $i = j+m, j+m-1$ if $\beta = \alpha^{(l)}$ for $l = 2,4,\dots,2s$.

Observe that in all the remaining cases the equalities $(*)_{\beta,j;i}$ follow trivially, since by the definitions all the nonzero components of $\varphi(x_j)$ and $\varphi(y_{j+d_\beta})$ appearing in the formula are identities (respectively, all the components of $\tilde{N}(\tilde{\beta}_j)$ appearing essentially in the formula are equal to the corresponding ones for $\tilde{N}'(\tilde{\beta}_j)$).

Case 1°. Note first that it suffices to show $(*)_{\beta,j;j+m,j+m+d_\beta+1}$ (the remaining ones are evidently of the shape "$0=0$"). We consider three subcases:

If $x,y \notin Q_0(\omega)$ then $(*)_{\beta,j;j+m,j+m+d_\beta+1}$ is trivially satisfied.

If $x \in Q_0(\omega)$ and $y \notin Q_0(\omega)$ (resp. $x \notin Q_0(\omega)$ and $y \in Q_0(\omega)$), then the equality holds, since from assumption (iii)$'$ and Remark 3.2 we deduce that $N(\tilde{\beta}_{j+m+1})N(\widetilde{\omega(x)}_{j+m}) = 0$ (resp. $N(\widetilde{\omega(y)}_{j+m+d_\beta})N(\tilde{\beta}_{j+m}) = 0$).

Assume now that $x,y \in Q_0(\omega)$. Then $\omega(x)\beta - \beta\omega(y)$ belongs to I and $(*)_{\beta,j;j+m,j+m+d_\beta+1}$ follows from the equality $N(\tilde{\beta}_{j+m+1})N(\widetilde{\omega(x)}_{j+m}) = N(\widetilde{\omega(y)}_{j+m+d_\beta})N(\tilde{\beta}_{j+m})$ (see again (iii)$'$ and Remark 3.2; note that $\omega(x)\beta = \beta\omega(y)$ if $\beta \in Q_1(\omega)$).

Case 2°. We have to verify only the equalities $(*)_{\alpha^{(l)},j;j+m,i'}$ for $i' = j+m+d_l, j+m+d_l+1, j+m+d_l+2$ and $(*)_{\alpha^{(l)},j;j+m-1,i''}$ for $i'' = j+m-1+d_l, j+m+d_l, j+m+d_l+1$. Note first that $(*)_{\alpha^{(l)},j;j+m,j+m+d_l}$ is trivially satisfied. The equality $(*)_{\alpha^{(l)},j;j+m,j+m+d_l+2}$ holds, since $\deg\omega(l)v_{\alpha^{(l)}} \geq \deg v_{\alpha^{(l)}} + 1$

and, by (iv), $\omega(l)v_{\alpha^{(l)}}$ belongs to I, so $N((\widetilde{v_{\alpha^{(l)}}})_{j+m+1})N(\widetilde{\alpha^{(l)}}_{j+m})=0$. Finally, since $v_{\alpha^{(l)}}=\omega(l)\alpha^{(l)}$, by Corollary 3.4 we have

$$\begin{aligned}&(a_j^{(l)}-a_{j+n_m+r}^{(l)})N((\widetilde{v_{\alpha^{(l)}}})_{j+m})+cN(\widetilde{\alpha^{(l)}}_{j+m+1})N(\widetilde{\omega(l)}_{j+m})\\&-(a_j^{(l)}-a_{j+n_m}^{(l)})N((\widetilde{v_{\alpha^{(l)}}})_{j+m})=(-a_{j+n_m+r}^{(l)}+c+a_{j+n_m}^{(l)})N((\widetilde{v_{\alpha^{(l)}}})_{j+m})=0.\end{aligned}$$

Consequently, we obtain $(*)_{\alpha^{(l)},j;j+m,j+m+d_l+1}$. The remaining three equalities (for $i=j+m-1$) are trivially satisfied, or follow from assumption (iv) and Remark 3.2.

Case 3°. If $i=j+m$, we have to show only $(*)_{\alpha^{(l)},j;j+m,j+m+d_l}$ and $(*)_{\alpha^{(l)},j;j+m,j+m+d_l+1}$. The first equality is trivially satisfied, the second one follows from the equality

$$\begin{aligned}&(a_j^{(l)}-a_{j+n_m+r}^{(l)})N((\widetilde{v_{\alpha^{(l)}}})_{j+m})-(a_j^{(l)}-a_{j+n_m}^{(l)})N((\widetilde{v_{\alpha^{(l)}}})_{j+m})\\&-cN(\widetilde{\omega(l)}_{j+d_l+m})N(\widetilde{\alpha^{(l)}}_{j+m})=(-a_{j+n_m+r}^{(l)}+a_{j+n_m}^{(l)}-c)N((\widetilde{v_{\alpha^{(l)}}})_{j+m})=0.\end{aligned}$$

(We apply the equality $v_{\alpha^{(l)}}=\omega(l)\alpha^{(l)}$ and again Corollary 3.4.)

