

*SINCERE POSETS OF FINITE PRINJECTIVE TYPE  
WITH THREE MAXIMAL ELEMENTS AND THEIR  
SINCERE PRINJECTIVE REPRESENTATIONS*

BY

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**Abstract.** Assume that  $K$  is an arbitrary field. Let  $(I, \preceq)$  be a poset of finite prinjective type and let  $KI$  be the incidence  $K$ -algebra of  $I$ . A classification of all sincere posets of finite prinjective type with three maximal elements is given in Theorem 2.1. A complete list of such posets consisting of 90 diagrams is presented in Tables 2.2. Moreover, given any sincere poset  $I$  of finite prinjective type with three maximal elements, a complete set of pairwise non-isomorphic sincere indecomposable prinjective modules over  $KI$  is presented in Tables 8.1. The list consists of 723 modules.

**0. Introduction.** Throughout this paper  $I = (I, \preceq)$  is a finite poset (i.e. partially ordered set) with partial order  $\preceq$ . We write  $i \prec j$  if  $i \preceq j$  and  $i \neq j$ . For simplicity we write  $I$  instead of  $(I, \preceq)$ . The poset  $I$  is said to be *connected* if  $I$  is not the union of two proper subposets  $I_1, I_2$  such that  $I_1 \cap I_2 = \emptyset$ . Throughout the paper all posets are assumed to be connected. Following [21] we denote by  $\max I$  the set of all maximal elements of  $I$  (called *peaks* of  $I$ ). The poset  $I$  is called an  *$r$ -peak poset* if  $|\max I| = r$ , where  $|X|$  denotes the cardinality of the set  $X$ . A subposet  $J$  of  $I$  is said to be a *peak subposet* if  $J \cap \max I = \max J$ .

Throughout the paper,  $K$  is a field. We denote by  $KI$  the *incidence  $K$ -algebra* of the poset  $I$ , that is,  $KI$  is the  $K$ -subalgebra of the full  $I \times I$  matrix algebra  $\mathbb{M}_I(K)$  consisting of all matrices  $[\lambda_{ij}]$  in  $\mathbb{M}_I(K)$  such that  $\lambda_{ij} = 0$  if  $i \not\prec j$  in  $(I, \preceq)$  (see [20], [21]). Given  $j \in I$  we denote by  $e_j \in KI$  the standard primitive idempotent corresponding to  $j$ .

A right  $KI$ -module  $X$  is identified with a system

$$X = (X_i; {}_j h_i)_{i,j \in I}$$

where  $X_i = X e_i$  is a finite-dimensional vector space over  $K$  and  ${}_j h_i : X_i \rightarrow X_j$ ,  $i \prec j$ , are  $K$ -linear maps such that  ${}_i h_i = \text{id}$  and  ${}_t h_j \cdot {}_j h_i = {}_t h_i$  for all  $i \prec j \prec t$  in  $I$ . Such systems are called  *$K$ -linear representations of*

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the poset  $I$ . Throughout this paper we often use the word “representation” instead of “module”.

We denote by  $\text{mod}(KI)$  the category of finitely generated right  $KI$ -modules and by  $\text{prin}(KI)$  the full subcategory of  $\text{mod}(KI)$  consisting of prinjective modules in the sense of the following definition. The right  $KI$ -module  $X$  is called *prinjective* if  $X$  is finitely generated and the right module  $Xe_-$  over the algebra  $KI^- \cong e_-(KI)e_-$  is projective, where  $I^- = I \setminus \max I$  and  $e_- = \sum_{j \in I^-} e_j$ . It is easy to prove that a module  $X$  in  $\text{mod}(KI)$  is prinjective if and only if there exists a short exact sequence

$$(0.1) \quad 0 \rightarrow P_1 \rightarrow P_0 \rightarrow X \rightarrow 0$$

in  $\text{mod}(KI)$ , where  $P_0, P_1$  are projective  $KI$ -modules and  $P_1$  is semisimple of the form  $P_1 = \bigoplus_{p \in \max I} (e_p KI)^{t_p}$ ,  $t_p \geq 0$  (see [17] and [25]).

It follows from [17] that the category  $\text{prin}(KI)$  is additive, has the finite unique decomposition property, is closed under extensions in  $\text{mod}(KI)$ , has Auslander–Reiten sequences, source maps and sink maps, and has enough relative projective and relative injective objects.

Following [21] the poset  $I$  is said to be of *finite prinjective type* if the category  $\text{prin}(KI)$  is of finite representation type, that is, the number of isomorphism classes of indecomposable modules in  $\text{prin}(KI)$  is finite. It follows from [21, Theorem 3.1] that the definition does not depend on the choice of  $K$ .

Let us recall that a complete set of 14 sincere one-peak posets of finite prinjective type and their 42 sincere prinjective representations was given by M. Kleiner in [12], [13] (see also [20, Theorem 10.2 and Tables 10.7] for a correction of Kleiner’s list of [13]). A complete list of 60 sincere two-peak posets of finite prinjective type and their 328 sincere prinjective representations is presented in [14].

In the present paper we continue the study of sincere posets started in [14], where the motivation for the study of prinjective modules is presented. For the motivation see also [2], [28]–[30].

The main aim of this paper is to give a complete set of sincere three-peak posets  $I$  of finite prinjective type (Tables 2.2) and a complete set of indecomposable modules in  $\text{prin}(KI)$  and their coordinate vectors (Tables 8.1). We prove in Theorem 2.1 that there are precisely 90 sincere three-peak posets  $\mathcal{F}_1^{(3)}, \dots, \mathcal{F}_{90}^{(3)}$  of finite prinjective type. Moreover we prove in Theorem 2.1 that every sincere indecomposable prinjective module over the  $K$ -algebra  $K\mathcal{F}_j^{(3)}$ ,  $1 \leq j \leq 90$ , is isomorphic to one of the sincere prinjective modules  $M_j^{(i,3)}$  listed in Tables 8.1. The total number of sincere indecomposable prinjective representations of three-peak posets of finite prinjective type is 723. Finally, we show in Theorem 7.4 that the incidence  $K$ -algebra  $KI$  of

any sincere poset of finite prinjective type is a tilted algebra and therefore  $\text{gl.dim } KI \leq 2$ .

Our construction of posets is presented in Section 6 and applies two computer implementable algorithms (Algorithms 6.7 and 6.8). We construct inductively sincere posets  $I$  of finite prinjective type such that  $|I| = n + 1$  from sincere posets  $J$  of finite prinjective type such that  $|J| = n$ . Here we develop the technique introduced in [14].

The main results of this paper were presented to the International Conference “Representations Theory of Algebras” in Beijing (August 2000). A similar problem in the setting of weakly positive quadratic forms was studied by M. V. Zeldich in [27].

The author would like to express her sincere thanks and appreciation to Professor Daniel Simson for formulating the problem, for his careful reading of the manuscript and for helpful suggestions and comments. The author wishes to thank Dr. Stanisław Kasjan for remarks on the proof of Theorem 7.4.

**1. Preliminaries and notation.** Following [17] and [21] we define the coordinate vector  $\mathbf{cdn } X \in \mathbb{N}^I$  of a module  $X$  in  $\text{mod}(KI)$  by

$$(1.1) \quad (\mathbf{cdn } X)(j) = \begin{cases} \dim_K(X_j) & \text{for } j \in \max I, \\ \dim_K(\text{top } X)e_j & \text{for } j \in I \setminus \max I, \end{cases}$$

where  $\text{top } X = X/XJ(KI)$  and  $J(KI)$  is the Jacobson radical of the algebra  $KI$ . We view  $\mathbf{cdn } X$  as a map  $\mathbf{cdn } X : I \rightarrow \mathbb{N}$ .

We recall that a vector  $x \in \mathbb{N}^n$  is said to be *sincere* if  $x_j \neq 0$  for  $j \in \{1, \dots, n\}$ . Moreover the poset  $I$  is said to be *sincere* if there exists an indecomposable module  $X$  in  $\text{prin}(KI)$  with sincere coordinate vector.

Following [6] and [17] we associate with any poset  $I$  the Tits quadratic form  $q_I : \mathbb{Z}^I \rightarrow \mathbb{Z}$ ,

$$(1.2) \quad q_I(x) = \sum_{i \in I} x_i^2 + \sum_{\substack{i < j \\ j \in I \setminus \max I}} x_i x_j - \sum_{p \in \max I} \left( \sum_{i < p} x_i \right) x_p.$$

We frequently use the following characterisation of posets of finite prinjective type given in [21, Theorem 3.1] and completed in [16] (cf. also [4]).

**THEOREM 1.3.** *For a finite poset  $I$  the following conditions are equivalent.*

- (a)  $I$  is of finite prinjective type.
- (b) The quadratic form  $q_I$  is weakly positive, i.e.  $q_I(v) > 0$  for any non-zero vector  $v \in \mathbb{N}^I$ .
- (c) There exists a preprojective component  $\tilde{\mathcal{P}}(I)$  of the Auslander–Reiten translation quiver  $\Gamma(\text{prin}(KI))$  of the category  $\text{prin}(KI)$ , and  $\Gamma(\text{prin}(KI)) = \tilde{\mathcal{P}}(I)$ .

- (d) The set  $\mathcal{R}_{q_I}^+ = \{v \in \mathbb{N}^f : q_I(v) = 1\}$  of positive roots of  $q_I$  is finite.
- (e)  $q_I(\mathbf{cdn} X) = 1$  for every indecomposable module  $X$  in  $\text{prin}(KI)$ .
- (f) The map  $X \mapsto \mathbf{cdn} X$  defines a bijection  $\Gamma(\text{prin}(KI)) \rightarrow \mathcal{R}_{q_I}^+$ . ■

The reader is referred to [2], [4], [18] and [20] for the definition of an Auslander–Reiten translation quiver of  $\text{mod}(KI)$ ,  $\text{prin}(KI)$  and a preprojective component.

**2. The main classification theorem.** One of the main aims of this paper is the following result.

**THEOREM 2.1.** *Let  $K$  be an arbitrary field.*

(a) *A three-peak poset  $I$  of finite prinjective type is sincere if and only if  $I$  has one of the 90 forms in Tables 2.2 below.*

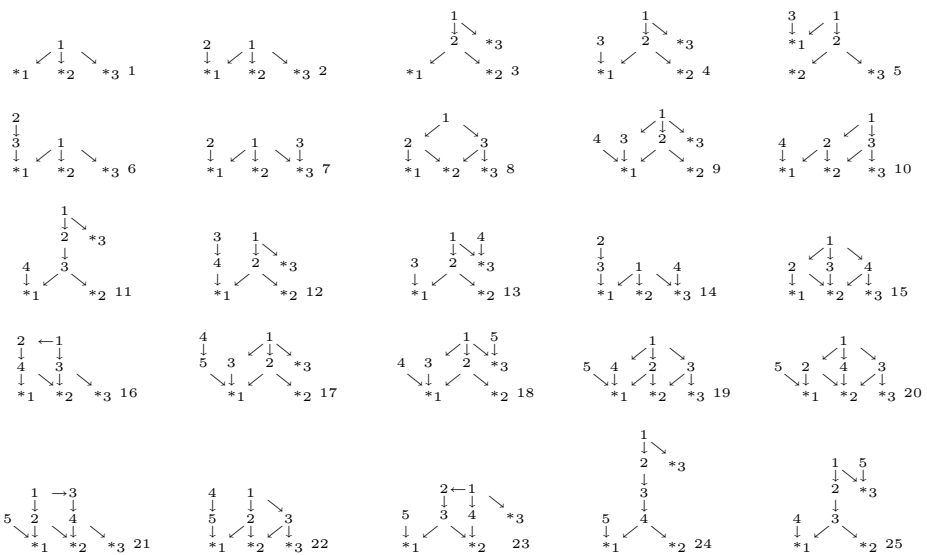
(b) *Let  $\mathcal{F}_i^{(3)}$  be a poset of Tables 2.2 and let  $q_{\mathcal{F}_i^{(3)}}$  be the associated quadratic form. A sincere positive vector  $z \in \mathbb{N}^{\mathcal{F}_i^{(3)}}$  satisfies  $q_{\mathcal{F}_i^{(3)}}(z) = 1$  if and only if  $z$  is one of the vectors  $z_i^{(j,3)}$  in Tables 8.1 of Section 8.*

(c) *An indecomposable prinjective  $K\mathcal{F}_i^{(3)}$ -module  $X$  is sincere if and only if  $X$  is isomorphic to a module  $M_i^{(j,3)}$  in Tables 8.1.*

We prove Theorem 2.1 in Section 7 by developing the methods introduced in [14] and in Section 6 of the present paper.

**Tables 2.2**

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**3. The peak reduction.** The peak reduction described in [11] allows us to reduce some problems for  $r$ -peak posets to corresponding problems for  $(r - 1)$ -peak posets. In Section 7 we apply this idea and its generalisation given below to construct sincere indecomposable prinjective modules.

If  $|\max I| \geq 2$ , then  $q \in \max I$  is said to be a *weak reducible peak* if there exists  $c \in I$  such that

- (i) there is no  $t \in I$  such that  $c \prec t \prec q$ ,
- (ii) the subposet  $I_q := q^\nabla \setminus \{c\}$  of  $I$  is linearly ordered.

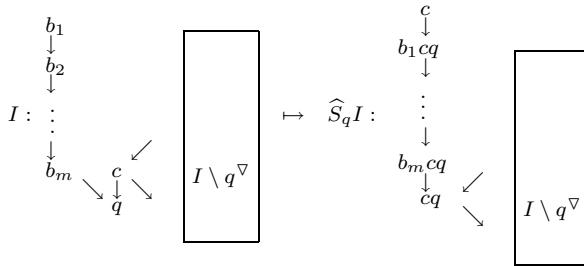
The poset  $I$  is said to be *weak peak reducible* if there exists a weak reducible peak in  $I$ ; otherwise  $I$  is said to be *weak peak irreducible*. If  $q \in I$  is a weak reducible peak, then we define the poset  $\widehat{S}_q I$  as follows. Let  $I_q = \{b_1 \prec \dots \prec b_m\}$ . We set

$$(3.1) \quad \widehat{S}_q I = \{c, b_1 c q, \dots, b_m c q, c q\} \cup (I \setminus q^\nabla)$$

where  $b_1 c q, b_2 c q, \dots, b_m c q, c q$  are new points. The partial order in  $\widehat{S}_q I$  is generated by the partial order  $\preceq$  in  $I \setminus q^\nabla$  and the following relations:

- (i)  $c \prec b_1 c q \prec \dots \prec b_m c q \prec c q$ ;
- (ii)  $c q \prec i$  if  $c \prec i$  in  $I$  and  $i \in I \setminus q^\nabla$ .
- (iii)  $c q \succ i$ , if  $c \succ i$  in  $I$  and  $i \in I \setminus q^\nabla$ .

The construction  $I \mapsto \widehat{S}_q I$  can be visualised as follows:



Following [11, Section 5] we prove some properties of the weak peak reduction. We define a pair of functors

$$\widetilde{G}_q : \text{prin}(K\widehat{S}_q I) \rightarrow \text{prin}(KI), \quad \widetilde{F}_q : \text{prin}(KI) \rightarrow \text{mod}(K\widehat{S}_q I)$$

as follows. Given a prinjective  $K\widehat{S}_q I$ -module  $X = (X_{i, j} h_l)_{i, j, l \in \widehat{S}_q I, l \prec j}$  we set  $\widetilde{G}_q(X) = (\overline{X}_{i, j} \overline{h}_l)_{i, j, l \in I, l \prec j}$ , where

$$\overline{X}_i = \begin{cases} X_i & \text{if } i \in I \setminus q^\nabla, \\ X_{c q} & \text{if } i = c, \\ X_{c q} / \text{Im } c q h_c & \text{if } i = q, \\ X_{i c q} / \text{Im } i c q h_c & \text{if } i \in I_q, \end{cases}$$

$\overline{h}_t = j h_t$  for  $j, t \in I \setminus q^\nabla$ ,  ${}_q \overline{h}_c$  is a natural projection and  ${}_b \overline{h}_b, {}_q \overline{h}_b, b, b' \in I_q$ , are induced by  ${}_b c q h_{b c q}, {}_q c q h_{b c q}$ , respectively. Given a prinjective  $KI$ -module

$X = (X_i, jh_l)_{i,j,l \in I, l \prec j}$  we set  $\tilde{F}_q(X) = (\hat{X}_i, j\hat{h}_l)_{i,j,l \in \hat{S}_q I, l \prec j}$ , where

$$\hat{X}_i = \begin{cases} X_i & \text{if } i \in I \setminus q^\nabla, \\ X_c & \text{if } i = cp, \\ {}_q h_c^{-1}(\text{Im } {}_q h_b) & \text{if } i = bcq, b \in I_q, \\ {}_q h_c^{-1}(0) & \text{if } i = c, \end{cases}$$

$j\hat{h}_t = jh_t$  for  $t \in I \setminus q^\nabla$ ,  $j\hat{h}_{cq} = jh_c$  for  $j \in I \setminus q^\nabla$  and  $j\hat{h}_t$  are natural embeddings in the remaining cases. The functors  $\tilde{G}_q$  and  $\tilde{F}_q$  are defined on morphisms in a natural way.

Now we show that the functor  $\tilde{G}_q$  carries prinjective modules to prinjective ones. Let  $Y = (Y_i, jf_i) \in \text{prin}(K\hat{S}_q I)$ . Then the  $K(\hat{S}_q I)^-$ -module  $Ye_-$  is projective and therefore the  $K(\hat{S}_q I)^-$ -module  $Ye_-$  has the decomposition  $Ye_- \simeq \bigoplus_{i \in (\hat{S}_q I)^-} (e_i K(\hat{S}_q I)^-)^{t_i}$  for some  $t_i \geq 0$ . We can write

$$\begin{aligned} Ye_- &\simeq \bigoplus_{i=1}^m (e_{b_i c q} K(\hat{S}_q I)^-)^{t_{b_i c q}} \oplus (e_c K(\hat{S}_q I)^-)^{t_c} \oplus (e_{c q} K(\hat{S}_q I)^-)^{t_{c q}} \\ &\quad \oplus \bigoplus_{i \in I \setminus q^\nabla} (e_i K(\hat{S}_q I)^-)^{t_i}. \end{aligned}$$

Note that

$$(\tilde{G}_q(Y))e_- \simeq \bigoplus_{i=1}^m (e_{b_i c q} K I^-)^{t_{b_i c q} - t_c} \oplus (e_c K I^-)^{t_{c q}} \oplus \bigoplus_{i \in I \setminus q^\nabla} (e_i K I^-)^{t_i}$$

is a projective  $KI^-$ -module and therefore the  $KI$ -module  $\tilde{G}_q(Y)$  is prinjective.

