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REPRESENTATION-DIRECTED ALGEBRAS FORM AN OPEN SCHEME

BY

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Dedicated to Professor Idun Reiten
on the occasion of her sixtieth birthday

Abstract. We apply van den Dries's test to the class of algebras (over algebraically closed fields) which are not representation-directed and prove that this class is axiomatizable by a positive quantifier-free formula. It follows that the representation-directed algebras form an open $\mathbb{Z}$-scheme.

1. Introduction. The well-known result of Gabriel [5] asserts that the representation finite algebras induce a Zariski-open subset in the variety of all associative algebras of fixed dimension over a fixed algebraically closed field. This fact is connected with finite axiomatizability of the class of all representation finite algebras of fixed dimension over algebraically closed fields [6]. Standard model-theoretical techniques applied to the axioms for representation finite algebras allow one to prove that these algebras form a constructible $\mathbb{Z}$-scheme (see [7, Remark 12.60]). In the context of the latter result the following question arises:

QUESTION 1.1. *Do the representation finite algebras form an open $\mathbb{Z}$-scheme?*

Following [7] we say that a class $\mathcal{C}$ of algebras forms an *open $\mathbb{Z}$-scheme* if given a natural number d there exist finitely many polynomials $H_1, \ldots, H_r$ with integral coefficients such that for every algebraically closed field K the set induced by $\mathcal{C}$ in the variety of d-dimensional K-algebras is defined by non-vanishing of at least one of H_i (see Corollary 1.4 below). The answer to Question 1.1 is still not known. The natural way to solve the problem is to apply van den Dries's test (see Theorem 3.1, [7, Theorem 12.6, Corollary 12.8]).

In this paper we give an answer to an analogous, easier question. Namely, we restrict our attention to the class of representation-directed algebras and

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solve the corresponding problem in Corollary 1.4. Recall that an algebra is *representation finite* if it has finitely many isomorphism classes of indecomposable modules, and it is *representation-directed* if in addition the associated Auslander–Reiten quiver [1, Chapter VII] has no oriented cycles. Equivalently, there is no cycle of non-isomorphisms between indecomposable modules, or: every indecomposable module is directing.

To apply van den Dries's test we need to deal with lattices over orders.

Let $v : K \to G \cup \{\infty\}$ be a valuation of an algebraically closed field K with values in an ordered group G and V the corresponding valuation ring with the maximal ideal $\mathfrak{p}$. Recall that for every non-zero $x \in K$ either $x \in V$ or $x^{-1} \in V$. Every finitely generated ideal in V is principal [2]. Every torsion-free V module is flat and every finitely generated torsion-free V-module is free.

Denote the residue field $V/\mathfrak{p}$ of V by k.

Assume that A is a V-*order*, that is, A is a V-algebra which is finitely generated and free as a V-module. Let d be the V-rank of A. We denote by $A^{(K)}$ and $\bar{A}$ the K-algebras $A \otimes_V K$ and $A \otimes_V k$ respectively. There is a canonical ring homomorphism $A \to \bar{A}$ with kernel $\mathfrak{p}A$. The value of this homomorphism on an element $a \in A$ will be denoted by $\overline{a}$.

A finitely generated A-module X is a *lattice* over A if X is free as a V-module. Let $\mathrm{rk}_V(X)$ denote the V-rank of a V-module X. The right A-lattices form a full subcategory $\mathrm{latt}(A)$ in the category $\mathrm{mod}(A)$ of finitely generated right A-modules.

There are canonical functors

$$\overline{(-)} : \mathrm{mod}(V) \to \mathrm{mod}(k), \quad (-)^{(K)} : \mathrm{mod}(V) \to \mathrm{mod}(K) \tag{1.2}$$

defined by $\overline{X} = X \otimes_V k$ and $X^{(K)} = X \otimes_V K$. They are right exact and the latter one is exact. If X is endowed with a structure of a V-algebra (or a module over a V-algebra A) then $\overline{X}$ (resp. $X^{(K)}$) has the induced structure of a k-algebra (resp. K-algebra), or a module over the algebra $\bar{A}$ (resp. $A^{(K)}$). Note that $\overline{X} \cong X/\mathfrak{p}X$; we identify the two k-modules.