Now assume $i=j+m-1$. We have to verify the following three equalities: $(*)_{\alpha^{(l)},j;j+m-1,j+m-1+d_l}$, $(*)_{\alpha^{(l)},j;j+m-1,j+m+d_l}$, $(*)_{\alpha^{(l)},j;j+m-1,j+m+1+d_l}$. The first two are trivially satisfied. The third one follows from assumption (iv) and Remark 3.2, since $\deg(v_{\alpha^{(l)}}\omega(l))\ge\deg v_{\alpha^{(l)}}+1$, so $N(\widetilde{\omega(l)}_{j+d_l+m})$ $N((\widetilde{v_{\alpha^{(l)}}})_{j+m-1})=0$.

In this way the proof is complete. ∎

3.6. Let $\underline{n}=(n_i)\in\mathbb{Z}^{\mathbb{Z}}$ be as above. Then for any $m,q\in\mathbb{Z}$, we denote by $\underline{n}_m^q$ the collection $(n_i')\in\mathbb{Z}^{\mathbb{Z}}$ such that $n_m'=n_m+qr$ and $n_i'=n_i$ for all $i\neq m$. Then, keeping the notation from the proof above, we denote by $\varphi_{\underline{n},m}^q(N)$ the $R(\tilde{Q})$-isomorphism

$$\varphi^+_{\underline{n}_m^{q-1},m}(N)\circ\cdots\circ\varphi^+_{\underline{n}_m^0,m}(N):\tilde{N}_{\underline{n}}\to\tilde{N}_{\underline{n}_m^q}.$$

It is easily seen that the components $\varphi(x_j)=[\varphi(x_j)^{(i',i)}]$, $(x,j)\in Q_0\times\mathbb{Z}$, of the $R(\tilde{Q})$-isomorphism $\varphi_{\underline{n},m}^q(N)$ are equal to $\bigoplus_{i\in\mathbb{Z}}\mathrm{id}_{N(i)}$ if $x\notin Q_0(\omega)$, and otherwise they are given by the formula

$$\varphi(x_j)^{(i',i)}=\begin{cases}\mathrm{id}_{N(i)} & \text{if } i'=i,\\ qcN(\widetilde{\omega(x)}_i) & \text{if } i=j+m,\ i'=j+m+1,\\ 0 & \text{otherwise.}\end{cases}$$

Proposition. *Given $\underline{n}\in\mathbb{Z}$, the composition*

$$\varphi_{\underline{n}}^q(N)=\cdots\circ\varphi^{(m+1)}\circ\varphi^{(m)}\circ\cdots:\tilde{N}_{\underline{n}}\to\tilde{N}_{\underline{n}^q}$$

is well defined for any $q=(q_m)\in\mathbb{Z}^{\mathbb{Z}}$, and it yields an $R(\tilde{Q})$-isomorphism $\tilde{N}_{\underline{n}}\cong\tilde{N}_{\underline{n}^q}$, where $\underline{n}^q=(n_i+q_ir)$, $\varphi^{(m)}=\varphi^{q_m}_{\underline{n}_{(m-1)},m}(N):\tilde{N}_{\underline{n}_{(m-1)}}\to\tilde{N}_{\underline{n}_{(m)}}$

and $\underline{n}_{(m)} = (n'_i) \in \mathbb{Z}^{\mathbb{Z}}$ *with*

$$n'_i = \begin{cases} n_i + q_i r & \text{if } i \le m, \\ n_i & \text{if } i > m, \end{cases}$$

for any $m \in \mathbb{Z}$.

Proof. To prove that the composition $\varphi^q_{\underline{n}} = \varphi^q_{\underline{n}}(N)$ is well defined we show that the composition

$$\varphi^q_{\underline{n}}(x_j) = \cdots \circ \varphi^{(m+1)}(x_j) \circ \varphi^{(m)}(x_j) \circ \cdots : \bigoplus_{i \in \mathbb{Z}} N(x_i) \to \bigoplus_{i' \in \mathbb{Z}} N(x_{i'})$$

is well defined for any $(x, j) \in Q_0 \times \mathbb{Z}$. We have to consider only the case $x \in Q_0(\omega)$, since otherwise $\varphi^{(m)}(x_j) = \mathrm{id}_{\bigoplus_{i\in\mathbb{Z}} N(x_i)}$ for any m.

Let $(x, j) \in Q_0 \times \mathbb{Z}$ with x as above. It suffices to check that $\varphi^q_{\underline{n}}(x_j)_{|N(x_i)}$ is well defined for every $i \in \mathbb{Z}$. Fix i and set $m = i - j$. Note that given $\underline{n}' \in \mathbb{Z}^{\mathbb{Z}}$, we have $\varphi^{q'}_{\underline{n}',m'}(x_j)(N(x_i)) \subseteq N(x_i)$, and $\varphi^{q'}_{\underline{n}',m'}(x_j)_{|N(x_i)} = \mathrm{id}_{N(x_i)}$ if $m' \ne m$; $\varphi^{q'}_{\underline{n}',m'}(x_j)(N(x_i)) \subseteq N(x_i) \oplus N(x_{i+1})$ if $m' = m$, for any $q' \in \mathbb{Z}$. Moreover, by the assumptions,

$$(\varphi^{q''}_{\underline{n}'^{q'},m'+1}(x_j) \circ \varphi^{q'}_{\underline{n}',m'}(x_j))_{|N(x_i)} = \varphi^{q'}_{\underline{n}',m'}(x_j)_{|N(x_i)}$$