Let us warn the reader that the functor  $\tilde{F}_q$  does not carry prinjective modules to prinjective ones, in general.

LEMMA 3.2. *The functors  $\tilde{G}_q$  and  $\tilde{F}_q$  induce equivalences of categories*

$$(3.3) \quad (\text{prin}(KI))'_q \begin{matrix} \xrightarrow{\tilde{F}'_q} \\ \xleftarrow{\tilde{G}'_q} \end{matrix} \text{prin}(K\hat{S}_q I)$$

inverse to each other, where  $(\text{prin}(KI))'_q$  is the full subcategory of  $\text{prin}(KI)$  consisting of the objects  $X$  without direct summands from  $\text{prin}(K(q^\nabla \setminus c^\nabla))$  and such that  $\tilde{F}_q(X)$  is a prinjective  $\hat{S}_q I$ -module.

*Proof.* Note that it is enough to prove the following four statements:

- (1) if  $X \in (\text{prin}(KI))'_q$ , then  $\tilde{G}'_q(\tilde{F}'_q(X)) \simeq X$ ,
- (2) if  $Y \in \text{prin}(K\hat{S}_q I)$ , then  $\tilde{F}'_q(\tilde{G}'_q(Y)) \simeq Y$ ,
- (3) if  $f : X \rightarrow Y$  is a morphism in  $(\text{prin}(KI))'_q$ , then  $\tilde{G}'_q(\tilde{F}'_q(f)) \simeq f$ ,
- (4) if  $f : X \rightarrow Y$  is a morphism in  $\text{prin}(K\hat{S}_q I)$ , then  $\tilde{F}'_q(\tilde{G}'_q(f)) \simeq f$ .

For (1), let  $X = (X_i, {}_j h_i) \in (\text{prin}(KI))'_q$ . Since  $X$  is prinjective, the linear maps  ${}_j h_{b_i}$  are injective. Moreover  ${}_q h_{b_m}$  is injective and  ${}_q h_c$  is surjective, because  $X \in (\text{prin}(KI))'_q$ . This yields the isomorphisms

$$\begin{aligned} \widetilde{G}'_q(\widetilde{F}'_q(X))_{b_i} &= {}_q h_c^{-1}(\text{Im } {}_q h_{b_i}) / \text{Im } {}_{b_i c q} \widehat{h}_c \simeq {}_q h_c^{-1}(\text{Im } {}_q h_{b_i}) / {}_q h_c^{-1}(0) \\ &\simeq \text{Im } {}_q h_{b_i} \simeq X_{b_i}, \\ \widetilde{G}'_q(\widetilde{F}'_q(X))_q &= X_c / \text{Im } {}_{c q} \widehat{h}_c \simeq X_c / {}_q h_c^{-1}(0) \simeq X_q. \end{aligned}$$

Moreover immediately from the definitions of the functors  $\widetilde{G}'_q$  and  $\widetilde{F}'_q$  it follows that  $\widetilde{G}'_q(\widetilde{F}'_q(X))_c = X_c$  and  $\widetilde{G}'_q(\widetilde{F}'_q(X))_i = X_i$  for the remaining  $i \in I$ . Hence (1) follows easily.

The proofs of (2), (3) and (4) are similar. ■

**4. Tits forms and their positive roots.** Tits forms and their positive roots are fundamental in our considerations of Sections 5 and 6. Below we collect some properties of positive roots of integral quadratic forms. For the proofs the reader is referred to [10], [14] and [18].

Let  $q : \mathbb{Z}^n \rightarrow \mathbb{Z}$  be an *integral quadratic form*, that is,

$$(4.1) \quad q(x_1, \dots, x_n) = \sum_{i=1}^n x_i^2 + \sum_{i < j} \alpha_{ij} x_i x_j,$$

where  $x = (x_1, \dots, x_n) \in \mathbb{Z}^n$  and  $\alpha_{ij} \in \mathbb{Z}$ .

We call the vector  $x$  *positive* if  $x \neq 0$  and  $x_i \geq 0$  for all  $i = 1, \dots, n$ . The integral quadratic form (4.1) is said to be *weakly positive* provided  $q(x) > 0$  for all positive vectors  $x \in \mathbb{Z}^n$ . A vector  $x \in \mathbb{Z}^n$  satisfying  $q(x) = 1$  is called a *root* of  $q$ . If in addition  $x$  is positive we call it a *positive root* of  $q$ . It is well known that a weakly positive integral quadratic form has only finitely many positive roots (see [18, Section 1]).

A weakly positive integral quadratic form is said to be *sincere* provided there exists a sincere positive root  $x$  of  $q$ .

For each  $i = 1, \dots, n$  we denote by  $e(i)$  the vector of  $\mathbb{Z}^n$  having 1 at the  $i$ th position and zeros elsewhere.

Let  $(-, -)_q$  be the symmetric  $\mathbb{Z}$ -bilinear form corresponding to  $q$ , that is,

$$2(x, y)_q = q(x + y) - q(x) - q(y).$$

Following [18] we define the  $\mathbb{Z}$ -linear form  $D_i q : \mathbb{Z}^n \rightarrow \mathbb{Z}$  by

$$D_i q(x) := 2(e(i), x) = 2x_i + \sum_{i \neq j} \alpha_{ij} x_j.$$

Let  $q$  be the quadratic form (4.1). The following proposition is proved in [18].



PROPOSITION 4.2. (a) If  $x$  is a root of an integral quadratic form (4.1), then  $\sum_{i=1}^n x_i D_i q(x) = 2$ .

(b) Let  $x = (x_1, \dots, x_n)$  be a positive root of a weakly positive integral quadratic form (4.1) and let  $i \in \{1, \dots, n\}$  be such that  $x_i \neq 0$ . If  $x \neq e(i)$ , then  $|D_i q(x)| \leq 1$ .

(c) Let  $x = (x_1, \dots, x_n)$  be a positive root of a weakly positive integral quadratic form (4.1). If  $x \neq e(j)$  for  $j = 1, \dots, n$ , then there exists  $i \in \{1, \dots, n\}$  such that  $x_i \neq 0$  and  $D_i q(x) = 1$ .

(d) If  $x = (x_1, \dots, x_n)$  is a root of an integral quadratic form (4.1), then  $x - D_i q(x)e(i)$  is a root of  $q$  for each  $i = 1, \dots, n$ . ■

The following fact, whose proof may be found in [14], is essentially used throughout this paper.

PROPOSITION 4.3. (a) If there is a sincere positive root  $x = (x_1, \dots, x_n)$  of a weakly positive integral quadratic form (4.1), then there exists a sincere positive root  $y = (y_1, \dots, y_n)$  of  $q$  such that

$$(4.4) \quad D_i q(y) = 1 \quad \text{implies} \quad y_i = 1,$$

for any  $i = 1, \dots, n$ .

(b) Let  $q : \mathbb{Z}^n \rightarrow \mathbb{Z}$ ,  $n > 0$ , be a quadratic form and let  $z$  be a root of  $q$ . Then  $z \pm e(i)$  is a root of  $q$  if and only if  $D_i q(z) = \mp 1$ . ■

**5. Properties of sincere posets of finite prinjective type.** Fix an integer  $r \geq 1$ . The following lemma shows that there are only finitely many  $r$ -peak posets of finite prinjective type. A similar result, with a weaker estimate, was presented by S. Kasjan in [8].

LEMMA 5.1. Let  $r > 1$  be an integer and let  $I$  be a sincere connected  $r$ -peak poset of finite prinjective type. Then  $|I| \leq 13r - 1$ .

*Proof.* Let  $z$  be a sincere positive root of the quadratic form (1.2). From Proposition 4.2(b) it follows that  $|D_i q_I(z)| \leq 1$  for any  $i \in I$ . We can assume that  $D_i q_I(z) \geq 0$  for each  $i \in I$ . Indeed, if there exists  $i_1 \in I$  such that  $D_{i_1} q_I(z) = -1$ , then  $z^{(1)} = z + e(i_1) \neq z$  is a positive root of  $q_I$  such that  $z_{i_1}^{(1)} > z_{i_1}$  (see Proposition 4.2(d)). If there exists  $i_2 \in I$  such that  $D_{i_2} q_I(z^{(1)}) = -1$ , then  $z^{(2)} = z^{(1)} + e(i_2) \neq z^{(1)}$  is a positive root of  $q_I$  and  $z_{i_2}^{(2)} > z_{i_2}^{(1)}$ . In this way we construct a chain  $z, z^{(1)}, z^{(2)}, \dots$  of pairwise different positive roots of  $q_I$ . Since  $I$  is of finite prinjective type, according to the Theorem of Drozd (see [18, Section 1]) the form  $q_I$  has only finitely many positive roots and we conclude that there exists  $j \in \mathbb{N}$  such that  $D_i q_I(z^{(j)}) > -1$  for all  $i \in I$ . It follows that there exists a positive root  $z$  satisfying  $D_i q_I(z) \geq 0$  for each  $i \in I$ .

In particular,  $D_p q_I(z) \geq 0$  for each  $p \in \max I$ . On the other hand, Theorem 1 of Ovsienko [18, Section 1] yields  $D_p q_I(z) = 2z_p - \sum_{i \prec_p} z_i \leq 12 - \sum_{i \prec_p} z_i$  for  $p \in \max I$ . Hence there exist at most 12 elements  $i \in I$  such that  $i \prec p$ . Since  $I$  is connected, there exists at least one  $i \in I$  satisfying  $i \prec p_1$  and  $i \prec p_2$  for some  $p_1 \neq p_2, p_1, p_2 \in \max I$ . It follows that  $|I| \leq 12r - 1 + r = 13r - 1$ . ■

LEMMA 5.2. *Let  $I$  be a sincere connected  $r$ -peak poset and let  $z$  be a sincere positive root of the Tits form  $q_I$ . Let  $i \in I$  be such that  $D_i q_I(z) = 1$  and  $z_i = 1$ . Then the poset  $J = I \setminus \{i\}$  is sincere and connected.*

*Proof.* Let  $x \in \mathbb{N}^J$  be a vector such that  $x_j = z_j$  for  $j \in J$ . By Proposition 4.3(b),  $x$  is a sincere root of  $q_J$ . Consequently,  $q_J$  is sincere, and therefore so is  $J$ . To show that  $J$  is connected, assume that  $J = J_1 \cup J_2$ , where  $J_1 \neq \emptyset, J_2 \neq \emptyset$  are peak subposets such that  $J_1 \cap J_2 = \emptyset$ . Then  $x^{(1)}, x^{(2)}$  are sincere roots of  $q_{J_1}, q_{J_2}$ , respectively, where  $x_j^{(i)} = x_j$  for  $i = 1, 2$  and  $j \in J_i$ . Hence

$$1 = q_J(x) = q_{J_1}(x^{(1)}) + q_{J_2}(x^{(2)}) = 1 + 1 = 2,$$

which is a contradiction. ■

Let  $I$  be a poset. For any  $i \in I$  we define the following two subposets of  $I$ :

$$i^\Delta = \{s \in I^- : s \succeq i\}, \quad i^\nabla = \{s \in I^- : s \preceq i\}.$$

The following theorem provides us with a main tool for our algorithms of Section 6 (compare with [14, Theorem 5.2]).

THEOREM 5.3. *Let  $I$  be a connected sincere  $r$ -peak poset of finite prinjective type and let  $r \geq 3$ . Then at least one of the following conditions is satisfied:*

(a) *There exists  $j \in I^-$  such that  $J = I \setminus \{j\}$  is a sincere connected  $r$ -peak poset of finite prinjective type.*

(b) *There exist  $p, q \in \max I$  such that  $p \neq q, |p^\nabla| = 1, |q^\nabla| = 1$  and  $J_1 = I \setminus \{p\}, J_2 = I \setminus \{q\}$  are sincere connected  $(r - 1)$ -peak posets of finite prinjective type.*

*Proof.* Let  $z$  be a sincere positive root of  $q_I$  satisfying condition (4.4) for any  $i \in I$ . Proposition 4.2(c) shows that there exists  $p \in I$  such that  $D_p q_I(z) = 1$ . By Lemma 5.2, the poset  $J = I \setminus \{p\}$  is sincere, connected and of finite prinjective type. If  $p \in I^-$ , then  $J$  is an  $r$ -peak poset and (a) is satisfied. Let  $p \in \max I$ . Then

$$1 = D_p q_I(z) = 2z_p - \sum_{i \prec_p} z_i = 2 - \sum_{i \prec_p} z_i,$$

because  $z_p = 1$ . It follows that there exists exactly one  $i \in I^-$  such that  $i \in p^\nabla$  and the set  $p^\nabla$  consists of one element. By Proposition 4.2(a), there

exists  $p \neq q \in I$  such that  $D_{qI}(z) = 1$ . It follows that the poset  $J = I \setminus \{q\}$  is sincere and connected. If  $q \in I^-$ , then (a) is satisfied. If  $q \in \max I$ , then we can show similarly that  $q^\vee$  consists of one element and (b) is satisfied. ■

In view of Theorem 5.3 we introduce the following definition.

DEFINITION 5.4. Let  $r > 1$  be an integer. A sincere  $r$ -peak poset  $(J, \preceq_J)$  of finite prinjective type is said to be  $r$ -minimal if there exists an  $(r-1)$ -peak poset  $(I, \preceq_I)$  satisfying the following conditions:

(1) There exists  $p \in \max I$  such that  $p^\vee$  consists of one element.

(2)  $J = I \cup \{c\}$ , where  $c \in (\max J) \setminus I$ ,  $c^\vee = \{i\}$ ,  $i \in \min I^-$  and there exists a sincere positive root  $z$  of  $q_I$  such that  $z_i = 1$ . ■

Let  $I$  be a sincere  $r$ -peak poset of finite prinjective type. Let  $z$  be a sincere positive root of  $q_I$  satisfying (4.4) for any  $i \in I$ .

In view of Theorem 5.3, if  $I$  is not  $r$ -minimal, then there exists  $i \in I^-$  such that  $I \setminus \{i\}$  is a sincere  $r$ -peak poset. For simplicity such an  $i$  will be chosen in a special way. For this we need the following two lemmata proved in [14].

LEMMA 5.5. Let  $I$  and  $z$  be as above and let  $i \in I^- = I \setminus \max I$  be such that  $D_{iI}(z) = 1$ . Assume that the subset  $i^\vee = \{c_1 \prec \dots \prec c_k = i\}$  of  $I$  is linearly ordered and satisfies the condition

$$(5.6) \quad c_1 \prec p \Rightarrow i \prec p$$

for any  $p \in \max I$ . Then  $D_{c_j I}(z) = 1$  for  $j = 1, \dots, k$ . ■

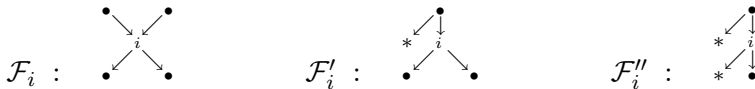
LEMMA 5.7. Let  $I$  and  $z$  be as above and let  $i \in I^-$  be such that  $D_{iI}(z) = 1$ . Assume that the subset  $i^\Delta = \{i = c_1 \prec \dots \prec c_k\}$  of  $I$  is linearly ordered and satisfies the condition

$$(5.8) \quad i \prec p \Rightarrow c_k \prec p$$

for any  $p \in \max I$ . Then  $D_{c_j I}(z) = 1$  for  $j = 1, \dots, k$ . ■

The following lemma is a generalisation of Lemma 5.7 in [14].

LEMMA 5.9. Let  $I$  and  $z$  be as above. Assume that for all  $i \in I^-$  such that  $D_{iI}(z) = 1$  the assumptions of Lemmata 5.5 and 5.7 are not satisfied. Then  $I$  contains a proper peak subposet  $J$  containing one of the following three posets:



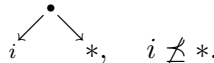
where  $* \in \max I$  and  $\bullet \in I^-$ .

Proof. Let  $z$  be a sincere root of  $q_I$  and let  $i \in I$  be such that  $D_{iI}(z) = 1$ . If the assumptions of Lemma 5.5 are not satisfied, then either  $i^\vee$  is not lin-

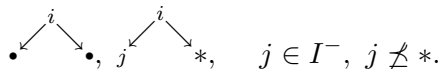
early ordered or (5.6) is not satisfied. In the first case,  $I$  contains the poset



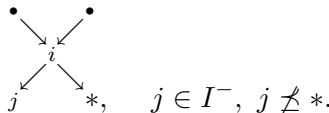
and in the second case, it contains



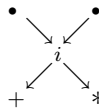
Similarly, if the assumptions of Lemma 5.7 are not satisfied, then  $I$  contains one of the following posets:



Therefore  $I$  contains  $\mathcal{F}_i, \mathcal{F}'_i, \mathcal{F}''_i$  or



Note that in the last case the poset  $I$  contains



for some  $+ \in \max I, + \neq *$  and therefore is of infinite prinjective type (see [21, Theorem 3.1]).

From Proposition 4.2(a) it follows that there exists  $j \in I, j \neq i$ , such that  $D_j q_I(z) = 1$ . Since  $z$  satisfies (4.4) for any  $i \in I$ , we have  $z_j = 1$ . Therefore the poset  $J = I \setminus \{j\}$  is sincere. Now we can show easily that  $J = I \setminus \{j\}$  contains  $\mathcal{F}_i, \mathcal{F}'_i$  or  $\mathcal{F}''_i$ . ■

**6. An algorithmic construction of sincere multi-peak posets of finite prinjective type.** In this section we present algorithms for the construction of sincere  $r$ -peak posets of finite prinjective type (for  $r > 1$ ). We generalise the ideas of [14].

DEFINITION 6.1. We say that a sincere poset  $J = (J, \preceq_J)$  is *dominated by a poset*  $I = (I, \preceq_I)$  (or  $I$  *dominates*  $J$ ) if  $|J| = |I| + 1$  and:

- (a)  $I$  is sincere.
- (b) There exists  $j \in J^-$  such that  $J \setminus \{j\} = I$  and the relation  $\preceq_I$  is the restriction of  $\preceq_J$ .
- (c) There exists a sincere positive root  $z$  of  $q_I$  such that  $\tilde{z}$  is a sincere positive root of  $q_J$ , where  $\tilde{z}_i = z_i$  for  $i \neq j$  and  $\tilde{z}_j = 1$ .