We investigate the relationships between the categories $\mathrm{mod}(A^{(K)})$ and $\mathrm{mod}(\bar{A})$ to support the opinion that the latter is at least as complicated as the former. The proof of the following theorem is given in the next section.

THEOREM 1.3. *Assume that V is a valuation ring in an algebraically closed field K with residue field k. Let A be a V-order. If the algebra $\bar{A} = A \otimes_V k$ is representation-directed then so is $A^{(K)} = A \otimes_V K$.*

In Section 4 we show that in the above setting the Auslander–Reiten quivers of $A^{(K)}$ and $\bar{A}$ are isomorphic.

Together with finite axiomatizability of the class of representation-directed algebras (Lemma 3.3) the above theorem allows one to apply van den Dries's test to the class of algebras of fixed dimension over algebraically closed fields which are not representation-directed. We shall prove that this class admits a set of axioms which are equivalent to positive quantifier-free formulas. Such formulas express vanishing of polynomials with integral coefficients. The proof of the following result is given in Section 3.

COROLLARY 1.4. *There is an open $\mathbb{Z}$-scheme of representation-directed algebras of dimension d, that is, there exist polynomials*

$$H_1, \ldots, H_r \in \mathbb{Z}[X_{ijl};\, i,j,l = 1, \ldots, d]$$

such that for every algebraically closed field L, a d-dimensional L-algebra R defined by a system $\gamma = (\gamma_{ijl})_{i,j,l=1,\ldots,d} \in L^{d^3}$ of structure constants is representation-directed if and only if $H_i(\gamma) \neq 0$ for some $i = 1, \ldots, r$.

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2. Proof of Theorem 1.3

LEMMA 2.1. *If R is a finite-dimensional algebra over an algebraically closed field K and $K \subseteq L$ is a field extension then R is representation-directed if and only if $R \otimes_K L$ is representation-directed.*

Proof. Since the extension $K \subseteq L$ is MacLane separable, it follows that R is representation finite if and only if $R \otimes_K L$ is (by [6]). Moreover if this is the case every indecomposable $R \otimes_K L$ module is a direct summand of $X \otimes_K L$ for some R-module X. But since K is algebraically closed the module $X \otimes_K L$ is indecomposable if X is indecomposable. Now the assertion follows easily. ■

Observe first that without loss of generality we can assume that V is maximally complete (see [7]), which guarantees the possibility of lifting idempotents from $\bar{A}$ to A ([7, Theorem 12.28]). Recall that a valued field $(K, v : K \to G \cup \{\infty\})$, or the corresponding valuation ring V, is *maximally complete* if there is no proper extension of v to a field extension of K with the same residue field and value group. Indeed, by [8], [12, Corollary 6] there exists a field $\widetilde{K}$ containing K and a valuation $\widetilde{v} : \widetilde{K} \to G \cup \{\infty\}$ extending v such that $(\widetilde{K}, \widetilde{v})$ is maximally complete with same residue field k. More precisely, if $\widetilde{V}$ is the corresponding valuation ring then the embedding $V \to \widetilde{V}$ induces an isomorphism of residue fields. Since K is algebraically closed the residue field is also algebraically closed and the value group of v is divisible. Therefore $\widetilde{K}$ is also algebraically closed by [12, Proposition 6]. Let $\widetilde{A} = A \otimes_V \widetilde{V}$.

LEMMA 2.2. (a) *The k-algebras $\widetilde{A} \otimes_{\widetilde{V}} k$ and $\bar{A}$ are isomorphic.*

(b) *If the $\widetilde{K}$-algebra $\widetilde{A} \otimes_{\widetilde{V}} \widetilde{K}$ is representation-directed (resp. representation finite) then the algebra $A^{(K)}$ has the same property.*

Proof. The assertion (a) is trivial; in order to prove (b) observe that

$$\widetilde{A} \otimes_{\widetilde{V}} \widetilde{K} \cong A \otimes_V \widetilde{V} \otimes_{\widetilde{V}} \widetilde{K} \cong A \otimes_V K \otimes_K \widetilde{K}.$$

The assertion follows from Lemma 2.1. ∎

ASSUMPTION. From now on we assume that V is a maximally complete valuation ring in an algebraically closed field K and $\bar{A}$ is representation-directed.

Thanks to Lemma 2.2, to prove Theorem 1.3 it is enough to prove that $A^{(K)}$ is representation-directed under the above assumption. We collect some preparatory facts.