for any $q', q'' \in \mathbb{Z}$. Then $\varphi^q_{\underline{n}}(x_j)(N(x_i)) \subseteq N(x_i) \oplus N(x_{i+1})$, and

$$\varphi^q_{\underline{n}}(x_j)_{|N(x_i)} : N(x_i) \to N(x_i) \oplus N(x_{i+1})$$

is the composition of the maps

$$\varphi^{(m')}(x_j)_{|N(x_i)} = \mathrm{id}_{N(x_i)} : N(x_i) \to N(x_i)$$

for all $m' < m$, the map

$$(\varphi^{(m+1)}(x_j) \circ \varphi^{(m)}(x_j))_{|N(x_i)} = \varphi^{(m)}(x_j)_{|N(x_i)} : N(x_i) \to N(x_i) \oplus N(x_{i+1})$$

and the maps

$$\varphi^{(m')}(x_j)_{|N(x_i) \oplus N(x_{i+1})} = \mathrm{id}_{N(x_i) \oplus N(x_{i+1})}$$

for all $m' \ge m + 2$. Consequently, $\varphi^q_{\underline{n}}(x_j) = \varphi^q_{\underline{n}}(N)(x_j)$ is well defined, $\varphi^q_{\underline{n}}(x_j)_{|N(x_i)} = \varphi^{q_m}_{\underline{n}_{(m-1)},m}(x_j)_{|N(x_i)}$ for every $i \in \mathbb{Z}$, and so is $\varphi^q_{\underline{n}} = \varphi^q_{\underline{n}}(N)$. (Note that $\varphi^q_{\underline{n}}(x_j)_{|N(x_i)} = \varphi^{q_m}_{\underline{n}_{(m-1)},m}(x_j)_{|N(x_i)}$.) Now it is clear that $\varphi^q_{\underline{n}}(N) : \tilde{N}_{\underline{n}} \to \tilde{N}_{\underline{n}^q}$ is an $R(\tilde{Q})$-homomorphism, so an isomorphism. ■

Corollary. *The family* $\varphi = (\varphi^q_{\underline{n}}(N))_{N \in \mathrm{MOD}\,\tilde{R}}$ *of* $R(\tilde{Q})$-*homomorphisms defines an isomorphism between the functors* $\Phi_{\underline{n}}$ *and* $\Phi_{\underline{n}^q}$.

Proof. An easy check on definitions. ■

3.7. *Proof of Theorem 3.2.* (a) We apply the last corollary for $\underline{n} = \mathrm{id}_{\mathbb{Z}}$ and $q = (-\mathrm{quo}_r(i))_{i\in\mathbb{Z}}$. Then $\underline{n}^q = \mathrm{rem}$, where $\mathrm{rem} = (\mathrm{rem}_r(i))_{i\in\mathbb{Z}}$, and we obtain an isomorphism $F'_\bullet F'_\lambda \cong \Phi_{\mathrm{rem}}$. We show that a decomposition

$$(*)\qquad \tilde{N}_{\mathrm{rem}} \cong \bigoplus_{p\in\mathbb{Z}} {}^{pr}\dot{N}$$

in $\operatorname{Mod} R(\tilde{Q})$ holds for any N in $\operatorname{Mod}\tilde{R}$.

To prove $(*)$ it suffices to check that $(\bigoplus_{p\in\mathbb{Z}} {}^{pr}\dot{N})(\tilde{\beta}_j) = \bigoplus_{p\in\mathbb{Z}} {}^{pr}\dot{N}(\tilde{\beta}_j)$ and $\tilde{N}_{\mathrm{rem}}(\tilde{\beta}_j)$ coincide for all arrows $\tilde{\beta}_j$ such that $\beta\in Q_1(x,y)$ and $j\in\mathbb{Z}$, if we identify the spaces $(\bigoplus_{p\in\mathbb{Z}} {}^{pr}\dot{N})(x_j) = \bigoplus_{p\in\mathbb{Z}}(\bigoplus_{i=j-pr-r+1}^{j-pr} N(x_i))$ and $\tilde{N}_{\mathrm{rem}}(x_j) = \bigoplus_{i\in\mathbb{Z}} N(x_i)$ in a natural way.

Fix a pair $(x,j)\in Q_0\times\mathbb{Z}$. In case $\beta\neq\alpha^{(1)},\dots,\alpha^{(2s)}$ the required equality is trivially satisfied since $b_{\beta,i} = b_\beta$ for all $i\in\mathbb{Z}$, and therefore the maps under consideration have the form

$$\bigoplus_{i\in\mathbb{Z}} N(\beta_i) : \bigoplus_{i\in\mathbb{Z}} N(x_i) \to \bigoplus_{i\in\mathbb{Z}} N(y_{i+d_\beta}).$$

Assume that $\beta = \alpha^{(l)}$ for some $l = 1,\dots,2s$. Then $\tilde{N}_{\mathrm{rem}}(\tilde{\beta}_j) : \bigoplus_{i\in\mathbb{Z}} N(x_i) \to \bigoplus_{i'\in\mathbb{Z}} N(y_{i'})$ is given by

$$\tilde{N}_{\mathrm{rem}}(\tilde{\beta}_j)^{(i',i)} = \begin{cases} N(\alpha_i^{(l)}) & \text{if } i' = i + d_\beta,\\ (a_j^{(l)} - a_{j+\mathrm{rem}_r(i-j)}^{(l)})N((\widetilde{v_{\alpha^{(l)}}})_i) & \text{if } i' = i + d_\beta + 1,\\ 0 & \text{otherwise.}\end{cases}$$