If in addition  $j \in J^-$  is a minimal (resp. maximal) element in  $J^-$ , then we say that  $J$  is *min-dominated* (resp. *max-dominated*) by  $I$ . ■

DEFINITION 6.2. We call a poset  $J$  *iteratively dominated* by a poset  $I$  if:

- (a) The forms  $q_I, q_J$  are weakly positive and sincere, or equivalently,  $I, J$  are sincere posets of finite prinjective type.
- (b) There exists a chain of posets  $J_{(0)}, \dots, J_{(m)}$  such that  $J_{(0)} = I, I_{(m)} = J$  and  $J_{(j)}$  is min- or max-dominated by  $J_{(j-1)}$  for  $j = 1, \dots, m$ . ■

THEOREM 6.3. *Let  $J$  be a sincere  $r$ -peak poset of finite prinjective type, where  $r > 2$ . If  $J$  is not  $r$ -minimal, then there exists a sincere  $r$ -peak poset  $I$  which dominates  $J$ .*

*Proof.* Let  $\tilde{z}$  be a sincere positive root of  $q_J$  satisfying (4.4) for any  $i \in J$ . We show the existence of  $i \in J^-$  such that  $D_i q_J(\tilde{z}) = 1$ . Proposition 4.2(c) shows that there exists  $i \in J$  such that  $D_i q_J(\tilde{z}) = 1$ . If  $i \in \max J$ , then  $|i^\nabla| = 1$ . Moreover there exists  $i \neq j \in J$  such that  $D_j q_J(\tilde{z}) = 1$ . If  $j \in \max J$ , then  $|j^\nabla| = 1$  (cf. the proof of Theorem 5.3). Hence  $I = J \setminus \{j\}$  satisfies conditions (1)–(3) of Definition 5.4, so  $J$  is  $r$ -minimal. We get a contradiction. Therefore there exists  $k \in J^-$  such that  $D_k q_J(\tilde{z}) = 1$  and the poset  $J$  is dominated by the sincere  $r$ -peak poset  $I = J \setminus \{k\}$ . ■

Following [14] we define two families  $\mathcal{X}_I, \mathcal{X}^I$  of posets.

A *family*  $\mathcal{X}_I$ . Let  $I = (I, \preceq)$  be an  $r$ -peak poset such that  $|I| = n, \max I = \{*_1, \dots, *_r\}$  and let  $X_1, \dots, X_k$  be all pairwise different sets consisting of pairwise incomparable elements of  $I^-$ . We define new posets  $(I_1, \preceq_1), \dots, (I_k, \preceq_k), (I_{*_1}, \preceq_{*_1}), \dots, (I_{*_r}, \preceq_{*_r})$  in the following way:

- For  $j = 1, \dots, k$  we set  $I_j = I \cup \{c\}$ , where  $c \notin I$  and  $\preceq_j$  is the smallest partial order relation satisfying:

- (a)  $c \preceq_j i$  for all  $i \in X_j$ ,
- (b)  $i \preceq_j s$  if and only if  $i \preceq s$ , for  $i, s \in I$ .

- For  $j \in \max I$  we set  $I_j = I \cup \{c\}$ , where  $c \notin I$  and  $\preceq_j$  is the smallest partial order relation satisfying:

- (a)  $c \preceq_j j$  ( $j \in \max I$ ),
- (b)  $i \preceq_j s$  if and only if  $i \preceq s$ , for  $i, s \in I$ .

We set  $\mathcal{X}_I = \{I_1, \dots, I_k, I_{*_1}, \dots, I_{*_r}\}$ .

A *family*  $\mathcal{X}^I$ . We define dually a finite family  $\mathcal{X}^I$  as follows. We form new  $r$ -peak posets  $(I^1, \preceq^1), \dots, (I^k, \preceq^k), (I^{*_1}, \preceq^{*_1}), \dots, (I^{*_r}, \preceq^{*_r}), (I^{1,*_l}, \preceq^{1,*_l}), \dots, (I^{k,*_l}, \preceq^{k,*_l}), l = 1, \dots, r, (I^{1,*_i^*j}, \preceq^{1,*_i^*j}), \dots, (I^{k,*_i^*j}, \preceq^{k,*_i^*j}), i \neq j, (i, j) \in \{1, \dots, r\} \times \{1, \dots, r\}$ , in the following way:

- For  $j = 1, \dots, k$  we set  $I^j = I \cup \{c\}$ , where  $c \notin I, c \notin \max I^j$  and  $\preceq^j$  is the smallest partial order relation satisfying:

- (a)  $i \preceq^j c$  for all  $i \in X_j$ ,
- (b)  $i \preceq^j s$  if and only if  $i \preceq s$ , for  $i, s \in I$ .

(c)  $c \preceq^j *$ ,  $*$   $\in \max I$ , if  $i \preceq *$  for all  $i \in X_j$ .

• For  $j = *_1, \dots, *_r$  we set  $I^j = I \cup \{c\}$ , where  $c \notin I$ ,  $c \notin \max I^j$  and  $\preceq^j$  is the smallest partial order relation satisfying:

(a)  $c \preceq^j j$  ( $j = *_1, \dots, *_r$ ),

(b)  $i \preceq^j s$  if and only if  $i \preceq s$ , for  $i, s \in I$ .

• For  $j = 1, \dots, k$  and  $p = *_1, \dots, *_r$  we set  $I^{j,p} = I \cup \{c\}$ , where  $c \notin I$ ,  $c \notin \max I^{j,p}$  and  $\preceq^{j,p}$  is the smallest partial order relation satisfying:

(a)  $i \preceq^{j,p} c$  for all  $i \in X_j$ ,

(b)  $i \preceq^{j,p} s$  if and only if  $i \preceq s$ , for  $i, s \in I$ .

(c)  $c \preceq^{j,p} p$  if  $i \preceq p$  for all  $i \in X_j$ .

• For  $j = 1, \dots, k$  and  $p \neq q$ ,  $p, q \in \{*_1, \dots, *_r\}$  we set  $I^{j,p,q} = I \cup \{c\}$ , where  $c \notin I$ ,  $c \notin \max I^{j,p,q}$  and  $\preceq^{j,p,q}$  is the smallest partial order relation satisfying:

(a)  $i \preceq^{j,p,q} c$  for all  $i \in X_j$ ,

(b)  $i \preceq^{j,p,q} s$  if and only if  $i \preceq s$ , for  $i, s \in I$ .

(c)  $c \preceq^{j,p,q} p$  and  $c \preceq^{j,p,q} q$  if  $i \preceq p$  and  $i \preceq q$  for all  $i \in X_j$ .

REMARK 6.4. Note that not for all  $i = 1, \dots, k$ ,  $p, q \in \max I$  there exist posets  $I^i$ ,  $I^{i,p}$  and  $I^{i,p,q}$  satisfying the required conditions. ■

We set  $\mathcal{X}^I = \{I^1, \dots, I^k, I^{*_1}, \dots, I^{*_r}, I^{1,*_1}, \dots, I^{k,*_1}, \dots, I^{1,*_r}, \dots, I^{k,*_r}, I^{1,*_i,*_j}, \dots, I^{k,*_i,*_j}, i \neq j\}$ .

LEMMA 6.5. *Let  $J$  and  $I$  be connected posets of finite prinjective type.*

(a) *If  $J$  is min-dominated by  $I$ , then  $J \in \mathcal{X}_I$ .*

(b) *If  $J$  is max-dominated by  $I$ , then  $J \in \mathcal{X}^I$ .*

*Proof.* Let  $J$  be a poset of finite prinjective type min-dominated (resp. max-dominated) by a poset  $I$ . By Definition 6.1, there exists a minimal (resp. maximal) element  $c$  of  $J^-$  such that  $I = J \setminus \{c\}$  and  $\preceq_I$  is the restriction of  $\preceq_J$ . Let  $X$  be the set of all  $i \in J^-$ ,  $i \neq c$ , such that the relation  $c \preceq_J i$  (resp.  $i \preceq_J c$ ) is minimal. Note that elements of  $X$  are pairwise incomparable in  $J^-$ .

(a) The proof is similar to that of Lemma 6.3 in [14], so we only sketch it.

First we consider the case  $X \neq \emptyset$ . Since  $X$  consists of pairwise incomparable elements of  $I^-$  we conclude that  $X = X_l$ , for some  $l \geq 1$ , is one of the sets  $X_1, \dots, X_k$  associated with  $I$  in the definition of  $\mathcal{X}_I$ . In this case we have  $J = I_l$ .

In the case  $X = \emptyset$  we have  $J = I_p$  for some  $p \in \max I$  if  $c \preceq_J p$  and  $c \not\preceq_J q$  for  $p \neq q \in \max I$ . Moreover if  $c \preceq_J p$  and  $c \preceq_J q$  for some  $p \neq q$ ,  $p, q \in \max I$ , then the poset  $J$  is of infinite prinjective type, contrary to assumption. Therefore  $c \not\preceq p^\nabla \cap q^\nabla$  for all  $p \neq q$ ,  $p, q \in \max I$ . Hence  $J \in \mathcal{X}_I$ .

(b) First consider the case  $X = \emptyset$ . Then there exists exactly one  $* \in \max I$  such that  $c \preceq *$ . Indeed, if there existed  $* \neq + \in \max I$  such that  $c \preceq +$ , then the subposet  $*^\nabla \cap +^\nabla$  of  $J$  would contain two incomparable elements (because  $I = J \setminus \{c\}$  is connected and  $c$  is the maximal element of  $J^-$ ). Therefore from [21, Theorem 3.1] it follows that  $J$  is of infinite prinjective type, contrary to assumptions. Therefore there exists exactly one  $* \in \max I$  such that  $c \preceq *$ . In this case  $J = I^*$ .

Let  $X \neq \emptyset$ . Consider the following cases.

- (1) There exists exactly one  $* \in \max I$  such that  $c \succ *.$
- (2) There exist exactly two  $* \in \max I$  such that  $c \succ *.$
- (3) There exist exactly three  $* \in \max I$  such that  $c \succ *.$
- (4) There exist at least four  $* \in \max I$  such that  $c \succ *.$

In case (4),  $J$  is of infinite prinjective type (because  $\mathcal{P}_{2,0} \subseteq J$ , see [21, Theorem 3.1]), and therefore (4) does not hold. In case (3) the poset  $J$  is of infinite prinjective type (because  $\mathcal{P}'_{3,0} \subseteq J$ , see [21, Theorem 3.1]), so (3) does not hold.

*Case (1).* Let  $c \prec * \in \max I$  and let  $c \not\prec p \in \max I$  for  $p \neq *.$  If some  $i \in X = X_I$  satisfies:  $i \prec * \in \max I$  and  $i \not\prec p \in \max I$  for  $p \neq *.$  then  $J = I^l \in \mathcal{X}^I$ ; otherwise  $J = I^{l,*} \in \mathcal{X}^I.$

*Case (2).* Let  $c \prec * \in \max I, c \prec + \in \max I, * \neq +,$  and let  $c \not\prec p \in \max I$  for  $p \neq *, +.$  If some  $i \in X = X_I$  satisfies:  $i \prec * \in \max I, i \prec + \in \max I$  and  $i \not\prec p \in \max I$  for  $p \neq *, +,$  then  $J = I^l \in \mathcal{X}^I$ ; otherwise  $J = I^{l,*,+} \in \mathcal{X}^I. \blacksquare$

We set  $\mathcal{X}(I) = \mathcal{X}^I \cup \mathcal{X}_I.$

**COROLLARY 6.6.** *If  $J$  is a connected poset of finite prinjective type min- or max-dominated by a poset  $I,$  then  $J \in \mathcal{X}(I).$   $\blacksquare$*

Denote by  $\mathcal{S}^{(r)}$  the set of all  $r$ -minimal posets and by  $\mathcal{S}^{(r,m)}$  the set of all  $r$ -minimal posets  $J$  such that  $|J| = m.$

**ALGORITHM 6.7** (Construction of sincere posets min- or max-dominated by a given poset).

*Input:* A sincere  $r$ -peak poset  $I$  of finite prinjective type.

*Output:* Sincere  $r$ -peak posets of finite prinjective type min- or max-dominated by  $I.$

*Description of the algorithm:*

STEP 1. Form the sets  $X_1, \dots, X_k$  defined above.

STEP 2. Set  $\mathcal{X}(I) = \mathcal{X}^I \cup \mathcal{X}_I.$

STEP 3. Form the set  $\mathcal{SR}_{q_I}^+$  of all sincere positive roots of  $q_I.$  This set is non-empty and finite because  $I$  is sincere of finite prinjective type.

STEP 4. The output is the set  $\mathcal{Y}(I)$  of all posets  $J$  in  $\mathcal{X}(I)$  which satisfy:

(a)  $J$  is of finite prinjective type.

(b) There exists a sincere positive root  $z$  of  $q_I$  such that  $D_c q_J(\tilde{z}) = 1$ , where  $\{c\} = J \setminus I$ ,  $\tilde{z}_i = z_i$  for  $i \in I$  and  $\tilde{z}_c = 1$ . ■

ALGORITHM 6.8 (Construction of sincere posets iteratively dominated by minimal posets).

*Input:* The set  $\mathcal{S}^{(r)}$  of all  $r$ -minimal posets.

*Output:* All sincere  $r$ -peak posets of finite prinjective type iteratively dominated by posets in  $\mathcal{S}^{(r)}$ .

*Description of the algorithm:*

STEP 1. Apply Algorithm 6.7 to all posets in  $\mathcal{S}^{(r)}$  to get the family  $\mathcal{Y}_{(0)} = \bigcup_{J \in \mathcal{S}^{(r)}} \mathcal{Y}(J)$ .

STEP 2. For  $i \in \mathbb{N}$  define inductively  $\mathcal{Y}_{(i)}$  in the following way. If  $\mathcal{Y}_{(n)}$  is defined for some  $n \in \mathbb{N}$ , apply Algorithm 6.7 to any poset  $I \in \mathcal{Y}_{(n)}$  to get  $\mathcal{Y}(I)$ . Set  $\mathcal{Y}_{(n+1)} = \bigcup_{I \in \mathcal{Y}_{(n)}} \mathcal{Y}(I)$ .

STEP 3. The output is  $\mathcal{Y} = \bigcup_{i \in \mathbb{N}} \mathcal{Y}_{(i)}$ . ■

THEOREM 6.9. (a) *Let  $I$  and  $\mathcal{Y}(I)$  be as in Algorithm 6.7. Then  $\mathcal{Y}(I)$  is the set of all sincere  $r$ -peak posets of finite prinjective type min- or max-dominated by  $I$ .*

(b) *Let  $\mathcal{S}^{(r)}$  and  $\mathcal{Y}$  be as in Algorithm 6.8. The set  $\mathcal{Y}$  is finite and consists of all  $r$ -peak posets of finite prinjective type iteratively dominated by  $r$ -minimal posets.*

*Proof.* (a) Let  $J$  be a sincere  $r$ -peak poset of finite prinjective type which is min- or max-dominated by  $I$  and let  $J \setminus I = \{c\}$ . Lemma 6.5 shows that  $J \in \mathcal{X}(I)$ , where  $\mathcal{X}(I)$  is defined in Step 2 of Algorithm 6.7. By Definition 6.1, there exists a sincere positive root  $z$  of  $q_I$  such that  $\tilde{z}$  is a sincere positive root of  $q_J$ , where  $\tilde{z}_j = z_j$  for  $j \in I$  and  $\tilde{z}_c = 1$ . Proposition 4.3(b) shows that  $D_c q_J(\tilde{z}) = 1$ . This implies that  $J \in \mathcal{Y}(I)$ . Conversely, the posets  $J \in \mathcal{X}(I)$  of finite prinjective type satisfying condition (b) in Step 4 of Algorithm 6.7 are min- or max-dominated by  $I$  (cf. Definition 6.1). Now (a) follows easily.

(b)  $\mathcal{Y}$  consists of sincere  $r$ -peak posets of finite prinjective type. Therefore it is finite by Lemma 5.1. The remaining assertion is an easy consequence of the definition of an iteratively dominated poset, and Algorithms 6.7 and 6.8. ■

**7. Proof of the main result.** The proof of Theorem 2.1 is preceded by two lemmata.

LEMMA 7.1. *Let  $K$  be a field,  $I$  be one of the posets of Tables 2.2 and let  $M$  be one of the  $KI$ -modules in Tables 8.1 of Section 8. Then:*





According to Lemma 3.2 the module  $X \simeq \widetilde{F}'_q \widetilde{G}'_q(X) \simeq \widetilde{F}'_q(Y)$  is an indecomposable prinjective  $KI$ -module. Moreover the definition of the functor  $\widetilde{F}'_q$  shows that  $X = (X_i, i\psi_j)_{j \preceq i \in I}$ , where  $X_i = Y_i$ ,  $i\psi_j = i\varphi_j$  for  $i \neq *_1$ ,  $j \neq 2$  and the  $K$ -linear map  $*_1\psi_2$  is given in the standard basis by the matrix  $*_1\psi_2 = [1 \ 0]$ . Note that  $X = M_{73}^{(29,3)}$ , where  $M_{73}^{(29,3)}$  is given in Tables 8.1. Consequently,  $M_{73}^{(29,3)}$  is an indecomposable prinjective  $KI$ -module.

In a similar way we construct the remaining sincere indecomposable prinjective representations of weak peak reducible posets. The details can be obtained from the author upon request. ■

Denote by  $\mathcal{PW}^{(r,n)}$  the set of all sincere  $r$ -peak posets of finite prinjective type of cardinality  $n$ . Moreover let  $\mathcal{W}^{(r,n)}$  be the set of all sincere posets  $J$  of finite prinjective type, which are min- or max-dominated by some poset  $I$  in  $\mathcal{PW}^{(r,n-1)}$ , and let  $\mathcal{S}^{(r,n)}$  be the set of all  $r$ -minimal posets of cardinality  $n$ .

LEMMA 7.2. *Let  $m \in \mathbb{N}$  and  $r > 2$ . If no poset from  $\mathcal{PW}^{(r,m)}$  contains  $\mathcal{F}_i$ ,  $\mathcal{F}'_i$  or  $\mathcal{F}''_i$  as a peak subposet, then  $\mathcal{W}^{(r,m+1)} \cup \mathcal{S}^{(r,m+1)} = \mathcal{PW}^{(r,m+1)}$ .*

*Proof.* The proof is analogous to the proof of Lemma 7.1 in [14]. ■

*Proof of Theorem 2.1.* (c) By Lemma 7.1 any  $K\mathcal{F}_i^{(3)}$ -module  $M_i^{(j,3)}$  in Tables 8.1 is indecomposable, prinjective and  $\mathbf{cdn} M_i^{(j,3)} = z_i^{(j,3)}$ , where  $z_i^{(j,3)}$  is given in Tables 8.1. Conversely, let  $X$  be an indecomposable prinjective  $K\mathcal{F}_i^{(3)}$ -module satisfying  $\mathbf{cdn} X = z_i^{(j,3)}$ . Theorem 1.3(f) yields  $X \simeq M_i^{(j,3)}$  and (c) is proved.