If Q is a finite quiver and T is a commutative ring then TQ denotes the path T-algebra of Q. In this paper we consider only directed quivers Q. Given a number n let TQ_n denote the ideal generated by all paths of length at least n. A two-sided ideal I of TQ is *admissible* provided $I \subseteq TQ_2$. Given a vertex x we denote by e_x the idempotent of TQ corresponding to x. We use the same notation for cosets of the idempotents in quotients of TQ. Given an arrow α of Q let $s(\alpha)$ and $t(\alpha)$ be the source and sink of α respectively.

LEMMA 2.3. *Assume that $R = LQ/I$ is a bound quiver L-algebra over an algebraically closed field L, where Q is directed and R is schurian, that is, $\dim_L(e_x R e_y) \le 1$ for all vertices x, y of Q. (We do not assume that I is admissible.) If there exist paths u, v with a common starting vertex and a common ending vertex in Q such that $u \notin I$ and $v \in I$ then there exists an indecomposable non-directing R-module.*

Proof (cf. [17]). Let v be the path $y_0 \xrightarrow{\beta_1} y_1 \xrightarrow{\beta_2} \ldots \xrightarrow{\beta_s} y_s$. Let $e = e_{y_0} + \ldots + e_{y_s}$ and $S = eRe$. Let Q' be the ordinary Gabriel quiver of S and I' an admissible ideal in LQ' such that $S \cong LQ'/I'$. The quiver Q' contains the arrows $\beta_1, \ldots, \beta_s$ and, since $e_{y_0} S e_{y_s} \ne 0$, at least one "new" arrow γ from y_i to y_j, where $0 \le i < j \le s$. Chose γ in such a way that $j - i$ is minimal (it is at least 2 as S is schurian and I' admissible). Let $e' = e_{y_i} + e_{y_{i+1}} + \ldots + e_{y_j}$ and $S' = e'Se'$. Then the ordinary Gabriel quiver Q'' of S' is isomorphic to

$$\begin{array}{ccccc} \circ & & \longrightarrow & & \circ \\ \big\downarrow{\scriptstyle \beta_{i+1}} & & & & \big\uparrow{\scriptstyle \beta_j} \\ \circ & \xrightarrow{\beta_{i+2}} & \cdots & \xrightarrow{\beta_{j-1}} & \circ \end{array}$$

and since S' is schurian we have $S' \cong LQ''/I''$, where I'' contains the path $\beta_{i+1} \ldots \beta_j$.

It is easy to construct a non-directing indecomposable S'-module X and then induce a non-directing indecomposable R-module $X \otimes_{S'} e'R$. ■

Since V is maximally complete there exist pairwise orthogonal idempotents $\varepsilon_1, \ldots, \varepsilon_n$ of A such that $\varepsilon_1 A \oplus \ldots \oplus \varepsilon_n A$ and $\overline{\varepsilon}_1 \bar{A} \oplus \ldots \oplus \overline{\varepsilon}_n \bar{A}$ are decompositions of A and $\bar{A}$ respectively into a direct sum of indecomposable modules [7, Theorem 12.28]. Without loss of generality we can assume there exists a number m such that $\varepsilon_1 A, \ldots, \varepsilon_m A$ are pairwise non-isomorphic and if $m < j \leq n$ then $\varepsilon_j A \cong \varepsilon_i A$ for some $i \leq m$. It follows by [7, Theorem 12.28] that also $\overline{\varepsilon}_1 \bar{A}, \ldots, \overline{\varepsilon}_m \bar{A}$ are pairwise non-isomorphic and if $m < j \leq n$ then $\overline{\varepsilon}_j \bar{A} \cong \overline{\varepsilon}_i \bar{A}$ for some $i \leq m$. Let $\varepsilon = \varepsilon_1 + \ldots + \varepsilon_m$ and $B = \varepsilon A \varepsilon$. The algebra $\overline{B} \cong \overline{\varepsilon} \bar{A} \overline{\varepsilon}$ is basic and Morita equivalent to $\bar{A}$. Note that B is finitely generated and free as a V-module. Let Q be the ordinary quiver of $\overline{B}$ with vertices $1, \ldots, n$. Let

$$\pi : kQ \to \overline{B}$$

be the canonical surjection such that $\pi(e_i) = \varepsilon_i$ for $i = 1, \ldots, n$ with an admissible kernel I. Since $\overline{B}$ is representation-directed the quiver Q is directed.