Now we fix $p\in\mathbb{Z}$. Then the components of the map

$${}^{pr}\dot{N}(\tilde{\beta}_j) = \dot{N}(\beta_{j-pr}) : \bigoplus_{i=j-pr-r+1}^{j-pr} N(x_i) \to \bigoplus_{i=j-pr-r+1+d_l}^{j-pr} N(x_i)$$

are as follows:

$${}^{pr}\dot{N}(\tilde{\beta}_j)^{(i',i)} = \begin{cases} N(\alpha_i^{(l)}) & \text{if } i' = i + d_\beta,\ j-pr-r+1 \le i \le j-pr,\\ (a_{j-pr}^{(l)} - a_{i+r}^{(l)})N((\widetilde{v_{\alpha^{(l)}}})_i) & \text{if } i' = i + d_\beta + 1,\ j-pr-r+1 \le i < j-pr,\\ 0 & \text{otherwise.}\end{cases}$$

Observe that $a_{j-pr}^{(l)} - a_{i+r}^{(l)} = a_j^{(l)} - a_{j+\mathrm{rem}_r(i-j)}^{(l)}$ for any $j-pr-r+1 \le i < j-pr$, since by Corollary 3.4 we have $a_{j-pr}^{(l)} - a_{i+r}^{(l)} = a_j^{(l)} - a_{i+pr+r}^{(l)} = a_j^{(l)} - a_{j+(i-j)+pr+r}^{(l)}$, and $i - j + pr + r = \mathrm{rem}_r(i-j)$. Note also that $a_j^{(l)} - a_{j+\mathrm{rem}_r(i-j)}^{(l)} = 0$ for $i = j - pr$. In this way the maps $\tilde{N}_{\mathrm{rem}}(\tilde{\beta}_j)$ and $(\bigoplus_{p\in\mathbb{Z}} {}^{pr}\dot{N})(\tilde{\beta}_j)$ coincide and thus $(*)$ is proved.

It is clear that the proved decomposition yields a canonical isomorphism $\Phi_{\mathrm{rem}} \cong \bigoplus_{i\in\mathbb{Z}} {}^{ir}\Phi$ of functors and in this way the proof of the isomorphism $F'_\bullet F'_\lambda \cong \bigoplus_{i\in\mathbb{Z}} {}^{ir}\Phi$ is complete. Notice that since for any N in $\operatorname{mod}\tilde{R}$, $F'_\bullet F'_\lambda(N)$ belongs to $\operatorname{Mod}\tilde{R}$, $\Phi(N)$ also belongs to $\operatorname{mod}\tilde{R}$ and Φ is an endofunctor of the category $\operatorname{mod}\tilde{R}$.

To prove the second assertion of (a) fix N in $\operatorname{mod}\tilde{R}$ and consider the ascending chain

$$\dot{N}^{(0)} \subseteq \dot{N}^{(1)} \subseteq \cdots \subseteq \dot{N}^{(r)} = \dot{N}$$

of $\tilde{R}$-submodules of $\dot{N}$ defined in 3.2 such that $\dot{N}^{(l)}/\dot{N}^{(l-1)} \cong {}^{l-1}N$ for all l (see Lemma 3.2). Now, if $\operatorname{Ext}^1_{\tilde{R}}({}^{l}N, N) = 0$ for all l such that $1 \le l \le r-1$, then the decomposition $F'_\bullet F'_\lambda(N) \cong \bigoplus_{i\in\mathbb{Z}} {}^{i}N$ follows from the isomorphism $F'_\bullet F'_\lambda(N) \cong \bigoplus_{p\in\mathbb{Z}} {}^{pr}\dot{N}$ and basic properties of Ext-functors.

(b) For a fixed M in $\operatorname{mod} R$ consider the family

$$\psi(M) = \{\psi(x_j) : M(x) \to M(x)\}_{(x,j)\in Q_0\times\mathbb{Z}}$$

defined by setting $\psi(x_j) = \mathrm{id}_{M(x)}$ if $x \notin Q_0(\omega)$, and $\psi(x_j) = \mathrm{id}_{M(x)} + cM(\omega(x))$ if $x \in Q_0(\omega)$. Note that each $\psi(x_j)$ is a k-isomorphism since $M(\omega(x))$ is nilpotent. We show that $\psi(M) \in \operatorname{Hom}_{\tilde{R}}(F'_\bullet M, {}^{r}F'_\bullet M)$. We have to check, for any $\beta_j : x_j \to y_{j+d_\beta}$, $(x,j) \in Q_0 \times \mathbb{Z}$, the equality

$$(**)\qquad ({}^{r}F'_\bullet M)(\tilde{\beta}_j)\psi(x_j) = \psi(y_{j+d_\beta})(F'_\bullet M)(\tilde{\beta}_j),$$

or equivalently,

$$(**)'\qquad (M(\beta) + b_{\beta,j-r}M(\upsilon_\beta))\psi(x_j) - \psi(y_{j+d_\beta})(M(\beta) + b_{\beta,j}M(\upsilon_\beta)) = 0.$$

Fix $(\beta, j) \in Q_1 \times \mathbb{Z}$. We consider several cases.