(b) It is easy to see that any vector  $z_i^{(j,3)}$  in Tables 8.1 is a sincere positive vector such that  $z_i^{(j,3)} \in \mathbb{N}^{\mathcal{F}_i^{(3)}}$  and  $q_{\mathcal{F}_i^{(3)}}(z_i^{(j,3)}) = 1$ . Conversely, let  $I = \mathcal{F}_i^{(3)}$  be any of the posets of Tables 2.2. In [18, Section 1] and [14, Remark 7.3] one can find a construction of (sincere) positive roots of a weakly positive quadratic form (reflections of roots, construction of Weyl roots). Applying these methods we can prove that any positive sincere vector  $z \in \mathbb{N}^I$  such that  $q_I(z) = 1$  is one of the vectors  $z_i^{(j,3)}$  in Tables 8.1 below. A detailed proof can be obtained from the author upon request.

Now we prove (a) by a case-by-case inspection of the posets of Tables 2.2.

According to [21, Theorem 3.1] any poset  $\mathcal{F}_i^{(3)}$  in Tables 2.2 is a three-peak poset of finite prinjective type. Moreover the indecomposable prinjective  $K\mathcal{F}_i^{(3)}$ -modules  $M_i^{(j,3)}$  in Tables 8.1 satisfy  $(\mathbf{cdn} M_i^{(j,3)})(k) \neq 0$  for all  $k \in \mathcal{F}_i^{(3)}$  and therefore  $\mathcal{F}_i^{(3)}$  is sincere.

Conversely, let  $I$  be a three-peak sincere poset of finite prinjective type. We show that  $I$  is one of the posets in Tables 2.2. We use the following notations:

- $X_1, \dots, X_k$  denote pairwise different sets consisting of pairwise incomparable elements of  $I^-$ .
- $I_i, I^i, I_*, I_+, I^{i,*}, I^{i,*,+}, i = 1, \dots, k, *, + \in \max I, * \neq +$ , are the posets which are dominated by  $I$  and defined in Section 6.
- We denote by  $c \notin I$  an element such that  $I \cup \{c\} = I_i$  (resp.  $I^j, I_*, I_+, I^{i,*}, I^{i,*,+}$ ).
- Let  $\mathcal{X}(I) = \mathcal{X}_I \cup \mathcal{X}^I$  (see Section 6, Algorithm 6.7).
- Let  $\mathcal{Y}(I)$  denote the output set of Algorithm 6.7 with input  $I$ .
- We denote by  $\mathcal{SR}_{q_I}^+$  the set of all sincere positive roots of  $q_I$ .

Now we are ready to apply Algorithm 6.8 to produce a complete set of three-peak sincere posets of finite prinjective type which are iteratively dominated by 3-minimal posets. For any  $i \in \{1, \dots, 90\}$ , applying Algorithm 6.7, we construct the set  $\mathcal{Y}(\mathcal{F}_i^{(3)})$  of all posets min- or max-dominated by  $\mathcal{F}_i^{(3)}$ .

First we find the set of all 3-minimal posets. For this purpose we have to analyse [14, Tables 3.2 and 3.3] all sincere two-peak posets of finite prinjective type and sincere positive roots of their Tits forms. It is easy to check that  $\mathcal{S}^{(3)} = \{\mathcal{F}_i^{(3)} : i = 1, 2, 44, 47, 54, 55, 56, 70, 73, 76, 77\}$  is the set of all 3-minimal posets.

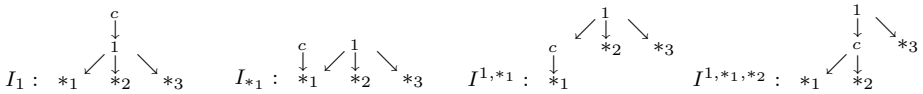
Now we describe our procedure:

(1) Applying Algorithm 6.7 we construct all posets min- or max-dominated by the poset



STEP 1. The set  $X_1 = \{1\}$  is the only set required in Step 1 of Algorithm 6.7.

STEP 2. The set  $\mathcal{X}(I)$  associated with  $I$  in Step 2 of Algorithm 6.7 consists of the following posets:



because  $I^1 = I_1, I_{*1} = I_{*2} = I_{*3}, I^{1,*1} = I^{1,*2} = I^{1,*3}$  and  $I^{1,*1,*2} = I^{1,*2,*3} = I^{1,*1,*3}$ .

STEP 3. It is easy to see that the set of all sincere positive roots of  $q_I$  is  $\mathcal{SR}_{q_I}^+ = \{z_1^{(1,3)} = (1, 1, 1, 1), z_1^{(2,3)} = (2, 1, 1, 1)\}$  (see Tables 8.1), where the coordinates of the vectors  $z_1^{(i,3)} = (z_1, z_{*1}, z_{*2}, z_{*3})$  are indexed by elements of  $I$ .

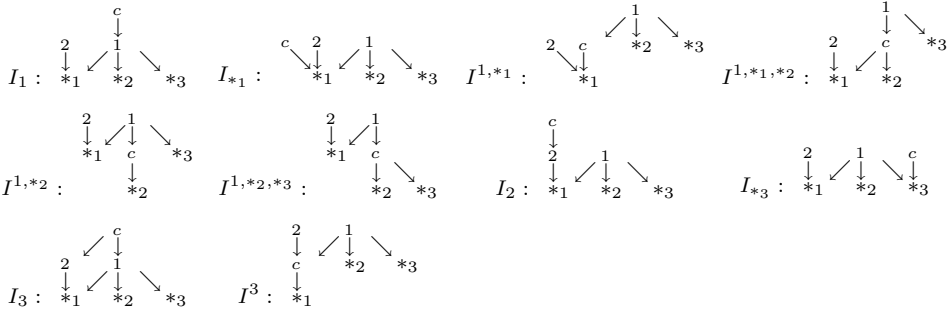
STEP 4. We note that  $D_c q_{I^{1,*1}}(z_1^{(1,3)}) = 2 \neq 1, D_c q_{I^{1,*1}}(z_1^{(2,3)}) = 3 \neq 1$  and  $D_c q_{I_1}(z_1^{(2,3)}) = D_c q_{I_{*1}}(z_1^{(1,3)}) = D_c q_{I^{1,*1,*2}}(z_1^{(1,3)}) = 1$ . The poset  $I_1$  is of infinite prinjective type and  $I_{*1}, I^{1,*1,*2}$  are of finite prinjective type (see [21,

Theorem 3.1]). Therefore the latter posets satisfy the conditions required in Step 4 of Algorithm 6.7 and they are all posets min- or max-dominated by  $I$ . We note that  $I_{*1} = \mathcal{F}_2^{(3)}$  and  $I^{1,*1,*2} = \mathcal{F}_3^{(2)}$ .

(2)  $I = \mathcal{F}_2^{(3)}$ .

STEP 1.  $X_1 = \{1\}$ ,  $X_2 = \{2\}$ ,  $X_3 = \{1, 2\}$ .

STEP 2. The set  $\mathcal{X}(I)$  in Step 2 of Algorithm 6.7 consists of the following posets:



STEP 3.  $\mathcal{SR}_{qI}^+ = \{z_2^{(1,3)} = (1, 1, 1, 1, 1), z_2^{(2,3)} = (2, 1, 1, 1, 1), z_2^{(3,3)} = (2, 1, 2, 1, 1)\}$  (see Tables 8.1).

STEP 4. Note that  $D_cq_{I^1,*1}(z_2^{(i,3)}) \neq 1$ ,  $D_cq_{I^1,*1}(z_2^{(i,3)}) \neq 1$ ,  $D_cq_{I^3}(z_2^{(i,3)}) \neq 1$  for  $i = 1, 2, 3$ , the posets  $I_1, I^{*1}, I_3$  are of infinite prinjective type (see [21, Theorem 3.1]) and  $D_cq_{I^1,*1,*2}(z_2^{(1,3)}) = D_cq_{I_1,*2,*3}(z_2^{(1,3)}) = D_cq_{I_2}(z_2^{(3,3)}) = D_cq_{I^3}(z_2^{(1,3)}) = 1$ . The posets  $I^{1,*1,*2} = \mathcal{F}_4^{(3)}$ ,  $I^{1,*2,*3} = \mathcal{F}_5^{(3)}$ ,  $I_2 = \mathcal{F}_6^{(3)}$ ,  $I^{*3} = \mathcal{F}_7^{(3)}$  satisfy the conditions in Step 4 of Algorithm 6.7 and they are all posets min- or max-dominated by the poset  $I$ .

In the remaining cases we only list the output sets  $\mathcal{Y}(\mathcal{F}_i^{(3)})$ ,  $i = 3, \dots, 90$ . The procedures described above work well. The details are left to the reader. A complete proof can be obtained from the author upon request.

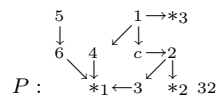
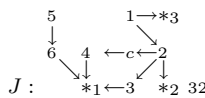
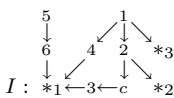
- |   |   |
|---|---|
| $\mathcal{Y}(\mathcal{F}_3^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 4, 5, 8\},$       | $\mathcal{Y}(\mathcal{F}_4^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 9, \dots, 13\},$      |
| $\mathcal{Y}(\mathcal{F}_5^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 10, 13\},$        | $\mathcal{Y}(\mathcal{F}_6^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 12, 14\},$            |
| $\mathcal{Y}(\mathcal{F}_7^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 13, 14\},$        | $\mathcal{Y}(\mathcal{F}_8^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 10, 15, 16\},$        |
| $\mathcal{Y}(\mathcal{F}_9^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 17, 18\},$        | $\mathcal{Y}(\mathcal{F}_{10}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 19, 20, 21, 22\},$ |
| $\mathcal{Y}(\mathcal{F}_{11}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 23, 24, 25\},$ | $\mathcal{Y}(\mathcal{F}_{12}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 17, 22, 26\},$     |
| $\mathcal{Y}(\mathcal{F}_{13}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 25, 27\},$     | $\mathcal{Y}(\mathcal{F}_{14}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 27, 28\},$         |
| $\mathcal{Y}(\mathcal{F}_{15}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 20\},$         | $\mathcal{Y}(\mathcal{F}_{16}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 21, 29, 30\},$     |
| $\mathcal{Y}(\mathcal{F}_{17}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 31, 32, 33\},$ | $\mathcal{Y}(\mathcal{F}_{18}^{(3)}) = \emptyset,$                                    |
| $\mathcal{Y}(\mathcal{F}_{19}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 34, 35, 36\},$ | $\mathcal{Y}(\mathcal{F}_{20}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 37\},$             |
| $\mathcal{Y}(\mathcal{F}_{21}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 38, 39\},$     | $\mathcal{Y}(\mathcal{F}_{22}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 36, 37\},$         |

- $\mathcal{Y}(\mathcal{F}_{23}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 40\},$
- $\mathcal{Y}(\mathcal{F}_{25}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 40\},$
- $\mathcal{Y}(\mathcal{F}_{27}^{(3)}) = \emptyset,$
- $\mathcal{Y}(\mathcal{F}_{29}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 38\},$
- $\mathcal{Y}(\mathcal{F}_j^{(3)}) = \emptyset, j = 31, \dots, 43,$
- $\mathcal{Y}(\mathcal{F}_{45}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 16, 48, \dots, 52\},$
- $\mathcal{Y}(\mathcal{F}_{47}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 15, 50, 54, 56\},$
- $\mathcal{Y}(\mathcal{F}_{49}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 58, 61, 62, 63\},$
- $\mathcal{Y}(\mathcal{F}_{51}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 29, 59, 61, 66, 67\},$
- $\mathcal{Y}(\mathcal{F}_{53}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 66, 69\},$
- $\mathcal{Y}(\mathcal{F}_{55}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 19, 67, 69, 73, 74, 75\},$
- $\mathcal{Y}(\mathcal{F}_{57}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 42\},$
- $\mathcal{Y}(\mathcal{F}_{59}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 78, 90\},$
- $\mathcal{Y}(\mathcal{F}_{61}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 80, 90\},$
- $\mathcal{Y}(\mathcal{F}_{63}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 79, 81\},$
- $\mathcal{Y}(\mathcal{F}_{65}^{(3)}) = \emptyset,$
- $\mathcal{Y}(\mathcal{F}_{67}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 78, 83\},$
- $\mathcal{Y}(\mathcal{F}_{69}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 84, 86, 87\},$
- $\mathcal{Y}(\mathcal{F}_{71}^{(3)}) = \emptyset,$
- $\mathcal{Y}(\mathcal{F}_{73}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 34, 35, 83\},$
- $\mathcal{Y}(\mathcal{F}_{75}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 87\},$
- $\mathcal{Y}(\mathcal{F}_{77}^{(3)}) = \{\mathcal{F}_i^{(3)} : yi = 89\},$
- $\mathcal{Y}(\mathcal{F}_{24}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 41\},$
- $\mathcal{Y}(\mathcal{F}_{26}^{(3)}) = \{\mathcal{F}_i^{(3)} ; i = 33\},$
- $\mathcal{Y}(\mathcal{F}_{28}^{(3)}) = \emptyset,$
- $\mathcal{Y}(\mathcal{F}_{30}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 42, 43\},$
- $\mathcal{Y}(\mathcal{F}_{44}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 8, 45, 46, 47\},$
- $\mathcal{Y}(\mathcal{F}_{46}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 10, 51, \dots, 55\}$
- $\mathcal{Y}(\mathcal{F}_{48}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 30, 57, \dots, 60\},$
- $\mathcal{Y}(\mathcal{F}_{50}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 60, 63, 64, 65, 71\},$
- $\mathcal{Y}(\mathcal{F}_{52}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 21, 57, 62, 64, 66, 68\},$
- $\mathcal{Y}(\mathcal{F}_{54}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 20, 64, 70, 71, 72\},$
- $\mathcal{Y}(\mathcal{F}_{56}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 63, 65, 72, 76, 77\},$
- $\mathcal{Y}(\mathcal{F}_{58}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 43\}$
- $\mathcal{Y}(\mathcal{F}_{60}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 79\},$
- $\mathcal{Y}(\mathcal{F}_{62}^{(3)}) = \emptyset,$
- $\mathcal{Y}(\mathcal{F}_{64}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 39, 82\},$
- $\mathcal{Y}(\mathcal{F}_{66}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 38\},$
- $\mathcal{Y}(\mathcal{F}_{68}^{(3)}) = \emptyset,$
- $\mathcal{Y}(\mathcal{F}_{70}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 37\},$
- $\mathcal{Y}(\mathcal{F}_{72}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 82, 89\},$
- $\mathcal{Y}(\mathcal{F}_{74}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 86\},$
- $\mathcal{Y}(\mathcal{F}_{76}^{(3)}) = \{\mathcal{F}_i^{(3)} : i = 81, 88\},$
- $\mathcal{Y}(\mathcal{F}_j^{(3)}) = \emptyset, j = 78, \dots, 90.$

It is easy to check that  $\bigcup_{i=1}^{90} \mathcal{Y}(\mathcal{F}_i^{(3)})$  is the set of all posets in Tables 2.2. From the constructions and remarks given above it follows that the posets of Tables 2.2 are all the three-peak sincere posets of finite prinjective type iteratively dominated by 3-minimal posets. We note that  $\mathcal{F}_{32}^{(3)}$  contains  $\mathcal{F}_2''$  and that no other poset ( $\neq \mathcal{F}_{32}^{(3)}$ ) of Tables 2.2 contains  $\mathcal{F}_i, \mathcal{F}_i',$  or  $\mathcal{F}_i''$  as a peak subposet.

Since  $\mathcal{F}_{32}^{(3)}$  contains  $\mathcal{F}_2''$ , Lemma 7.2 does not work in this case. We consider the set  $\mathcal{Y}_{32}$  of posets which are dominated (but not min- or max-dominated) by  $\mathcal{F}_{32}^{(3)}$ . By Lemma 7.2 and Theorem 6.3 the set  $\bigcup_{i=1}^{90} \mathcal{Y}(\mathcal{F}_i^{(3)}) \cup \mathcal{Y}_{32}$  gives all sincere three-peak posets of finite prinjective type.

Consider the following three posets:



Note that  $P$  is of infinite prinjective type. Moreover

$$\mathcal{SR}_{\mathcal{F}_{32}^{(3)}}^+ = \{z = (1, 1, 1, 1, 1, 1, 3, 1, 1)\}$$

(see Tables 2.2 and 8.1). We can check easily that  $D_cq_I(z) = D_cq_J(z) = 2 \neq 1$ . Therefore  $\mathcal{F}_{32}^{(3)}$  does not dominate  $I, J, P$ . Hence  $\mathcal{Y}_{32} = \emptyset$ . This finishes the proof of (a). ■

The results in [12], [13], [14], [15] and in the present paper provide a complete classification of sincere posets of finite prinjective type and their sincere prinjective indecomposable representations. Let us make some concluding remarks about this classification.

REMARK 7.3. (a) A complete set of 14 sincere one-peak posets of finite prinjective type and their 42 (up to isomorphism) sincere prinjective indecomposable representations was given by M. Kleiner in [12], [13] (see also [20, Theorem 10.2 and Tables 10.7] for a correction of Kleiner's list [13]). A complete set of 60 sincere two-peak posets of finite prinjective type and their 328 (up to isomorphism) sincere prinjective indecomposable representations is given in [14]. A complete set of 90 sincere three-peak posets of finite prinjective type and their 723 (up to isomorphism) sincere prinjective indecomposable representations is given in the present paper. We finish this classification in [15], where we present a complete set of sincere  $r$ -peak posets  $I$  of finite prinjective type, for  $r \geq 4$ , and a complete set of indecomposable modules in  $\text{prin}(KI)$  and their coordinate vectors. It follows that there are precisely 40 sincere 4-peak posets of finite prinjective type, and  $6r + 2$  sincere  $r$ -peak posets of finite prinjective type for  $r \geq 5$ . The total number of sincere indecomposable prinjective representations of 4-peak posets of finite prinjective type (up to isomorphism) is 364, and for any  $r \geq 5$  the total number of sincere indecomposable prinjective representations of  $r$ -peak posets of finite prinjective type (up to isomorphism) is  $18r - 23$ .

(b) It follows from the tables in [20], [14], [15] and Tables 8.1 in the present paper that more than half of the sincere posets of finite prinjective type have only one (up to isomorphism) indecomposable sincere prinjective representation. So the corresponding Tits quadratic form has only one sincere positive root.