For every arrow α of Q let b'_α be an element of B such that $\overline{b'_\alpha} = \pi(\alpha)$ and $b_\alpha = \varepsilon_{s(\alpha)} b'_\alpha \varepsilon_{t(\alpha)}$. Then there exists a V-algebra homomorphism

$$\widetilde{\pi} : VQ \to B \tag{2.4}$$

defined by $\widetilde{\pi}(e_x) = \varepsilon_x$ and $\widetilde{\pi}(\alpha) = b_\alpha$ for every vertex x and arrow α of Q. Observe that $\mathrm{Im}(\widetilde{\pi}) + \mathfrak{p}B = B$ and $\widetilde{\pi}$ is surjective by the Nakayama Lemma. Let J be the kernel of $\widetilde{\pi}$. This homomorphism induces a surjective K-algebra homomorphism

$$\widetilde{\pi}^{(K)} : KQ \to B^{(K)}$$

whose kernel is $J^{(K)}$, since B is a torsion-free V-module. It is easy to prove that $B^{(K)}$ is Morita equivalent to $A^{(K)}$. Note also that $\bar{J} = I$.

LEMMA 2.5. *The kernel J of $\widetilde{\pi}$ is an admissible ideal of VQ and therefore $J^{(K)}$ is an admissible ideal of KQ.*

Proof. First we prove that $J \subseteq VQ_1$. Otherwise, since Q has no oriented cycles, $\varrho e_i \in \operatorname{Ker} \widetilde{\pi}$ for some $\varrho \in V$ and i, $1 \leq i \leq m$. Since $\overline{\varepsilon}_i \neq 0$ in $\overline{B}$ it follows that $e_i \notin \operatorname{Ker} \widetilde{\pi}$, which contradicts the fact that B is torsion-free. (Observe that till now we have not used the assumption that $\overline{B}$ is representation-directed, only that Q is a directed quiver.)

Now suppose that J is not contained in VQ_2. Since $\overline{B}$ is representation finite, for every i, j the V-module $\varepsilon_i B \varepsilon_j$ has rank at most 1. The quiver Q has no multiple arrows. Then there exist an arrow α and a path u in Q with common sink and common source and such that $\varrho_1 u + \varrho_2 \alpha \in J$ for some non-zero $\varrho_1, \varrho_2 \in K$. Again since B is torsion-free we can assume without loss of generality that $\varrho_1, \varrho_2 \in V$ and one of them equals 1. Clearly $\overline{\varrho}_1 u + \overline{\varrho}_2 \alpha \in \operatorname{Ker} \pi$ and since $\operatorname{Ker} \pi$ is admissible in kQ it follows that $\overline{\varrho}_2 = 0$. But then $\overline{B}$ is not representation-directed by Lemma 2.3. ■

Let us remark that our way of presenting a V-lattice by a quiver and an admissible ideal of the path algebra, as well as the notion of an admissible ideal, differs from the construction in [13].

Thanks to Morita equivalences mentioned above and Lemma 2.5 we can reduce our problem to the following situation: having a V-order A of the form VQ/J with directed quiver Q and an admissible ideal J of VQ prove that $A^{(K)} \cong KQ/J^{(K)}$ is representation-directed if so is $\bar{A} \cong kQ/\bar{J}$.

Let us introduce the following notation. For a vertex i of Q let

$$d_i = \mathbf{dim}\, e_i \bar{A} = \mathbf{dim}\, e_i A^{(K)}.$$

Here $\mathbf{dim}$ denotes the dimension vector. If $\operatorname{rad} e_i \bar{A} \cong R_{i,1} \oplus \ldots \oplus R_{i,n_i}$ with $R_{i,j}$ indecomposable for $j = 1, \ldots, n_i$ (and pairwise non-isomorphic since $\bar{A}$ is representation finite) then denote by r^i_j the vector $\mathbf{dim}\, R_{i,j}$ for $j = 1, \ldots, n_i$. We refer to the system $((d_i, \{r^i_j\}_{j=1,\ldots,n_i})_{i=1,\ldots,m})$ as the *combinatorial data* of $\bar{A}$.