Assume first $\beta \neq \alpha^{(1)}, \ldots, \alpha^{(2s)}$. It is clear that if $x, y \notin Q_0(\omega)$ then $(**)'$ holds trivially, since $b_{\beta,j-r} = b_{\beta,j}$. If $x \in Q_0(\omega)$ and $y \notin Q_0(\omega)$ (resp. $y \in Q_0(\omega)$ and $x \notin Q_0(\omega)$) then $(**)'$ follows from $M(\beta)M(\omega(x)) = 0 = M(\upsilon_\beta)M(\omega(x))$ (resp. $M(\omega(y))M(\beta) = 0 = M(\omega(y))M(\upsilon_\beta)$, see assumptions (iv) and (iii)$'$). Finally, in case $x, y \notin Q_0(\omega)$, $(**)'$ follows by analogous arguments from the equality $M(\beta)M(\omega(x)) = M(\omega(y))M(\beta)$ (see (iii)$'$ if β does not belong to ω; otherwise it is obvious).

Next we assume that $\beta = \alpha^{(l)}$ for $l = 1, 3, \ldots, 2s-1$. Then $\upsilon_\beta = \omega(l)\alpha^{(l)}$, $b_{\beta,j'} = a^{(l)}_{j'}$ and by (iv), the verification of $(**)'$ reduces to the verification of the equality

$$a^{(l)}_{j-r}M(\omega(l)\alpha^{(l)}) + cM(\alpha^{(l)})M(\omega(l)) - a^{(l)}_{j}M(\omega(l)\alpha^{(l)}) = 0.$$

Consequently, $(**)'$ follows immediately from Corollary 3.4.

In the final case, $\beta = \alpha^{(l)}$ for $l = 2, 4, \ldots, 2s$, we proceed analogously. Now $(**)'$ reduces to the equality

$$a_{j-r}^{(l)}M(\alpha^{(l)}\omega(l)) - cM(\alpha^{(l)})M(\omega(l)) - a_j^{(l)}M(\alpha^{(l)}\omega(l)) = 0,$$

which again holds true by Corollary 3.4.

In this way we showed that $\psi(M)$ is an $\tilde{R}$-homomorphism. It is easily seen that $\psi = (\psi(M))_{M\in \mathrm{MOD}\,R}$ is a natural transformation, which yields an isomorphism $F'_{\bullet} \cong {}^rF'_{\bullet}$.

(c) We check the required formula for $\alpha = \beta + I'$, where $\beta \in Q_1(y,x)$. Fix i. From equality 3.3(∗) we infer that ${}_{x_i}\dot{\beta}_{y_{i+d_\beta}} = \tilde{\beta}_i$, ${}_{x_i}\dot{\beta}_{y_{i+\deg v_\beta}} = -b_{\beta,i}(\widetilde{v_\beta})_i$ and ${}_{x_i}\dot{\beta}_{y_j} = 0$ for the remaining cases. We also have at our disposal the equality

$$\beta + I' = F({}_i\tilde{\beta}) - b_{\beta,i-d_\beta}F({}_i(\widetilde{v_\beta}))$$

dual to 3.3(∗), where ${}_i\widetilde{w} = \widetilde{w}_{i-\deg w}$ for any $w \in \mathcal{P}(y,x)$. Consequently, ${}_{x_{i-d_\beta}}\dot{\beta}_{y_i} = \tilde{\beta}_{i-d_\beta}$, ${}_{x_{i-\deg v_\beta}}\dot{\beta}_{y_i} = -b_{\beta,i-d_\beta}(\widetilde{v_\beta})_{i-\deg v_\beta}$ and ${}_{x_i}\dot{\beta}_{y_j} = 0$ for the remaining cases. Now the required formula follows easily. The second assertion is an immediate consequence of the first, by [8, Lemma 3.8.1(a)]. ∎

3.8. To end this section we discuss the series of examples introduced in [8, 4.1]. In fact we show that they fit into the abstract scheme developed in 3.2.

Let $R_t^{(n)} = R(Q^{(n)}, I_t^{(n)})$, $t \in k$, $n = 1,\dots,7$, be algebras defined as follows:

$$Q^{(1)}:\ \begin{tikzcd} 1 \arrow[loop left, "\eta"] \arrow[r, bend left, "\alpha^{(1)}"] & 2 \arrow[l, bend left, "\alpha^{(2)}"] \end{tikzcd}$$

$I_t^{(1)} = (\nu^2\alpha^{(1)}, \alpha^{(2)}\nu^2, \alpha^{(1)}\alpha^{(2)}\alpha^{(1)}, \alpha^{(2)}\alpha^{(1)}\alpha^{(2)}, \alpha^{(1)}\alpha^{(2)} - t\alpha^{(1)}\nu\alpha^{(2)}$, $\alpha^{(2)}\alpha^{(1)} - \nu^3)$ (we also set $w_1 := \varepsilon_1$, $w_2 := \alpha^{(1)}$, $\omega := \nu$ and $r := 3$);

$$Q^{(2)}:\ \begin{tikzcd} 1 \arrow[loop left, "\nu"] \arrow[r, bend left, "\alpha^{(1)}"] & 2 \arrow[l, bend left, "\alpha^{(2)}"] \arrow[loop right, "\gamma"] \end{tikzcd}$$