(c) It follows from the tables in [20], [14], [15] and Tables 8.1 in the present paper that the canonical forms of indecomposable representations of sincere posets of finite prinjective type can be represented by  $\{0, 1\}$ -matrices. These canonical forms do not depend on the base field  $K$ . It seems that this is not accidental (see [19] and [10] for details).

(d) Let  $I$  be a sincere poset of finite prinjective type. The algebra  $KI$  is not, in general, of finite representation type. For instance  $I = \mathcal{F}_{35}^{(3)}$  is of finite prinjective type, but  $KI$  is of infinite representation type.

We finish this section with a theorem which says that the algebra  $KI$  is a tilted algebra for any sincere poset  $I$  of finite prinjective type (see [18, Section 4], [9] and [26]). Recall that a connected component  $\mathcal{P}$  of the Auslander–Reiten quiver  $\Gamma(\text{mod}(A))$  of a  $K$ -algebra  $A$  is called *preprojective* if  $\mathcal{P}$  contains no oriented cycle and each point of  $\mathcal{P}$  has only finitely many predecessors. Here  $U$  is a *predecessor* of  $V$  provided there is a path  $U = X_0 \rightarrow X_1 \rightarrow \dots \rightarrow X_n$  in  $\mathcal{P}$  with  $n \geq 0$  (see [2], [4], [18] and [20]).

**THEOREM 7.4.** *Let  $I$  be a sincere poset of finite prinjective type. Then the  $K$ -algebra  $KI$  is a tilted algebra. In particular  $\text{gl.dim } KI \leq 2$ .*

*Proof.* Without loss of generality we may assume that  $I$  is connected. Let  $I = \{1, \dots, n, p_1, \dots, p_r\}$ , where  $\max I = \{p_1, \dots, p_r\}$ . Moreover let  $P_1, \dots, P_n, P_{p_1}, \dots, P_{p_r}$  be a complete set of pairwise non-isomorphic indecomposable projective  $KI$ -modules. It is easy to see that this is a complete set of pairwise non-isomorphic indecomposable prin-projective modules (i.e. projective objects in the category  $\text{prin}(KI)$ ). Denote by  $Q_1^\diamond, \dots, Q_n^\diamond, Q_{p_1}^\diamond, \dots, Q_{p_r}^\diamond$  a complete set of pairwise non-isomorphic indecomposable prin-injective modules (i.e. injective objects in  $\text{prin}(KI)$ ). The reader is referred to [17] for details.

By Theorem 1.3 the Auslander–Reiten quiver  $\Gamma(\text{prin}(KI))$  is connected, finite and equal to its preprojective component  $\tilde{\mathcal{P}}(I)$ . Therefore we can define a partial order  $\rightarrow$  on the set  $\{Q_1^\diamond, \dots, Q_n^\diamond, Q_{p_1}^\diamond, \dots, Q_{p_r}^\diamond\}$ :  $Q_i^\diamond \rightarrow Q_j^\diamond$  if and only if  $Q_i^\diamond$  is a predecessor of  $Q_j^\diamond$  in  $\Gamma(\text{prin}(KI)) = \tilde{\mathcal{P}}(I)$ . Moreover we can choose  $i_0 \in I$  in such a way that  $Q_{i_0}^\diamond$  is minimal for  $\rightarrow$ . Denote by  $\mathcal{P}_0$  the full subquiver of  $\Gamma(\text{prin}(KI))$  consisting of all predecessors of  $Q_{i_0}^\diamond$ .

We claim that the Auslander–Reiten quiver  $\Gamma(\text{mod}(KI))$  has a preprojective component. Indeed, since  $I$  is of finite prinjective type it contains no subposet of the form  $\mathcal{P}_{n,1}$  ( $n \geq 2$ ) listed in [21, Theorem 3.1]. Therefore by [4, 2.1] any point  $x \in I$  is separating in the sense of [5], [4]. From [4, Theorem 2.5] it follows that  $\Gamma(\text{mod}(KI))$  has a preprojective component  $\mathcal{P}$ . Moreover there exists  $i \in I$  such that  $P_i$  belongs to  $\mathcal{P}$ .

Since  $I$  is sincere, there exists a sincere indecomposable prinjective representation  $M$  of  $I$ . We now show that  $M$  lies on  $\mathcal{P}$ .

Since  $M$  is sincere, [9, Lemma 2.16] shows that  $\text{Hom}_{KI}(P_i, M) \neq 0$  and  $\text{Hom}_{KI}(M, Q_i^\diamond) \neq 0$  for any  $i \in I$ . In particular the modules  $M$  and  $P_i$ , for  $i \in I$ , belong to  $\mathcal{P}_0$ .

Consider the following Auslander–Reiten sequence in  $\text{prin}(KI)$ :

$$(7.5) \quad 0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0,$$

where  $X, Y, Z$  are in  $\mathcal{P}_0$ . We note that

(1)  $\text{Hom}_{KI}(Z, P_j) = 0$  for  $j = p_1, \dots, p_r$ , because  $P_j$  are simple projective modules,

(2)  $\text{Hom}_{KI}(Q_j^\diamond, Y) = 0$  for  $j = 1, \dots, n$ , by the choice of  $i_0$ .

By (1), (2) and [9, Lemma 2.17] the sequence (7.5) is an Auslander–Reiten sequence in  $\text{mod}(KI)$ . Moreover we have shown that  $M, P_i \in \mathcal{P}_0, i \in I$ , and therefore it follows easily that all the modules  $M$  and  $P_i$ , for  $i \in I$ , belong to the same connected component of  $\Gamma(\text{mod}(KI))$ . Since there exists  $j \in I$  such that  $P_j \in \mathcal{P}$ , it follows that  $M$  and  $P_i$ , for  $i \in I$ , all lie on  $\mathcal{P}$ .

Therefore  $\mathcal{P}$  is a preprojective component of  $\Gamma(\text{mod}(KI))$  which contains a sincere module  $M$ . By [18, 4.2],  $KI$  is a tilted algebra and  $\text{gl.dim}(KI) \leq 2$ . ■

**8. Tables of sincere prinjective modules.** Below we present a set of canonical forms of sincere prinjective modules  $M_i^{(j,3)}$  over the incidence algebra  $K\mathcal{F}_j^{(3)}$ . From Lemma 7.1 it follows that the modules  $M_i^{(j,3)}$  are those modules in  $\text{prin}(K\mathcal{F}_i^{(3)})$  such that  $\mathbf{cdn} M_i^{(j,3)} = z_i^{(j,3)}$  (see Tables 2.2). By Lemma 7.1 the modules  $M_i^{(j,3)}$  are indecomposable. All sincere prinjective representations in Tables 8.1 are constructed by applying the methods described in the proof of Lemma 7.1.

In the tables below the  $K$ -linear maps  $a, b, c, e, f, \dots$  in the definition of  $M_i^{(j,3)}$  are defined by their matrices in the standard bases. Following [20] we denote by  $d$  the  $K$ -linear map defined by the matrix of the form  $[1, 1, \dots, 1]$  (or by its transpose). Moreover the  $K$ -linear map denoted by  $d_{i_1} \vee \dots \vee d_{i_n}$  is given by the matrix whose  $j$ th column is the  $i_j$ th standard vector (see [20]). The  $K$ -linear maps denoted by  $\Downarrow, \searrow, \swarrow, \circ\rightarrow, \leftarrow\circ$  are defined by matrices of the form

$$\begin{bmatrix} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \end{bmatrix}$$

or by their transposes. The remaining  $K$ -linear maps are defined by matrices of the form

$$\begin{bmatrix} 1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \end{bmatrix}$$

or by their transposes.

In the third row of the tables the coordinate vectors  $z_i^{(j,3)}$  of the respective  $K\mathcal{F}_i^{(3)}$ -modules  $M_i^{(j,3)}$  are given.



**Tables 8.1**  
Sincere prinjective  $KI$ -modules, where  
 $I$  is a sincere three-peak poset of finite prinjective type

$M_1^{(1,3)}$	$M_1^{(2,3)}$	$M_2^{(1,3)}$	$M_2^{(2,3)}$	$M_3^{(1,3)}$	$M_4^{(1,3)}$	$M_4^{(2,3)}$
$\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ 1 \\ 1 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ 2 \\ 1 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 2 \end{smallmatrix}$

$M_4^{(3,3)}$	$M_4^{(4,3)}$	$M_5^{(1,3)}$	$M_6^{(1,3)}$	$M_7^{(1,3)}$	$M_7^{(2,3)}$	$M_7^{(3,3)}$
$\begin{smallmatrix} 2 \\ 1 \\ 1 \\ 2 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ 1 \\ 1 \\ 2 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ 2 \\ 2 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ 2 \\ 1 \\ 1 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ 2 \\ 1 \\ 1 \end{smallmatrix}$
	$a = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$					

$M_7^{(4,3)}$	$M_7^{(5,3)}$	$M_7^{(6,3)}$	$M_7^{(7,3)}$	$M_8^{(1,3)}$	$M_8^{(2,3)}$	$M_9^{(1,3)}$
$\begin{smallmatrix} 1 \\ 2 \\ 1 \\ 2 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ 2 \\ 1 \\ 2 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ 3 \\ 1 \\ 2 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ 3 \\ 1 \\ 2 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 1 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 2 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{smallmatrix}$
			$a = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$			

$M_{10}^{(1,3)}$	$M_{10}^{(2,3)}$	$M_{10}^{(3,3)}$	$M_{10}^{(4,3)}$	$M_{10}^{(5,3)}$	$M_{10}^{(6,3)}$	$M_{10}^{(7,3)}$
$\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 2 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 2 \end{smallmatrix}$
				$a = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$a = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$	$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ , $b = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$M_{11}^{(1,3)}$	$M_{11}^{(2,3)}$	$M_{11}^{(3,3)}$	$M_{11}^{(4,3)}$	$M_{11}^{(5,3)}$
$\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{smallmatrix}$	$\begin{smallmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 2 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 2 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 3 \end{smallmatrix}$	$\begin{smallmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 3 \end{smallmatrix}$
	$a = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$a = d_1 \vee d_3$ , $b = d_1 \vee d_2 \vee d_4$	$a = d_1 \vee d_3$ , $b = d_1 \vee d_2 \vee d_4$	$a = d_1 \vee d_3$ , $b = d_1 \vee d_2 \vee d_4$
		$c = d_1 \vee 0 \vee 0 \vee d_2$	$c = d_1 \vee d_2 \vee 0 \vee d_3$	$c = d_1 \vee d_2 \vee 0 \vee d_3$
		$e = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$	$e = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$	$e = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

$M_{12}^{(1,3)}$	$M_{12}^{(2,3)}$	$M_{12}^{(3,3)}$	$M_{12}^{(4,3)}$	$M_{12}^{(5,3)}$	$M_{12}^{(6,3)}$	$M_{12}^{(7,3)}$
$\begin{matrix} 11 \\ 111 \\ 21 \end{matrix}$	$\begin{matrix} 12 \\ 111 \\ 21 \end{matrix}$	$\begin{matrix} 12 \\ 111 \\ 22 \end{matrix}$	$\begin{matrix} 12 \\ 111 \\ 31 \end{matrix}$	$\begin{matrix} 12 \\ 111 \\ 32 \end{matrix}$	$\begin{matrix} 13 \\ 111 \\ 32 \end{matrix}$	$\begin{matrix} 12 \\ 121 \\ 32 \end{matrix}$
		$a = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$		$a = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$a = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$	$a = d_2 \vee d_3 \quad b = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

$M_{13}^{(8,3)}$	$M_{13}^{(1,3)}$	$M_{13}^{(2,3)}$	$M_{13}^{(3,3)}$	$M_{13}^{(4,3)}$	$M_{13}^{(5,3)}$	$M_{13}^{(6,3)}$
$\begin{matrix} 13 \\ 112 \\ 32 \end{matrix}$	$\begin{matrix} 11 \\ 111 \\ 11 \end{matrix}$	$\begin{matrix} 11 \\ 111 \\ 21 \end{matrix}$	$\begin{matrix} 21 \\ 111 \\ 21 \end{matrix}$	$\begin{matrix} 21 \\ 111 \\ 22 \end{matrix}$	$\begin{matrix} 21 \\ 112 \\ 21 \end{matrix}$	$\begin{matrix} 21 \\ 112 \\ 22 \end{matrix}$
$a = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$				$a = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$		$a = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$M_{13}^{(7,3)}$	$M_{13}^{(8,3)}$	$M_{13}^{(9,3)}$	$M_{14}^{(1,3)}$	$M_{14}^{(2,3)}$	$M_{14}^{(3,3)}$	$M_{14}^{(4,3)}$
$\begin{matrix} 31 \\ 112 \\ 22 \end{matrix}$	$\begin{matrix} 31 \\ 112 \\ 32 \end{matrix}$	$\begin{matrix} 31 \\ 212 \\ 32 \end{matrix}$	$\begin{matrix} 1 \\ 121 \\ 211 \end{matrix}$	$\begin{matrix} 1 \\ 121 \\ 212 \end{matrix}$	$\begin{matrix} 1 \\ 131 \\ 212 \end{matrix}$	$\begin{matrix} 1 \\ 131 \\ 222 \end{matrix}$
$a = d_1 \vee d_2 \vee d_4$ $b = d_1 \vee 0 \vee 0 \vee d_2$ $c = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$	$a = d_1 \vee d_2 \vee d_4$ $b = d_1 \vee d_2 \vee 0 \vee d_3$ $c = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$	$a = d_1 \vee d_2 \vee d_4$ $b = d_1 \vee d_2 \vee 0 \vee d_3$ $c = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$				$a = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$M_{14}^{(5,3)}$	$M_{14}^{(6,3)}$	$M_{14}^{(7,3)}$	$M_{14}^{(8,3)}$	$M_{14}^{(9,3)}$	$M_{15}^{(1,3)}$	$M_{16}^{(1,3)}$
$\begin{matrix} 1 \\ 131 \\ 312 \end{matrix}$	$\begin{matrix} 1 \\ 131 \\ 322 \end{matrix}$	$\begin{matrix} 1 \\ 141 \\ 322 \end{matrix}$	$\begin{matrix} 1 \\ 141 \\ 323 \end{matrix}$	$\begin{matrix} 1 \\ 142 \\ 323 \end{matrix}$	$\begin{matrix} 1 \\ 111 \\ 121 \end{matrix}$	$\begin{matrix} 1 \\ 111 \\ 121 \end{matrix}$
	$a = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$a = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ $b = d_1 \vee 0 \vee 0 \vee d_2$	$a = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee 0 \vee d_3$	$a = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee 0 \vee d_3$		

$M_{16}^{(2,3)}$	$M_{17}^{(1,3)}$	$M_{17}^{(2,3)}$	$M_{17}^{(3,3)}$	$M_{17}^{(4,3)}$
$\begin{matrix} 11 \\ 11 \\ 221 \end{matrix}$	$\begin{matrix} 11 \\ 1111 \\ 21 \end{matrix}$	$\begin{matrix} 11 \\ 1111 \\ 31 \end{matrix}$	$\begin{matrix} 11 \\ 1121 \\ 31 \end{matrix}$	$\begin{matrix} 12 \\ 111 \\ 31 \end{matrix}$
$a = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$		$a = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$	$a = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	

$M_{17}^{(5,3)}$	$M_{17}^{(6,3)}$	$M_{17}^{(7,3)}$	$M_{17}^{(8,3)}$	$M_{17}^{(9,3)}$
$\begin{matrix} 11 \\ 1121 \\ 32 \end{matrix}$	$\begin{matrix} 11 \\ 1111 \\ 32 \end{matrix}$	$\begin{matrix} 11 \\ 1121 \\ 32 \end{matrix}$	$\begin{matrix} 12 \\ 1121 \\ 42 \end{matrix}$	$\begin{matrix} 12 \\ 2121 \\ 42 \end{matrix}$
$a = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$a = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$a = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$	$a = d_1 \vee d_4 \quad b = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	$a = d_1 \vee d_2 \vee d_4 \quad b = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

$M_{17}^{(10,3)}$	$M_{18}^{(1,3)}$	$M_{19}^{(1,3)}$	$M_{19}^{(2,3)}$	$M_{19}^{(3,3)}$
$\begin{matrix} 2 & 2 \\ 1 & 1 \\ 1 & 2 \\ 4 & 2 \end{matrix}$	$\begin{matrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 2 & 1 \end{matrix}$	$\begin{matrix} 1 \\ 1 & 1 \\ 1 & 2 \end{matrix}$	$\begin{matrix} 1 \\ 1 & 1 \\ 1 & 2 \end{matrix}$	$\begin{matrix} 1 \\ 1 & 1 \\ 1 & 2 \end{matrix}$
$a=d_1 \vee d_2 \vee d_4$ $b= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$				$a=d_1 \vee d_1 \vee d_2$

$M_{19}^{(4,3)}$	$M_{19}^{(5,3)}$	$M_{20}^{(1,3)}$	$M_{20}^{(2,3)}$	$M_{20}^{(3,3)}$
$\begin{matrix} 1 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{matrix}$	$\begin{matrix} 1 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{matrix}$	$\begin{matrix} 1 \\ 1 & 1 \\ 1 & 2 \end{matrix}$	$\begin{matrix} 1 \\ 1 & 1 \\ 1 & 2 \end{matrix}$	$\begin{matrix} 1 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{matrix}$
	$a= \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$			$a= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $b=d_1 \vee d_1 \vee d_2$

$M_{20}^{(4,3)}$	$M_{20}^{(5,3)}$	$M_{20}^{(6,3)}$	$M_{21}^{(1,3)}$	$M_{21}^{(2,3)}$	$M_{21}^{(3,3)}$
$\begin{matrix} 1 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{matrix}$	$\begin{matrix} 1 \\ 2 & 1 \\ 1 & 2 \\ 1 & 3 \end{matrix}$	$\begin{matrix} 1 \\ 2 & 1 \\ 2 & 3 \end{matrix}$	$\begin{matrix} 1 \\ 1 & 1 \\ 1 & 2 \end{matrix}$	$\begin{matrix} 1 \\ 1 & 1 \\ 1 & 2 \end{matrix}$	$\begin{matrix} 1 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{matrix}$
$a= \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$	$a= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$a= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$			$a= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$M_{21}^{(4,3)}$	$M_{21}^{(5,3)}$	$M_{21}^{(6,3)}$	$M_{21}^{(7,3)}$	$M_{21}^{(8,3)}$	$M_{21}^{(9,3)}$
$\begin{matrix} 1 \\ 1 & 1 \\ 1 & 2 \\ 1 & 2 \end{matrix}$	$\begin{matrix} 1 \\ 1 & 1 \\ 1 & 2 \\ 2 & 2 \end{matrix}$	$\begin{matrix} 1 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{matrix}$	$\begin{matrix} 1 \\ 1 & 1 \\ 1 & 2 \\ 2 & 2 \end{matrix}$	$\begin{matrix} 1 \\ 1 & 1 \\ 1 & 2 \\ 2 & 2 \end{matrix}$	$\begin{matrix} 1 \\ 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{matrix}$
$a= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$	$a= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$		$a= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ $b= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$a= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$ $b= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$	$a= \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ $b= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

$M_{21}^{(10,3)}$	$M_{21}^{(11,3)}$	$M_{21}^{(12,3)}$	$M_{21}^{(13,3)}$	$M_{21}^{(14,3)}$	$M_{21}^{(15,3)}$
$\begin{matrix} 1 \\ 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{matrix}$	$\begin{matrix} 1 \\ 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{matrix}$	$\begin{matrix} 1 \\ 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{matrix}$	$\begin{matrix} 1 \\ 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{matrix}$	$\begin{matrix} 1 \\ 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{matrix}$	$\begin{matrix} 1 \\ 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{matrix}$
$a=d_1 \vee d_2 \vee d_2 \vee d_1$ $b=d_1 \vee d_2 \vee d_3 \vee d_2$ $c= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ $e=d_1 \vee d_2 \vee d_2 \vee 0$	$a= \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ $b=d_1 \vee d_2 \vee d_2 \vee d_3$	$a= \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ $c= \begin{bmatrix} 0 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$ $b=d_1 \vee d_2 \vee d_2 \vee d_3$	$a= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$	$a= \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$a= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$













$M_{30}^{(1,3)}$	$M_{30}^{(2,3)}$	$M_{30}^{(3,3)}$	$M_{30}^{(4,3)}$
$\frac{11}{2121}$	$\frac{11}{2131}$	$\frac{11}{2131}$	$\frac{11}{2132}$
$a=d_1 \vee d_3, b=d_1 \vee d_2 \vee d_4$ $c=\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ $e=d_1 \vee 0 \vee 0 \vee d_2$	$a=d_1 \vee d_3, b=d_1 \vee d_2 \vee d_4$ $c=\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ $e=d_1 \vee d_2 \vee 0 \vee d_3$	$a=d_1 \vee d_3, b=d_1 \vee d_2 \vee d_4$ $c=\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ $e=d_1 \vee d_2 \vee 0 \vee d_3$	$a=d_1 \vee d_3, b=d_1 \vee d_2 \vee d_4$ $c=\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ $e=d_1 \vee d_2 \vee 0 \vee d_3, f=d_1 \vee 0 \vee d_2$

$M_{31}^{(1,3)}$	$M_{32}^{(1,3)}$	$M_{33}^{(1,3)}$	$M_{34}^{(1,3)}$	$M_{34}^{(2,3)}$
$\frac{11}{1121}$ $\frac{3}{2}$	$\frac{11}{1111}$ $\frac{311}{1}$	$\frac{112}{1121}$ $\frac{4}{2}$	$\frac{11}{1121}$ $\frac{321}{1}$	$\frac{11}{2121}$ $\frac{321}{1}$
$a=\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$a=\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$	$a=d_1 \vee d_4, b=\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$		$a=\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

$M_{34}^{(3,3)}$	$M_{34}^{(4,3)}$	$M_{34}^{(5,3)}$	$M_{34}^{(6,3)}$
$\frac{11}{2121}$ $\frac{421}{1}$	$\frac{11}{2131}$ $\frac{421}{1}$	$\frac{11}{2131}$ $\frac{431}{1}$	$\frac{11}{2132}$ $\frac{431}{1}$
$a=\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b=d_1 \vee d_3 \vee d_4$	$a=\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b=0 \vee d_1 \vee d_2 \vee 0$	$a=\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b=d_1 \vee d_3$	$a=\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

$M_{34}^{(7,3)}$	$M_{35}^{(1,3)}$	$M_{36}^{(1,3)}$	$M_{37}^{(1,3)}$	$M_{37}^{(2,3)}$
$\frac{11}{2132}$ $\frac{432}{1}$	$\frac{1111}{1122}$ $\frac{3}{3}$	$\frac{11}{1121}$ $\frac{321}{1}$	$\frac{11}{1111}$ $\frac{221}{1}$	$\frac{11}{1211}$ $\frac{221}{1}$
$a=\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$		$a=\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$		$a=\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

$M_{37}^{(3,3)}$	$M_{37}^{(4,3)}$	$M_{37}^{(5,3)}$	$M_{37}^{(6,3)}$	$M_{37}^{(7,3)}$
$\frac{11}{1211}$ $\frac{231}{1}$	$\frac{11}{1212}$ $\frac{231}{1}$	$\frac{11}{1212}$ $\frac{232}{1}$	$\frac{12}{1212}$ $\frac{332}{1}$	$\frac{11}{1211}$ $\frac{321}{1}$
$a=\begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$	$a=\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$a=\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$	$a=\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b=\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$ $c=d_1 \vee d_2 \vee 0 \vee d_3$ $e=\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$	$a=d_1 \vee d_2 \vee d_2$ $b=\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$



$M_{37}^{(29,3)}$	$M_{37}^{(30,3)}$	$M_{37}^{(31,3)}$	$M_{38}^{(1,3)}$
$\frac{1}{1423}$ $\frac{452}{452}$	$\frac{1}{2423}$ $\frac{452}{452}$	$\frac{2}{1423}$ $\frac{452}{452}$	$\frac{1}{1111}$ $\frac{1212}{1212}$
$a=d_1 \vee d_2 \vee d_3 \vee d_4$ $b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ $c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ $e=0 \vee d_1 \vee d_2$	$a=d_5 \vee d_1 \vee d_2 \vee d_3 \vee d_4$ $b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ $c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$	$a=d_5 \vee d_1 \vee d_2 \vee d_3 \vee d_4$ $b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ $c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$	$a = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

$M_{38}^{(2,3)}$	$M_{38}^{(3,3)}$	$M_{38}^{(4,3)}$	$M_{38}^{(5,3)}$	$M_{38}^{(6,3)}$
$\frac{1}{1111}$ $\frac{2212}{2212}$	$\frac{1}{1211}$ $\frac{2312}{2312}$	$\frac{1}{1211}$ $\frac{2312}{2312}$	$\frac{1}{1211}$ $\frac{2322}{2322}$	$\frac{1}{1221}$ $\frac{2312}{2312}$
$a = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$a = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$	$a = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$	$a = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$a = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

$M_{38}^{(7,3)}$	$M_{38}^{(8,3)}$	$M_{38}^{(9,3)}$	$M_{38}^{(10,3)}$	$M_{38}^{(11,3)}$
$\frac{1}{1211}$ $\frac{2323}{2323}$	$\frac{1}{1221}$ $\frac{2313}{2313}$	$\frac{1}{1221}$ $\frac{2323}{2323}$	$\frac{1}{1221}$ $\frac{2423}{2423}$	$\frac{1}{1321}$ $\frac{2423}{2423}$
$a=d_2 \vee d_3$ $b = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$	$a = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$	$a = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$	$a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ $b = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$	$a = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$ $b = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$

$M_{38}^{(12,3)}$	$M_{38}^{(13,3)}$	$M_{39}^{(1,3)}$	$M_{39}^{(2,3)}$	$M_{39}^{(3,3)}$
$\frac{1}{1321}$ $\frac{3423}{3423}$	$\frac{1}{2321}$ $\frac{3423}{3423}$	$\frac{1}{1211}$ $\frac{231}{231}$	$\frac{1}{1211}$ $\frac{232}{232}$	$\frac{1}{1211}$ $\frac{232}{232}$
$a = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$	$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ $c = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$		$a = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$	$a = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

$M_{39}^{(4,3)}$	$M_{39}^{(5,3)}$	$M_{39}^{(6,3)}$	$M_{39}^{(7,3)}$	$M_{40}^{(1,3)}$
$\frac{1}{1212}$ $\frac{242}{242}$	$\frac{1}{1312}$ $\frac{342}{242}$	$\frac{1}{1312}$ $\frac{342}{342}$	$\frac{1}{2312}$ $\frac{342}{342}$	$\frac{1}{1111}$ $\frac{122}{22}$
$a=d_1 \vee d_2$ $b=d_1 \vee d_4 \vee d_3$ $c = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	$a=0 \vee d_1 \vee d_2$ $b=d_1 \vee d_2 \vee d_4 \vee d_3$ $c = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	$a=d_1 \vee d_2 \vee d_3$ $b=d_1 \vee d_4 \vee d_2 \vee d_3$ $c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$	$a=d_1 \vee d_2 \vee d_3$ $b=d_1 \vee d_2 \vee d_4 \vee d_3$ $c = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$	$a = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$

$M_{41}^{(1,3)}$	$M_{42}^{(1,3)}$	$M_{43}^{(1,3)}$	$M_{44}^{(1,3)}$	$M_{45}^{(1,3)}$
$\begin{matrix} 11 \\ 31 \\ 12 \end{matrix}$	$\begin{matrix} 11 \\ 121 \\ 2132 \end{matrix}$	$\begin{matrix} 11 \\ 111 \\ 23 \end{matrix}$	$\begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}$	$\begin{matrix} 1 \\ 1 & 1 \\ 1 & 1 \end{matrix}$
$a=d_1 \vee d_3, c=d_1 \vee d_2 \vee d_4$ $b=d_1 \vee d_2 \vee 0 \vee d_3$ $e = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$	$a=d_1 \vee d_3, b=d_1 \vee d_2 \vee d_4$ $c=d_1 \vee 0 \vee d_2, f=d_1 \vee d_2 \vee 0 \vee d_3$ $e = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$	$a=d_1 \vee d_3, b=d_1 \vee d_2 \vee d_4$ $e=d_1 \vee d_2 \vee 0 \vee d_3$ $c = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$		

$M_{45}^{(2,3)}$	$M_{45}^{(3,3)}$	$M_{46}^{(1,3)}$	$M_{47}^{(1,3)}$	$M_{47}^{(2,3)}$	$M_{47}^{(3,3)}$	$M_{47}^{(4,3)}$	$M_{47}^{(5,3)}$
$\begin{matrix} 1 \\ 1 & 1 \\ 121 \end{matrix}$	$\begin{matrix} 1 \\ 1 & 2 \\ 121 \end{matrix}$	$\begin{matrix} 111 \\ 111 \end{matrix}$	$\begin{matrix} 111 \\ 111 \end{matrix}$	$\begin{matrix} 111 \\ 121 \end{matrix}$	$\begin{matrix} 211 \\ 121 \end{matrix}$	$\begin{matrix} 112 \\ 121 \end{matrix}$	$\begin{matrix} 212 \\ 121 \end{matrix}$

$M_{47}^{(6,3)}$	$M_{47}^{(7,3)}$	$M_{48}^{(1,3)}$	$M_{48}^{(2,3)}$	$M_{48}^{(3,3)}$	$M_{48}^{(4,3)}$	$M_{48}^{(5,3)}$
$\begin{matrix} 212 \\ 131 \end{matrix}$	$\begin{matrix} 222 \\ 131 \end{matrix}$	$\begin{matrix} 1 \\ 1 & 1 \\ 121 \end{matrix}$	$\begin{matrix} 1 \\ 1 & 2 \\ 121 \end{matrix}$	$\begin{matrix} 1 \\ 1 & 1 \\ 221 \end{matrix}$	$\begin{matrix} 1 \\ 1 & 2 \\ 221 \end{matrix}$	$\begin{matrix} 1 \\ 1 & 2 \\ 131 \end{matrix}$
	$a = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$			$a = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$a = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	

$M_{48}^{(6,3)}$	$M_{48}^{(7,3)}$	$M_{48}^{(8,3)}$	$M_{48}^{(9,3)}$	$M_{49}^{(1,3)}$	$M_{50}^{(1,3)}$	$M_{50}^{(2,3)}$
$\begin{matrix} 1 \\ 1 & 2 \\ 231 \end{matrix}$	$\begin{matrix} 1 \\ 2 & 2 \\ 231 \end{matrix}$	$\begin{matrix} 1 \\ 1 & 2 \\ 231 \end{matrix}$	$\begin{matrix} 2 \\ 1 & 2 \\ 231 \end{matrix}$	$\begin{matrix} 11 \\ 111 \\ 12 \end{matrix}$	$\begin{matrix} 1 \\ 111 \\ 121 \end{matrix}$	$\begin{matrix} 1 \\ 112 \\ 121 \end{matrix}$
$a = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$a=d_2 \vee d_3$ $b = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$	$a = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$	$a = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$			

$M_{50}^{(3,3)}$	$M_{50}^{(4,3)}$	$M_{50}^{(5,3)}$	$M_{51}^{(1,3)}$	$M_{51}^{(2,3)}$	$M_{51}^{(3,3)}$	$M_{51}^{(4,3)}$
$\begin{matrix} 1 \\ 112 \\ 131 \end{matrix}$	$\begin{matrix} 1 \\ 212 \\ 131 \end{matrix}$	$\begin{matrix} 1 \\ 212 \\ 231 \end{matrix}$	$\begin{matrix} 1 \\ 111 \end{matrix}$	$\begin{matrix} 1 \\ 111 \\ 121 \end{matrix}$	$\begin{matrix} 1 \\ 111 \\ 211 \end{matrix}$	$\begin{matrix} 1 \\ 112 \\ 121 \end{matrix}$
$a = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$	$a = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$a = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$				

$M_{51}^{(5,3)}$	$M_{51}^{(6,3)}$	$M_{51}^{(7,3)}$	$M_{51}^{(8,3)}$	$M_{51}^{(9,3)}$	$M_{51}^{(10,3)}$	$M_{51}^{(11,3)}$
$\frac{1}{111}$ $\frac{1}{221}$	$\frac{1}{112}$ $\frac{1}{221}$	$\frac{1}{121}$ $\frac{1}{221}$	$\frac{2}{111}$ $\frac{1}{221}$	$\frac{1}{122}$ $\frac{1}{221}$	$\frac{2}{112}$ $\frac{1}{221}$	$\frac{1}{122}$ $\frac{1}{231}$
			$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$		$a = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$	

$M_{51}^{(12,3)}$	$M_{51}^{(13,3)}$	$M_{51}^{(14,3)}$	$M_{51}^{(15,3)}$	$M_{52}^{(1,3)}$	$M_{52}^{(2,3)}$	$M_{52}^{(3,3)}$
$\frac{2}{112}$ $\frac{1}{231}$	$\frac{2}{122}$ $\frac{1}{231}$	$\frac{2}{122}$ $\frac{1}{331}$	$\frac{2}{222}$ $\frac{1}{331}$	$\frac{1}{111}$ $\frac{1}{111}$	$\frac{1}{111}$ $\frac{1}{121}$	$\frac{1}{121}$ $\frac{1}{121}$
$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$	$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee 0 \vee 0 \vee d_2$	$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee 0 \vee d_3$	$a = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee 0 \vee d_3$			

$M_{52}^{(4,3)}$	$M_{53}^{(1,3)}$	$M_{54}^{(1)}$	$M_{54}^{(2)}$	$M_{54}^{(3)}$	$M_{54}^{(4)}$
$\frac{1}{121}$ $\frac{1}{122}$	$\frac{1}{1111}$ $\frac{1}{111}$	$\frac{1}{1111}$ $\frac{1}{111}$	$\frac{1}{1111}$ $\frac{1}{121}$	$\frac{1}{1211}$ $\frac{1}{121}$	$\frac{1}{1112}$ $\frac{1}{121}$

$M_{54}^{(5)}$	$M_{54}^{(6)}$	$M_{54}^{(7)}$	$M_{54}^{(8)}$	$M_{54}^{(9)}$	$M_{54}^{(10)}$
$\frac{1}{1211}$ $\frac{1}{221}$	$\frac{1}{1212}$ $\frac{1}{121}$	$\frac{1}{1212}$ $\frac{1}{221}$	$\frac{1}{1212}$ $\frac{1}{131}$	$\frac{1}{1212}$ $\frac{1}{231}$	$\frac{1}{1232}$ $\frac{1}{131}$
					$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

$M_{54}^{(11)}$	$M_{54}^{(12)}$	$M_{54}^{(13)}$	$M_{54}^{(14)}$	$M_{54}^{(15)}$	$M_{54}^{(16)}$
$\frac{1}{1312}$ $\frac{1}{231}$	$\frac{1}{1222}$ $\frac{1}{231}$	$\frac{1}{1322}$ $\frac{1}{231}$	$\frac{1}{1322}$ $\frac{1}{241}$	$\frac{1}{1323}$ $\frac{1}{241}$	$\frac{1}{1323}$ $\frac{1}{242}$
	$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$	$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$	$b = d_1 \vee d_4$ $a = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$	$b = d_1 \vee d_2 \vee d_4$ $a = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $c = [0 \ 1 \ 0]$	$b = d_1 \vee d_2 \vee d_4$ $a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$

$M_{55}^{(1)}$	$M_{55}^{(2)}$	$M_{55}^{(3)}$	$M_{55}^{(4)}$	$M_{55}^{(5)}$	$M_{56}^{(1)}$
$\frac{1}{1111}$ $\frac{1}{111}$	$\frac{1}{1111}$ $\frac{1}{211}$	$\frac{1}{1121}$ $\frac{1}{211}$	$\frac{1}{1121}$ $\frac{1}{221}$	$\frac{1}{1122}$ $\frac{1}{221}$	$\frac{1}{1111}$ $\frac{1}{2 \ 1}$



$M_{57}^{(24)}$	$M_{57}^{(25)}$	$M_{57}^{(26)}$	$M_{57}^{(27)}$	$M_{57}^{(28)}$	$M_{57}^{(29)}$	$M_{57}^{(30)}$
$\frac{2}{1}$ $\frac{131}{242}$	$\frac{2}{1}$ $\frac{231}{242}$	$\frac{1}{2}$ $\frac{231}{242}$	$\frac{2}{2}$ $\frac{131}{242}$	$\frac{2}{1}$ $\frac{231}{342}$	$\frac{1}{2}$ $\frac{231}{342}$	$\frac{2}{2}$ $\frac{131}{342}$
$a = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_4$	$a = d_2 \vee d_3 \vee d_4$ $b = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$	$a = d_2 \vee d_3 \vee d_4$ $b = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$	$a = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$	$a = d_2 \vee d_3 \vee d_4$ $b = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$	$a = d_2 \vee d_3 \vee d_4$ $b = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}$	$a = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$