$I_t^{(2)} = I^{(2)}(h)_t = (\nu^4, \nu^2\alpha^{(1)}, \alpha^{(2)}\nu^2, \alpha^{(1)}\alpha^{(2)} - \nu^2 + t\nu^3, \alpha^{(2)}\alpha^{(1)} - h\gamma^2$, $\nu\alpha^{(1)} - \alpha^{(1)}\gamma, \alpha^{(2)}\nu - \gamma\alpha^{(2)})$, where $h \in k\setminus\{0,1\}$ is fixed ($w_1 := \nu$, $w_2 := \alpha^{(1)}$, $\omega := \nu$, $r := 2$);

$$Q^{(3)}:\ \begin{tikzcd} & 3 \arrow[dr, "\alpha^{(2)}"] & \\ 1 \arrow[ur, "\alpha^{(1)}"] \arrow[rr, shift left, "\nu"] & & 2 \arrow[ll, shift left, "\gamma"] \end{tikzcd}$$

$I_t^{(3)} = (\alpha^{(1)}\alpha^{(2)} - \nu\gamma\nu, (\gamma\nu)^3\gamma, \alpha^{(2)}\gamma\alpha^{(1)}\alpha^{(2)}, \alpha^{(1)}\alpha^{(2)}\gamma\alpha^{(1)}$, $\alpha^{(2)}\gamma\alpha^{(1)} - t\alpha^{(2)}\gamma\nu\gamma\alpha^{(1)})$ ($w_1 := \varepsilon_1$, $w_2 := \nu$, $w_3 := \alpha^{(1)}$, $\omega := \nu\gamma$, $r := 2$);

$$Q^{(4)}: \qquad \begin{tikzcd} 2 \arrow[r, shift left, "\alpha^{(2)}"] & 1 \arrow[l, shift left, "\alpha^{(1)}"] \arrow[loop above, "\nu"] \arrow[r, shift left, "\gamma"] & 3 \arrow[l, shift left, "\delta"] \end{tikzcd}$$

$I_t^{(4)} = (\alpha^{(1)}\alpha^{(2)} - \nu^2, \nu^3 - \gamma\delta, \alpha^{(2)}\gamma, \delta\alpha^{(1)}, \nu\gamma, \delta\nu, \alpha^{(1)}\alpha^{(2)}\alpha^{(1)}, \alpha^{(2)}\alpha^{(1)}\alpha^{(2)}$, $\alpha^{(2)}\alpha^{(1)} - t\alpha^{(2)}\nu\alpha^{(1)})$ ($w_1 := \nu$, $w_2 := \alpha^{(1)}$, $w_3 := \gamma$, $\omega := \nu$, $r := 2$);

$$Q^{(5)}: \qquad \begin{tikzcd} 2 \arrow[r, shift left, "\alpha^{(2)}"] & 1 \arrow[l, shift left, "\alpha^{(1)}"] \arrow[r, shift left, "\gamma"] & 3 \arrow[l, shift left, "\delta"] \end{tikzcd}$$

$I_t^{(5)} = (\alpha^{(2)}\gamma\delta\gamma, \delta\gamma\delta\alpha^{(1)}, \alpha^{(1)}\alpha^{(2)}\alpha^{(1)}, \alpha^{(2)}\alpha^{(1)}\alpha^{(2)}, \alpha^{(2)}\alpha^{(1)} - t\alpha^{(2)}\gamma\delta\alpha^{(1)}$, $\alpha^{(1)}\alpha^{(2)} - (\gamma\delta)^2)$ ($w_1 := \varepsilon_1$, $w_2 := \alpha^{(1)}$, $w_3 := \gamma$, $\omega := \gamma\delta$, $r := 2$);

$$Q^{(6)}: \qquad \begin{tikzcd} & 3 \arrow[dr, "\delta"] & \\ 1 \arrow[loop left, "\nu"] \arrow[ur, "\gamma"] \arrow[rr, shift left, "\alpha^{(1)}"] & & 2 \arrow[ll, shift left, "\alpha^{(2)}"] \end{tikzcd}$$

$I^{(6)} = (\alpha^{(2)}\alpha^{(1)} - t\alpha^{(2)}\nu\alpha^{(1)}, \nu\gamma, \nu\alpha^{(1)} - \gamma\delta, \delta\alpha^{(2)}\nu, \alpha^{(1)}\alpha^{(2)} - \nu^2, \delta\alpha^{(2)}\alpha^{(1)}$, $\alpha^{(1)}\alpha^{(2)}\alpha^{(1)}, \alpha^{(2)}\alpha^{(1)}\alpha^{(2)})$ ($w_1 := \varepsilon_1$, $w_2 := \alpha^{(1)}$, $w_3 := \gamma$, $\omega := \nu$, $r := 2$);

$$Q^{(7)}: \qquad \begin{tikzcd} & 3 \arrow[dl, "\gamma"'] & \\ 1 \arrow[loop left, "\nu"] \arrow[rr, shift right, "\alpha^{(1)}"'] & & 2 \arrow[ll, shift right, "\alpha^{(2)}"'] \arrow[ul, "\delta"'] \end{tikzcd}$$