$M_{57}^{(31)}$	$M_{57}^{(32)}$	$M_{57}^{(33)}$	$M_{57}^{(34)}$	$M_{57}^{(35)}$	$M_{58}^{(1)}$
$\frac{2}{2}$ $\frac{231}{342}$	$\frac{2}{2}$ $\frac{231}{352}$	$\frac{2}{2}$ $\frac{241}{352}$	$\frac{2}{2}$ $\frac{241}{353}$	$\frac{2}{2}$ $\frac{242}{353}$	$\frac{1}{11}$ $\frac{111}{12}$
$a = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}$	$a = d_2 \vee d_3 \vee d_4 \vee d_5$ $b = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \end{bmatrix}$	$a = d_2 \vee d_3 \vee d_4 \vee d_5$ $b = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$	$a = d_2 \vee d_3 \vee d_4 \vee d_5$ $b = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$	$a = d_2 \vee d_3 \vee d_4 \vee d_5$ $b = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$	

$M_{58}^{(2)}$	$M_{58}^{(3)}$	$M_{58}^{(4)}$	$M_{58}^{(5)}$	$M_{58}^{(6)}$	$M_{58}^{(7)}$	$M_{58}^{(8)}$
$\frac{1}{11}$ $\frac{111}{22}$	$\frac{1}{11}$ $\frac{111}{13}$	$\frac{1}{11}$ $\frac{111}{23}$	$\frac{1}{12}$ $\frac{111}{13}$	$\frac{1}{11}$ $\frac{211}{23}$	$\frac{1}{11}$ $\frac{111}{23}$	$\frac{2}{11}$ $\frac{111}{23}$
$a = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$		$a = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$		$a = d_2 \vee d_3$ $b = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$	$a = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$	$a = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

$M_{58}^{(9)}$	$M_{58}^{(10)}$	$M_{58}^{(11)}$	$M_{58}^{(12)}$	$M_{58}^{(13)}$	$M_{58}^{(14)}$	$M_{58}^{(15)}$
$\frac{1}{12}$ $\frac{111}{23}$	$\frac{1}{12}$ $\frac{211}{23}$	$\frac{1}{22}$ $\frac{111}{23}$	$\frac{2}{2}$ $\frac{111}{23}$	$\frac{1}{12}$ $\frac{211}{24}$	$\frac{1}{22}$ $\frac{111}{24}$	$\frac{2}{12}$ $\frac{111}{24}$
$a = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$	$a = d_2 \vee d_3$ $b = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$	$a = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$	$a = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$	$a = d_2 \vee d_3$ $b = d_1 \vee d_3$ $c = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ $e = d_1 \vee d_2 \vee d_4$	$a = d_1 \vee d_3$ $b = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ $c = d_1 \vee d_2 \vee d_4$	$a = d_1 \vee d_3$ $b = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ $c = d_1 \vee d_2 \vee d_4$





$M_{60}^{(3)}$	$M_{60}^{(4)}$	$M_{60}^{(5)}$	$M_{60}^{(6)}$	$M_{61}^{(1)}$	$M_{61}^{(2)}$
$\begin{matrix} 1 \\ 212 \\ 231 \end{matrix}$	$\begin{matrix} 1 \\ 212 \\ 241 \end{matrix}$	$\begin{matrix} 1 \\ 213 \\ 241 \end{matrix}$	$\begin{matrix} 1 \\ 213 \\ 242 \end{matrix}$	$\begin{matrix} 11 \\ 1111 \\ 12 \end{matrix}$	$\begin{matrix} 11 \\ 1111 \\ 22 \end{matrix}$
$a = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$	$a = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_4$	$a = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_4$ $c = [0 \ 1 \ 0]$	$a = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_4$		

$M_{61}^{(3)}$	$M_{61}^{(4)}$	$M_{61}^{(5)}$	$M_{61}^{(6)}$	$M_{61}^{(7)}$	$M_{61}^{(8)}$
$\begin{matrix} 11 \\ 1211 \\ 22 \end{matrix}$	$\begin{matrix} 21 \\ 1111 \\ 22 \end{matrix}$	$\begin{matrix} 11 \\ 1211 \\ 23 \end{matrix}$	$\begin{matrix} 21 \\ 1111 \\ 23 \end{matrix}$	$\begin{matrix} 12 \\ 1211 \\ 23 \end{matrix}$	$\begin{matrix} 21 \\ 1211 \\ 23 \end{matrix}$
	$a = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$		$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$		$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee 0 \vee 0 \vee d_2$

$M_{61}^{(9)}$	$M_{61}^{(10)}$	$M_{61}^{(11)}$	$M_{61}^{(12)}$	$M_{61}^{(13)}$	$M_{61}^{(14)}$
$\begin{matrix} 22 \\ 1111 \\ 23 \end{matrix}$	$\begin{matrix} 21 \\ 1211 \\ 33 \end{matrix}$	$\begin{matrix} 22 \\ 1211 \\ 23 \end{matrix}$	$\begin{matrix} 21 \\ 2211 \\ 33 \end{matrix}$	$\begin{matrix} 22 \\ 1211 \\ 33 \end{matrix}$	$\begin{matrix} 22 \\ 1211 \\ 24 \end{matrix}$
$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$	$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee 0 \vee d_3$	$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee 0 \vee 0 \vee d_2$	$a = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee 0 \vee d_3$	$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee 0 \vee d_3$	$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_3$ $c = d_1 \vee 0 \vee 0 \vee d_2$ $e = d_1 \vee d_2 \vee d_4$

$M_{61}^{(15)}$	$M_{61}^{(16)}$	$M_{61}^{(17)}$	$M_{61}^{(18)}$	$M_{61}^{(19)}$
$\begin{matrix} 22 \\ 2211 \\ 33 \end{matrix}$	$\begin{matrix} 22 \\ 1211 \\ 34 \end{matrix}$	$\begin{matrix} 22 \\ 2211 \\ 34 \end{matrix}$	$\begin{matrix} 22 \\ 1311 \\ 34 \end{matrix}$	$\begin{matrix} 32 \\ 1311 \\ 34 \end{matrix}$
$a = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee 0 \vee d_3$	$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_3$ $c = d_1 \vee d_2 \vee 0 \vee d_3$ $e = d_1 \vee d_2 \vee d_4$	$a = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_3$ $c = d_1 \vee d_2 \vee 0 \vee d_3$ $e = d_1 \vee d_2 \vee d_4$	$a = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$ $b = d_1 \vee d_2 \vee 0 \vee 0 \vee d_3$	$a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ $b = d_1 \vee d_2 \vee 0 \vee 0 \vee d_3$





$M_{64}^{(21)}$  $\begin{matrix} 1 \\ 3241 \\ 252 \end{matrix}$	$M_{64}^{(22)}$  $\begin{matrix} 1 \\ 3241 \\ 253 \end{matrix}$	$M_{64}^{(23)}$  $\begin{matrix} 1 \\ 3242 \\ 253 \end{matrix}$	$M_{65}^{(1)}$  $\begin{matrix} 11 \\ 1111 \\ 31 \end{matrix}$	$M_{66}^{(1)}$  $\begin{matrix} 1 \\ 1111 \\ 111 \end{matrix}$
$a = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$b = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$a = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$b = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$a = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
$M_{66}^{(2)}$  $\begin{matrix} 1 \\ 1111 \\ 121 \end{matrix}$	$M_{66}^{(3)}$  $\begin{matrix} 1 \\ 1111 \\ 211 \end{matrix}$	$M_{66}^{(4)}$  $\begin{matrix} 1 \\ 1121 \\ 121 \end{matrix}$	$M_{66}^{(5)}$  $\begin{matrix} 1 \\ 1121 \\ 221 \end{matrix}$	$M_{66}^{(6)}$  $\begin{matrix} 1 \\ 1121 \\ 122 \end{matrix}$
$M_{66}^{(7)}$  $\begin{matrix} 1 \\ 1121 \\ 221 \end{matrix}$	$M_{66}^{(8)}$  $\begin{matrix} 1 \\ 1211 \\ 221 \end{matrix}$	$M_{66}^{(9)}$  $\begin{matrix} 2 \\ 1111 \\ 221 \end{matrix}$	$M_{66}^{(10)}$  $\begin{matrix} 1 \\ 1121 \\ 222 \end{matrix}$	$M_{66}^{(11)}$  $\begin{matrix} 1 \\ 1221 \\ 221 \end{matrix}$
		$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$		
$M_{66}^{(12)}$  $\begin{matrix} 2 \\ 1121 \\ 221 \end{matrix}$	$M_{66}^{(13)}$  $\begin{matrix} 1 \\ 1221 \\ 231 \end{matrix}$	$M_{66}^{(14)}$  $\begin{matrix} 1 \\ 1221 \\ 222 \end{matrix}$	$M_{66}^{(15)}$  $\begin{matrix} 2 \\ 1121 \\ 231 \end{matrix}$	$M_{66}^{(16)}$  $\begin{matrix} 2 \\ 1121 \\ 222 \end{matrix}$
$a = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$			$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$	$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$
$M_{66}^{(17)}$  $\begin{matrix} 1 \\ 1221 \\ 232 \end{matrix}$	$M_{66}^{(18)}$  $\begin{matrix} 2 \\ 1221 \\ 231 \end{matrix}$	$M_{66}^{(19)}$  $\begin{matrix} 2 \\ 1121 \\ 232 \end{matrix}$	$M_{66}^{(20)}$  $\begin{matrix} 1 \\ 1231 \\ 232 \end{matrix}$	$M_{66}^{(21)}$  $\begin{matrix} 2 \\ 1221 \\ 331 \end{matrix}$
	$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$	$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$		$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$
	$b = d_1 \vee 0 \vee 0 \vee d_2$			$b = d_1 \vee d_2 \vee 0 \vee d_3$
$M_{66}^{(22)}$  $\begin{matrix} 2 \\ 1221 \\ 232 \end{matrix}$	$M_{66}^{(23)}$  $\begin{matrix} 2 \\ 1131 \\ 232 \end{matrix}$	$M_{66}^{(24)}$  $\begin{matrix} 2 \\ 2221 \\ 331 \end{matrix}$	$M_{66}^{(25)}$  $\begin{matrix} 2 \\ 1221 \\ 332 \end{matrix}$	$M_{66}^{(26)}$  $\begin{matrix} 2 \\ 1231 \\ 232 \end{matrix}$
$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$	$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$	$a = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$	$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$	$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$
$b = d_1 \vee 0 \vee 0 \vee d_2$		$b = d_1 \vee d_2 \vee 0 \vee d_3$	$b = d_1 \vee d_2 \vee 0 \vee d_3$	$b = d_1 \vee 0 \vee 0 \vee d_2$

$M_{66}^{(27)}$	$M_{66}^{(28)}$	$M_{66}^{(29)}$	$M_{66}^{(30)}$	$M_{66}^{(31)}$
$\begin{matrix} 2 \\ 2231 \\ 332 \end{matrix}$	$\begin{matrix} 2 \\ 1231 \\ 332 \end{matrix}$	$\begin{matrix} 2 \\ 1231 \\ 242 \end{matrix}$	$\begin{matrix} 2 \\ 2231 \\ 332 \end{matrix}$	$\begin{matrix} 2 \\ 1231 \\ 342 \end{matrix}$
$a = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_3$	$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_3$	$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_3$ $c = d_1 \vee d_2 \vee d_4$	$a = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_3$	$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_3$ $c = d_1 \vee d_2 \vee d_4$

$M_{66}^{(32)}$	$M_{66}^{(33)}$	$M_{66}^{(34)}$	$M_{66}^{(35)}$	$M_{66}^{(36)}$
$\begin{matrix} 2 \\ 2331 \\ 342 \end{matrix}$	$\begin{matrix} 2 \\ 1331 \\ 342 \end{matrix}$	$\begin{matrix} 3 \\ 1231 \\ 342 \end{matrix}$	$\begin{matrix} 2 \\ 2331 \\ 342 \end{matrix}$	$\begin{matrix} 3 \\ 2341 \\ 342 \end{matrix}$
$a = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_3$ $c = d_1 \vee d_2 \vee d_4$	$a = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_3$	$a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_3$	$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_3$	$a = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_3$

$M_{66}^{(37)}$	$M_{66}^{(38)}$	$M_{66}^{(39)}$	$M_{66}^{(40)}$	$M_{66}^{(41)}$
$\begin{matrix} 2 \\ 2331 \\ 442 \end{matrix}$	$\begin{matrix} 3 \\ 2231 \\ 442 \end{matrix}$	$\begin{matrix} 3 \\ 2331 \\ 442 \end{matrix}$	$\begin{matrix} 3 \\ 3331 \\ 452 \end{matrix}$	$\begin{matrix} 3 \\ 2341 \\ 452 \end{matrix}$
$a = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$ $b = d_2 \vee d_3$ $c = d_1 \vee d_2 \vee d_3 \vee d_4$	$a = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ $b = d_2 \vee d_3$ $c = d_1 \vee d_2 \vee d_3 \vee d_4$	$a = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_3 \vee d_4$	$a = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ $b = d_2 \vee d_3$ $c = d_1 \vee d_2 \vee d_3 \vee d_4$	$a = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ $b = d_2 \vee d_3$ $c = d_1 \vee d_2 \vee d_3 \vee d_4$

$M_{66}^{(42)}$	$M_{66}^{(43)}$	$M_{67}^{(1)}$	$M_{67}^{(2)}$	$M_{67}^{(3)}$
$\begin{matrix} 3 \\ 2341 \\ 453 \end{matrix}$	$\begin{matrix} 3 \\ 2342 \\ 453 \end{matrix}$	$\begin{matrix} 1 & 1 \\ 1 & 1 \\ 2 & 1 \end{matrix}$	$\begin{matrix} 1 & 1 \\ 1 & 1 \\ 2 & 1 \end{matrix}$	$\begin{matrix} 1 & 1 \\ 1 & 1 \\ 2 & 1 \end{matrix}$
$a = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ $b = d_2 \vee d_3$ $c = d_1 \vee d_2 \vee d_3 \vee d_4$	$a = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ $b = d_2 \vee d_3$ $c = d_1 \vee d_2 \vee d_3 \vee d_4$			

$M_{67}^{(4)}$	$M_{67}^{(5)}$	$M_{67}^{(6)}$	$M_{67}^{(7)}$	$M_{67}^{(8)}$
$\begin{matrix} 1 & 1 \\ 1 & 2 \\ 2 & 2 \end{matrix}$	$\begin{matrix} 1 & 1 \\ 1 & 2 \\ 2 & 2 \end{matrix}$	$\begin{matrix} 1 & 1 \\ 1 & 2 \\ 3 & 2 \end{matrix}$	$\begin{matrix} 1 & 1 \\ 1 & 2 \\ 2 & 3 \end{matrix}$	$\begin{matrix} 1 & 1 \\ 1 & 2 \\ 3 & 2 \end{matrix}$
$a = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$			$a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$	

$M_{67}^{(9)}$	$M_{67}^{(10)}$	$M_{67}^{(11)}$	$M_{67}^{(12)}$	$M_{67}^{(13)}$
$\begin{matrix} 1 \\ 2121 \\ 321 \end{matrix}$	$\begin{matrix} 1 \\ 1122 \\ 331 \end{matrix}$	$\begin{matrix} 1 \\ 2122 \\ 321 \end{matrix}$	$\begin{matrix} 2 \\ 1122 \\ 331 \end{matrix}$	$\begin{matrix} 1 \\ 1132 \\ 331 \end{matrix}$
$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$		$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$	$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $c = d_1 \vee d_2 \vee 0 \vee d_3$	$b = d_1 \vee d_1 \vee d_2 \vee d_3$

$M_{67}^{(14)}$	$M_{67}^{(15)}$	$M_{67}^{(16)}$	$M_{67}^{(17)}$	$M_{67}^{(18)}$
$\begin{matrix} 1 \\ 2122 \\ 331 \end{matrix}$	$\begin{matrix} 2 \\ 2122 \\ 331 \end{matrix}$	$\begin{matrix} 1 \\ 2132 \\ 331 \end{matrix}$	$\begin{matrix} 2 \\ 2122 \\ 431 \end{matrix}$	$\begin{matrix} 1 \\ 2132 \\ 431 \end{matrix}$
$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$	$a = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$ $c = d_1 \vee d_2 \vee 0 \vee d_3$	$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ $b = d_1 \vee d_1 \vee d_2 \vee d_3$	$a = d_1 \vee d_3 \quad b = d_1 \vee d_4$ $c = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $e = d_1 \vee d_2 \vee d_4$ $f = d_1 \vee d_2 \vee 0 \vee d_3$	$a = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_4$ $c = d_4 \vee d_1 \vee d_2 \vee d_3$

$M_{67}^{(19)}$	$M_{67}^{(20)}$	$M_{67}^{(21)}$	$M_{67}^{(22)}$	$M_{67}^{(23)}$
$\begin{matrix} 2 \\ 2132 \\ 431 \end{matrix}$	$\begin{matrix} 1 \\ 2232 \\ 431 \end{matrix}$	$\begin{matrix} 2 \\ 2132 \\ 441 \end{matrix}$	$\begin{matrix} 2 \\ 2133 \\ 441 \end{matrix}$	$\begin{matrix} 2 \\ 2133 \\ 442 \end{matrix}$
$a = d_1 \vee d_3$ $b = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $c = d_1 \vee d_2 \vee d_4$ $e = d_1 \vee d_4 \vee d_1 \vee d_2 \vee d_3$	$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_4$ $c = d_4 \vee d_1 \vee d_2 \vee d_3$	$a = d_1 \vee d_3$ $b = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $c = d_1 \vee d_2 \vee d_4$ $e = d_1 \vee d_4 \vee d_1 \vee d_2 \vee d_3$	$a = d_1 \vee d_3$ $b = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $c = d_1 \vee d_2 \vee d_4$ $e = d_1 \vee d_4 \vee d_1 \vee d_2 \vee d_3$	$a = d_1 \vee d_3$ $b = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $c = d_1 \vee d_2 \vee d_4$ $e = d_1 \vee d_4 \vee d_1 \vee d_2 \vee d_3$

$M_{68}^{(1)}$	$M_{69}^{(1)}$	$M_{69}^{(2)}$	$M_{69}^{(3)}$	$M_{69}^{(4)}$
$\begin{matrix} 1 & 1 \\ 121 \\ 122 \end{matrix}$	$\begin{matrix} 11111 \\ 111 \end{matrix}$	$\begin{matrix} 11111 \\ 211 \end{matrix}$	$\begin{matrix} 11211 \\ 211 \end{matrix}$	$\begin{matrix} 11211 \\ 221 \end{matrix}$

$M_{69}^{(5)}$	$M_{69}^{(6)}$	$M_{70}^{(1)}$	$M_{70}^{(2)}$	$M_{70}^{(3)}$
$\begin{matrix} 11221 \\ 221 \end{matrix}$	$\begin{matrix} 11221 \\ 222 \end{matrix}$	$\begin{matrix} 1 \\ 1211 \\ 221 \end{matrix}$	$\begin{matrix} 1 \\ 1212 \\ 221 \end{matrix}$	$\begin{matrix} 1 \\ 1211 \\ 231 \end{matrix}$



$M_{71}^{(1)}$	$M_{72}^{(1)}$	$M_{72}^{(2)}$	$M_{72}^{(3)}$	$M_{72}^{(4)}$
$\begin{matrix} 1 \\ 1212 \\ 231 \end{matrix}$	$\begin{matrix} 11 \\ 111 \\ 21 \end{matrix}$	$\begin{matrix} 11 \\ 1112 \\ 21 \end{matrix}$	$\begin{matrix} 11 \\ 1112 \\ 31 \end{matrix}$	$\begin{matrix} 12 \\ 1112 \\ 31 \end{matrix}$
$a = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$				

$M_{72}^{(5)}$	$M_{72}^{(6)}$	$M_{72}^{(7)}$	$M_{72}^{(8)}$	$M_{72}^{(9)}$
$\begin{matrix} 11 \\ 1122 \\ 31 \end{matrix}$	$\begin{matrix} 12 \\ 2112 \\ 31 \end{matrix}$	$\begin{matrix} 12 \\ 1122 \\ 31 \end{matrix}$	$\begin{matrix} 12 \\ 2122 \\ 31 \end{matrix}$	$\begin{matrix} 12 \\ 1122 \\ 41 \end{matrix}$
$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$		$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$	$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$	$a = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_4$

$M_{72}^{(10)}$	$M_{72}^{(11)}$	$M_{72}^{(12)}$	$M_{72}^{(13)}$	$M_{72}^{(14)}$
$\begin{matrix} 12 \\ 2122 \\ 41 \end{matrix}$	$\begin{matrix} 12 \\ 1123 \\ 41 \end{matrix}$	$\begin{matrix} 13 \\ 2122 \\ 41 \end{matrix}$	$\begin{matrix} 12 \\ 2123 \\ 41 \end{matrix}$	$\begin{matrix} 12 \\ 1123 \\ 42 \end{matrix}$
$a = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_4$	$a = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_4$ $c = [0 \ 1 \ 0]$	$a = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_4$	$a = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_4$ $c = [0 \ 1 \ 0]$	$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_4$

$M_{72}^{(15)}$	$M_{72}^{(16)}$	$M_{72}^{(17)}$	$M_{72}^{(18)}$	$M_{72}^{(19)}$
$\begin{matrix} 13 \\ 2123 \\ 41 \end{matrix}$	$\begin{matrix} 12 \\ 2123 \\ 42 \end{matrix}$	$\begin{matrix} 13 \\ 2123 \\ 51 \end{matrix}$	$\begin{matrix} 13 \\ 2123 \\ 42 \end{matrix}$	$\begin{matrix} 13 \\ 2133 \\ 51 \end{matrix}$
$a = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_4$ $c = [0 \ 1 \ 0]$	$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_4$	$a = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_5$ $c = [0 \ 1 \ 0]$	$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_4$	$a = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_5$ $c = [0 \ 1 \ 0]$

$M_{72}^{(20)}$	$M_{72}^{(21)}$	$M_{72}^{(22)}$	$M_{72}^{(23)}$	$M_{72}^{(24)}$
$\begin{matrix} 13 \\ 2123 \\ 52 \end{matrix}$	$\begin{matrix} 13 \\ 2133 \\ 52 \end{matrix}$	$\begin{matrix} 13 \\ 2124 \\ 52 \end{matrix}$	$\begin{matrix} 13 \\ 2134 \\ 52 \end{matrix}$	$\begin{matrix} 13 \\ 2134 \\ 62 \end{matrix}$
$a = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_5$	$a = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_5$	$a = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_3 \vee d_5$ $c = 0 \vee d_1 \vee d_2 \vee 0$	$a = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_3 \vee d_5$ $c = 0 \vee d_1 \vee d_2 \vee 0$	$a = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_5 \vee d_6$



$M_{72}^{(25)}$	$M_{72}^{(26)}$	$M_{72}^{(27)}$	$M_{72}^{(28)}$	$M_{73}^{(1)}$
$\frac{13}{2234} \frac{62}{62}$	$\frac{14}{2134} \frac{62}{62}$	$\frac{14}{3134} \frac{62}{62}$	$\frac{24}{3134} \frac{62}{62}$	$\frac{1}{1111} \frac{211}{211}$
$a = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_3 \vee d_6$ $c = 0 \vee d_1 \vee d_2 \vee 0$	$a = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_3 \vee d_6$ $c = 0 \vee d_1 \vee d_2 \vee 0$	$a = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_3 \vee d_6$ $c = 0 \vee d_1 \vee d_2 \vee 0$	$a = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_3 \vee d_6$ $c = 0 \vee d_1 \vee d_2 \vee 0$	

$M_{73}^{(2)}$	$M_{73}^{(3)}$	$M_{73}^{(4)}$	$M_{73}^{(5)}$	$M_{73}^{(6)}$	$M_{73}^{(7)}$
$\frac{1}{1121} \frac{211}{211}$	$\frac{1}{1121} \frac{221}{221}$	$\frac{1}{1121} \frac{311}{311}$	$\frac{1}{1122} \frac{221}{221}$	$\frac{1}{1121} \frac{321}{321}$	$\frac{1}{2121} \frac{311}{311}$
					$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

$M_{73}^{(8)}$	$M_{73}^{(9)}$	$M_{73}^{(10)}$	$M_{73}^{(11)}$	$M_{73}^{(12)}$
$\frac{1}{1122} \frac{321}{321}$	$\frac{1}{1131} \frac{321}{321}$	$\frac{1}{2121} \frac{321}{321}$	$\frac{1}{1132} \frac{321}{321}$	$\frac{1}{2122} \frac{321}{321}$
		$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$		$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

$M_{73}^{(13)}$	$M_{73}^{(14)}$	$M_{73}^{(15)}$	$M_{73}^{(16)}$	$M_{73}^{(17)}$
$\frac{1}{2131} \frac{321}{321}$	$\frac{1}{1132} \frac{331}{331}$	$\frac{1}{2132} \frac{321}{321}$	$\frac{1}{2131} \frac{421}{421}$	$\frac{1}{2132} \frac{331}{331}$
$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$		$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$	$a = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_4$	$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

$M_{73}^{(18)}$	$M_{73}^{(19)}$	$M_{73}^{(20)}$	$M_{73}^{(21)}$	$M_{73}^{(22)}$
$\frac{1}{2132} \frac{421}{421}$	$\frac{1}{2231} \frac{421}{421}$	$\frac{2}{2131} \frac{421}{421}$	$\frac{1}{2132} \frac{431}{431}$	$\frac{1}{2232} \frac{421}{421}$
$a = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_4$	$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_4$	$a = d_1 \vee d_3$ $b = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $c = d_1 \vee d_2 \vee d_4$	$a = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_4$	$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_4$

$M_{73}^{(23)}$	$M_{73}^{(24)}$	$M_{73}^{(25)}$	$M_{73}^{(26)}$	$M_{73}^{(27)}$
$\frac{2}{2132}$ $\frac{421}{421}$	$\frac{1}{2142}$ $\frac{431}{431}$	$\frac{1}{2232}$ $\frac{431}{431}$	$\frac{2}{2132}$ $\frac{431}{431}$	$\frac{1}{2242}$ $\frac{431}{431}$
$a=d_1 \vee d_3$ $b = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $c=d_1 \vee d_2 \vee d_4$	$a = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b=d_1 \vee d_4$ $c=d_1 \vee d_2 \vee d_3$	$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b=d_1 \vee d_2 \vee d_4$	$a=d_1 \vee d_3$ $b = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $c=d_1 \vee d_2 \vee d_4$	$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b=d_1 \vee d_3 \vee d_4$ $c=d_1 \vee d_2 \vee d_3$

$M_{73}^{(28)}$	$M_{73}^{(29)}$	$M_{73}^{(30)}$	$M_{73}^{(31)}$	$M_{73}^{(32)}$
$\frac{2}{2142}$ $\frac{431}{431}$	$\frac{1}{2242}$ $\frac{531}{531}$	$\frac{2}{2142}$ $\frac{531}{531}$	$\frac{1}{3242}$ $\frac{531}{531}$	$\frac{2}{2242}$ $\frac{531}{531}$
$a=d_1 \vee d_3$ $b = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $c=d_1 \vee d_2 \vee d_4$ $e=d_1 \vee d_2 \vee d_3$	$a = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b=d_1 \vee d_2 \vee d_5$	$a=d_1 \vee d_3$ $b = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $c=d_1 \vee d_2 \vee d_5$	$a = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $b=d_1 \vee d_2 \vee d_5$	$a=d_1 \vee d_4$ $b = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $c=d_1 \vee d_2 \vee d_3 \vee d_5$

$M_{73}^{(33)}$	$M_{73}^{(34)}$	$M_{73}^{(35)}$	$M_{73}^{(36)}$	$M_{73}^{(37)}$
$\frac{2}{3142}$ $\frac{531}{531}$	$\frac{2}{3242}$ $\frac{531}{531}$	$\frac{2}{3242}$ $\frac{631}{631}$	$\frac{2}{3252}$ $\frac{631}{631}$	$\frac{2}{3252}$ $\frac{641}{641}$
$a=d_1 \vee d_3$ $b = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $c=d_1 \vee d_2 \vee d_5$	$a=d_1 \vee d_4$ $b = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $c=d_1 \vee d_2 \vee d_3 \vee d_5$	$a = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $b=d_1 \vee d_2 \vee d_5 \vee d_6$	$a=d_1 \vee d_4$ $b = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $b=d_1 \vee d_2 \vee d_3 \vee d_6$	$a=d_1 \vee d_4$ $b = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $b=d_1 \vee d_2 \vee d_3 \vee d_6$

$M_{73}^{(38)}$	$M_{73}^{(39)}$	$M_{74}^{(1)}$	$M_{75}^{(1)}$	$M_{76}^{(1)}$
$\frac{2}{3253}$ $\frac{641}{641}$	$\frac{2}{3253}$ $\frac{642}{642}$	$\frac{1}{11211}$ $\frac{22}{22}$	$\frac{11}{11111}$ $\frac{2}{2}$	$\frac{1}{11}$ $\frac{112}{112}$ $\frac{31}{31}$
$a=d_1 \vee d_4$ $b = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $b=d_1 \vee d_2 \vee d_3 \vee d_6$	$a=d_1 \vee d_4$ $b = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $b=d_1 \vee d_2 \vee d_3 \vee d_6$			

$M_{76}^{(2)}$	$M_{76}^{(3)}$	$M_{76}^{(4)}$	$M_{76}^{(5)}$	$M_{76}^{(6)}$
$\begin{matrix} 1 \\ 11 \\ 122 \\ 31 \end{matrix}$	$\begin{matrix} 1 \\ 11 \\ 122 \\ 41 \end{matrix}$	$\begin{matrix} 2 \\ 11 \\ 122 \\ 41 \end{matrix}$	$\begin{matrix} 1 \\ 11 \\ 123 \\ 41 \end{matrix}$	$\begin{matrix} 2 \\ 11 \\ 123 \\ 41 \end{matrix}$
$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$	$a = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_4$	$a = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_4$	$a = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_4$ $c = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$a = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_4$ $c = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$M_{76}^{(7)}$	$M_{76}^{(8)}$	$M_{76}^{(9)}$	$M_{76}^{(10)}$	$M_{76}^{(11)}$
$\begin{matrix} 1 \\ 11 \\ 123 \\ 42 \end{matrix}$	$\begin{matrix} 2 \\ 11 \\ 123 \\ 51 \end{matrix}$	$\begin{matrix} 2 \\ 11 \\ 123 \\ 42 \end{matrix}$	$\begin{matrix} 2 \\ 11 \\ 133 \\ 51 \end{matrix}$	$\begin{matrix} 2 \\ 11 \\ 123 \\ 52 \end{matrix}$
$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_4$	$a = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_3 \vee d_5$ $c = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$a = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_4$	$a = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_3 \vee d_5$ $c = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$	$a = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_5$

$M_{76}^{(12)}$	$M_{76}^{(13)}$	$M_{76}^{(14)}$	$M_{76}^{(15)}$	$M_{76}^{(16)}$
$\begin{matrix} 2 \\ 11 \\ 133 \\ 52 \end{matrix}$	$\begin{matrix} 2 \\ 11 \\ 124 \\ 52 \end{matrix}$	$\begin{matrix} 2 \\ 11 \\ 134 \\ 52 \end{matrix}$	$\begin{matrix} 2 \\ 11 \\ 134 \\ 62 \end{matrix}$	$\begin{matrix} 3 \\ 11 \\ 134 \\ 62 \end{matrix}$
$a = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_5$	$a = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_3 \vee d_5$ $c = 0 \vee d_1 \vee d_2 \vee 0$	$a = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_3 \vee d_5$ $c = 0 \vee d_1 \vee d_2 \vee 0$	$a = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_3 \vee d_6$	$a = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_3 \vee d_6$ $c = 0 \vee d_1 \vee d_2 \vee 0$

$M_{76}^{(17)}$	$M_{76}^{(18)}$	$M_{76}^{(19)}$	$M_{77}^{(1)}$	$M_{78}^{(1)}$
$\begin{matrix} 2 \\ 11 \\ 234 \\ 62 \end{matrix}$	$\begin{matrix} 2 \\ 12 \\ 134 \\ 62 \end{matrix}$	$\begin{matrix} 3 \\ 21 \\ 134 \\ 62 \end{matrix}$	$\begin{matrix} 11 \\ 1112 \\ 31 \end{matrix}$	$\begin{matrix} 11 \\ 1122 \\ 331 \end{matrix}$
$a = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_3 \vee d_6$ $c = 0 \vee d_1 \vee d_2 \vee 0$	$a = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_3 \vee d_6$ $c = 0 \vee d_1 \vee d_2 \vee 0$	$a = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ $b = d_1 \vee d_2 \vee d_3 \vee d_6$ $c = 0 \vee d_1 \vee d_2 \vee 0$		$a = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} b = d_2 \vee d_4$ $a = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$

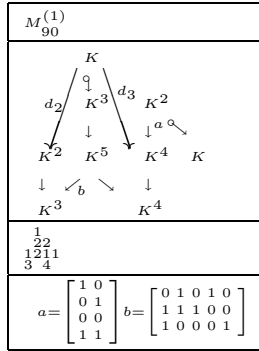
$M_{79}^{(1)}$	$M_{80}^{(1)}$	$M_{81}^{(1)}$	$M_{82}^{(1)}$	$M_{82}^{(2)}$
$\begin{matrix} 1 & 1 \\ 12 & 11 \\ 2 & 3 \end{matrix}$	$\begin{matrix} 2 \\ 32 & 1 \\ 25 \end{matrix}$	$\begin{matrix} 131 \\ 32 & 12 \\ 25 \end{matrix}$	$\begin{matrix} 11 \\ 112 & 1 \\ 321 \end{matrix}$	$\begin{matrix} 11 \\ 112 & 1 \\ 321 \end{matrix}$
	$a=0 \vee d_1 \vee d_2 \vee 0$ $b=d_1 \vee d_2 \vee d_4 \vee d_5$ $c = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$a=0 \vee d_1 \vee d_2 \vee 0$ $b=d_1 \vee d_2 \vee d_4 \vee d_5$ $c = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$		

$M_{82}^{(3)}$	$M_{82}^{(4)}$	$M_{82}^{(5)}$	$M_{82}^{(6)}$	$M_{82}^{(7)}$
$\begin{matrix} 11 \\ 1122 \\ 331 \end{matrix}$	$\begin{matrix} 11 \\ 1132 \\ 331 \end{matrix}$	$\begin{matrix} 11 \\ 1132 \\ 431 \end{matrix}$	$\begin{matrix} 11 \\ 1232 \\ 431 \end{matrix}$	$\begin{matrix} 21 \\ 1132 \\ 431 \end{matrix}$
	$a=d_1 \vee d_1 \vee d_2 \vee d_3$	$a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ $c=d_4 \vee d_1 \vee d_2 \vee d_3$	$a = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $c=d_4 \vee d_1 \vee d_2 \vee d_3$	$a = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ $c=d_4 \vee d_1 \vee d_2 \vee d_3$

$M_{82}^{(8)}$	$M_{82}^{(9)}$	$M_{82}^{(10)}$	$M_{82}^{(11)}$
$\begin{matrix} 21 \\ 1232 \\ 431 \end{matrix}$	$\begin{matrix} 21 \\ 1232 \\ 531 \end{matrix}$	$\begin{matrix} 21 \\ 1242 \\ 531 \end{matrix}$	$\begin{matrix} 21 \\ 1242 \\ 541 \end{matrix}$
$a = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $c=d_4 \vee d_1 \vee d_2 \vee d_3$	$a = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $c=d_5 \vee d_1 \vee d_2 \vee d_3$	$a = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $c=d_5 \vee d_1 \vee d_2 \vee d_3 \vee d_4$	$a = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $c=d_1 \vee d_2 \vee 0 \vee d_3 \vee d_4$

$M_{82}^{(12)}$	$M_{82}^{(13)}$	$M_{83}^{(1)}$	$M_{84}^{(1)}$
$\begin{matrix} 21 \\ 1243 \\ 541 \end{matrix}$	$\begin{matrix} 21 \\ 1243 \\ 542 \end{matrix}$	$\begin{matrix} 1122 & 1 \\ 222 \end{matrix}$	$\begin{matrix} 111 \\ 1112 \\ 31 \end{matrix}$
$a = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $c=d_5 \vee d_1 \vee d_2 \vee d_3 \vee d_4$	$a = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$ $b = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $c=d_5 \vee d_1 \vee d_2 \vee d_3 \vee d_4$		

$M_{85}^{(1)}$	$M_{86}^{(1)}$	$M_{87}^{(1)}$	$M_{88}^{(1)}$	$M_{89}^{(1)}$
$\begin{matrix} 11 \\ 11211 \\ 22 \end{matrix}$	$\begin{matrix} 111 \\ 11111 \\ 2 \end{matrix}$	$\begin{matrix} 1 \\ 211 \\ 1312 \\ 4 \end{matrix}$	$\begin{matrix} 111 \\ 1112 \\ 31 \end{matrix}$	$\begin{matrix} 112 \\ 2111 \\ 24 \end{matrix}$
		$a=d_1 \vee d_4 \quad b= \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$		$a=d_1 \vee d_4 \quad b= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$



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(4105)