$I_t^{(7)} = (\alpha^{(2)}\alpha^{(1)} - t\alpha^{(2)}\nu\alpha^{(1)}, \gamma\nu, \alpha^{(2)}\nu - \delta\gamma, \nu\alpha^{(1)}\delta, \alpha^{(1)}\alpha^{(2)} - \nu^2, \alpha^{(2)}\alpha^{(1)}\delta$, $\alpha^{(1)}\alpha^{(2)}\alpha^{(1)}, \alpha^{(2)}\alpha^{(1)}\alpha^{(2)})$ ($w_1 := \varepsilon_1$, $w_2 := \alpha^{(1)}$, $w_3 := \alpha^{(1)}\delta$, $\omega := \nu$, $r := 2$).

Set $\bar{R}_{(n)} = R_0^{(n)}$, $R'_{(n)} = R_1^{(n)}$ and $\tilde{R}_{(n)} = R(\tilde{Q}^{(n)}, \tilde{I}^{(n)})$, where $(\tilde{Q}^{(n)}, \tilde{I}^{(n)})$ is a universal covering of $(Q^{(n)}, I_0^{(n)})$ for $a_0 = 1$. Now we can prove the existence of almost Galois coverings of integral type for $R'_{(1)}, \ldots, R'_{(7)}$, which behave in a more regular way than usual (see [8, Theorem 4.3.1]).

THEOREM. *For each $n = 1, \ldots, 7$, the fundamental group $\Pi(Q^{(n)}, I_0^{(n)}) = (\Pi(Q^{(n)}, I_0^{(n)}), a_0)$ is an infinite cyclic group generated by $[\omega]$, where $\omega = \omega(n)$ is as above, and there exists an almost Galois G-covering $F'_{(n)} : \tilde{R}_{(n)} \to R'_{(n)}$ of integral type, with $G = \Pi(Q^{(n)}, I_0^{(n)})$ $(= [\omega]^{\mathbb{Z}})$, such that $F'_{(n)}$ has properties* 3.2(a)–(c). *More precisely, for $F' : \tilde{R} \to R'$ of the form $F' = F'_{(n)}$, for $n = 1, \ldots, 7$, we have the following:*

- *for any $x, y \in \operatorname{ob} R'$, $\alpha \in R'(y, x)$, and $x_i \in F'^{-1}(x)$, $y_{i'} \in F'^{-1}(y)$ we have $g({}_{x_i}\dot{\alpha}_{y_{i'}}) = {}_{gx_i}\dot{\alpha}_{gy_{i'}}$, where $g = [\omega]^{-1}$ $(= -1)$; consequently, $F'_\rho \circ {}^1(-) \cong F'_\lambda$;*

- *$F'_\bullet \cong {}^rF'_\bullet$, where $r = r(n)$ is as above;*
- *for N in $\operatorname{ind}\tilde{R}$, the R-isomorphism $F'_\bullet F'_\lambda(N) \cong \bigoplus_{j\in\mathbb{Z}} {}^iN$ holds, provided $\operatorname{Ext}^1_{\tilde{R}}({}^iN,N) = 0$ for all $i = 1,\dots,r-1$.*

Proof. We start by noticing that the proof of the first assertion is simply a case-by-case direct verification using only the definition of $\Pi((Q^{(n)}, I_0^{(n)}), a_0)$. To prove the main assertion we show that the functors $F'_{(n)}$ defined in [8, 4.2] satisfy the assumptions of Theorem 3.2. This clearly implies that they have the asserted properties.

In fact, the whole necessary information is contained in the table below:

n	s	$\kappa^{(1)}$	$\kappa^{(2)}$	m_1	m_2	j	$\nu_{j,1}$	$f_{j,1}$	$\nu_{j,2}$	$f_{j,2}$	$\underline{b}_{\alpha^{(1)},n}$	$\underline{b}_{\alpha^{(2)},n}$	c
1	1	ε_2	ε_1	1	1	1	ν^3	1	$\alpha^{(2)}\nu\alpha^{(1)}$	t	$-t\lfloor n+1/3\rfloor$	$t\lfloor n/3\rfloor$	t
2	1	ε_2	ε_1	2	1	1	ν^2	1	γ^2	u	$-t\lfloor n/2\rfloor$	$t\lfloor (n+1)/2\rfloor$	t
						2	ν^3	$-t$	$-$	$-$			
3	1	ε_2	γ	1	1	1	$\nu\gamma\nu$	1	$\alpha^{(2)}\gamma\nu\gamma\alpha^{(1)}$	t	$-t\lfloor n/2\rfloor$	$t\lfloor n/2\rfloor$	t
4	1	ε_2	ε_1	1	1	1	ν^2	1	$\alpha^{(2)}\nu\alpha^{(1)}$	t	$-t\lfloor n/2\rfloor$	$t\lfloor (n-1)/2\rfloor$	t
5	1	ε_2	ε_1	1	1	1	$(\gamma\delta)^2$	1	$\alpha^{(2)}\gamma\delta\alpha^{(1)}$	t	$-t\lfloor n/2\rfloor$	$t\lfloor n/2\rfloor$	t
6	1	ε_2	ε_1	1	1	1	ν^2	1	$\alpha^{(2)}\nu\alpha^{(1)}$	t	$-t\lfloor n/2\rfloor$	$t\lfloor n/2\rfloor$	t
7	1	ε_2	ε_1	1	1	1	ν^2	1	$\alpha^{(2)}\nu\alpha^{(1)}$	t	$-t\lfloor n/2\rfloor$	$t\lfloor n/2\rfloor$	t

The proof is very technical and therefore we only give an outline, providing a short hint how to use the data from the table. For the benefit of the reader we briefly recall the construction of the functors $F'_{(n)}$.

We first define functors $F_{(n)} : R(\tilde{Q}^{(n)}) \to R'_{(n)}$. We apply the notation established in 3.1, which refers to the choice of generators $[\omega]$ and the sets $\{w_a\}_{a\in Q_0^{(n)}}$ fixed above. For any arrow β in $Q^{(n)}$ different from $\alpha^{(1)}$ and $\alpha^{(2)}$, we set $F_{(n)}(\tilde{\beta}_i) = \beta + I_1^{(n)}$ for all $i \in \mathbb{Z}$. The values $F_{(n)}(\widetilde{\alpha^{(1)}}_i)$ and $F_{(n)}(\widetilde{\alpha^{(2)}}_i)$ for $i \in \mathbb{Z}$ are given by the formulas

$$(*)\quad F_{(n)}(\widetilde{\alpha^{(1)}}_i) := \alpha^{(1)} + b_{\alpha^{(1)},i}\omega\alpha^{(1)}, \qquad F_{(n)}(\widetilde{\alpha^{(2)}}_i) := \alpha^{(2)} + b_{\alpha^{(2)},i}\alpha^{(2)}\omega,$$

where $b_{\alpha^{(1)},i}$ and $b_{\alpha^{(1)},i}$ are equal, respectively, to the coefficients from the column with the labels "$\underline{b}_{\alpha^{(1)},i}$" and "$\underline{b}_{\alpha^{(1)},i}$" in the table, evaluated at $t = 1$. It is not hard to check that for every n, all the assumptions of Theorem 3.1 are satisfied, and that conditions 3.1(a)–(c) hold for $F_{(n)}$. Consequently, each $F_{(n)}$ induces a functor $F'_{(n)} : \tilde{R}_{(n)} \to R'_{(n)}$ which is an almost Galois G-covering of integral type, where $G = \Pi(Q^{(n)}, I_0^{(n)})$.

Now, returning to our main goal, observe that by the very construction the functors $F'_{(n)}$ satisfy condition 3.2(✠). Moreover, we can easily find that the collection $K = K(n) := (\omega; \alpha^{(1)}, \alpha^{(2)}; \kappa^{(1)}, \kappa^{(2)})$, where $\kappa^{(1)}, \kappa^{(2)}$ are as in the table, forms an s-flower in each $Q^{(n)}$, for $s=1$. Therefore, there only remains the most technical task: we have to check that all the conditions from (i)–(iv) in Theorem 3.2 hold. We again proceed by case-by-case inspection, applying the table. Note that all the necessary data are contained in the remaining columns and they can be easily recovered, since the names of column labels are adjusted precisely to the notation from Theorem 3.2. We leave all the computations to the reader. ∎

REMARK. (a) The algebras $R'_{(n)}$, $n=1,\dots,7$, belong to the ten-element list consisting of basic (nonstandard) selfinjective algebras socle equivalent to selfinjective algebras of tubular type, which themselves are are not of tubular type, given in [2]. One of the remaining three members of this list admits some other kind of covering, which is close to those discussed above. The other two seem to behave in a quite different way.

(b) The theorem remains valid if for $R'_{(n)}$ we take the algebras $R_t^{(n)}$ for $t \in k \setminus \{0\}$. More precisely, for any $t \in k$, replacing in the formulas $(*)$ the coefficients $b_{\alpha^{(1)},i} := \underline{b}_{\alpha^{(1)},i}(1)$ and $b_{\alpha^{(2)},i} := \underline{b}_{\alpha^{(2)},i}(1)$, respectively, by $\underline{b}_{\alpha^{(1)},i} = \underline{b}_{\alpha^{(1)},i}(t)$ and $\underline{b}_{\alpha^{(2)},i} = \underline{b}_{\alpha^{(2)},i}(t)$ from the table, we can construct almost Galois G-covering functors $F'^{(t)}_{(n)} : \tilde{R}_{(n)} \to R_t^{(n)}$ of integral type, with $G = \Pi(Q^{(n)}, I_0^{(n)})$. Clearly, $F'^{(1)}_{(n)} = F'_{(n)}$ and $F'^{(0)}_{(n)} = \bar{F}_{(n)}$, where $\bar{F}_{(n)} : \tilde{R}_{(n)} \to \bar{R}^{(n)}$ is a canonical Galois covering functor with group G. For each n, the functors $F'^{(t)}_{(n)}$, $t \in k$, form a geometric family of functors, which defines a degeneration of $F'_{(n)}$ to $\bar{F}_{(n)}$ (see [10, Definitions 2.3 and 2.4, Theorem 2.6]), and they all have the properties as in Theorem 3.8.

(c) In the last column of the table we provide for illustration the values of the constants c from Lemma 3.4 for all the functors $F'^{(t)}_{(n)}$.

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