Similarly let $\operatorname{rad} e_i A^{(K)} \cong T_{i,1} \oplus \ldots \oplus T_{i,m_i}$ with $T_{i,j}$ indecomposable and $t^i_j = \mathbf{dim}\, T_{i,j}$ for $j = 1, \ldots, m_i$ for $i = 1, \ldots, n$.

LEMMA 2.6. *If $\bar{A}$ is representation-directed then $n_i = m_i$ and*

$$\{r^i_1, \ldots, r^i_{n_i}\} = \{t^i_1, \ldots, t^i_{m_i}\}$$

for every vertex i of Q.

Proof. Since $\bar{R}$ is representation finite the module $\operatorname{rad} e_i \bar{A}$ (and hence $\operatorname{rad} e_i A^{(K)}$) is a *thin* module, that is, $\dim_k(\operatorname{rad} e_i \bar{A} e_x) \le 1$ (respectively: $\dim_K(\operatorname{rad} e_i A^{(K)} e_x) \le 1$) for every vertex x of Q. Therefore the decomposition data $\{r^i_1, \ldots, r^i_{n_i}\}$ of $\operatorname{rad} e_i \bar{A}$ depend only on the set $\overline{Z}_i$ of arrows $\alpha \in Q_1$ such that the associated map

$$(-) \cdot \alpha : \operatorname{rad} e_i \bar{A} \to \operatorname{rad} e_i \bar{A}$$

is zero. The same is true for the $A^{(K)}$-module $\operatorname{rad} e_i A^{(K)}$; denote the corresponding set of arrows by Z_i. Observe that $Z_i \subseteq \overline{Z}_i$, assume that the sets are not equal and let $\alpha \in \overline{Z}_i \setminus Z_i$. It follows that there exist in Q paths of positive length: u from i to $s(\alpha)$ and w from i to $t(\alpha)$ such that $w \notin \bar{J}$ and $\varrho w - u\alpha \in J$ for some $\varrho \in \mathfrak{p}$. But since $\bar{A}$ is representation-directed this leads to a contradiction with Lemma 2.3. ■

Given an algebra T let Γ_T denote its Auslander–Reiten quiver.

THEOREM 2.7. *Suppose that $(K, R), (L, S) \in \mathbf{Alg}(d)$ are basic, schurian and have the same ordinary quiver Q which is directed. Assume that the combinatorial data of (K, R) and (L, S) coincide. Let $\mathcal{P}_R(1), \ldots, \mathcal{P}_R(l)$ be* *all preprojective components in the Auslander–Reiten quiver Γ_R of R. Then Γ_S has l preprojective components $\mathcal{P}_S(1),\ldots,\mathcal{P}_S(l)$ and for $i=1,\ldots,l$ the components $\mathcal{P}_R(i)$ and $\mathcal{P}_S(i)$ are isomorphic as translation quivers and the isomorphism maps each vertex of $\mathcal{P}_R(i)$ to the vertex of $\mathcal{P}_S(i)$ corresponding to the same dimension vector.*

Proof. Let us remark that the algorithm for constructing the preprojective components [9] has as input data the combinatorial data only (note that they determine the dimension vectors of indecomposable injective modules). This is in fact enough to prove the theorem.

However let us present a more detailed proof by induction on the number of vertices of Q. If this number is 1 the assertion is clear. Assume that Q is a quiver with at least 2 vertices and let x be a source in Q, that is, there is no arrow ending at x. Denote the algebra R/Re_xR by R_x and similarly S/Se_xS by S_x. In what follows we use the name "dimension-preserving isomorphism" for an isomorphism of Auslander–Reiten components which preserves dimension vectors, as in the formulation of the theorem. Let $\mathcal{P}_{R_x}(1),\ldots,\mathcal{P}_{R_x}(n)$ be all preprojective components of the Auslander–Reiten quiver Γ_{R_x} of R_x and assume that a direct summand of the radical $\operatorname{rad} e_xR$ of e_xR belongs to $\mathcal{P}_{R_x}(i)$ if and only if $m<i\le n$ for some $m\le n$. Then $\mathcal{P}_{R_x}(1),\ldots,\mathcal{P}_{R_x}(m)$ are preprojective components of Γ_R as well. By the inductive hypothesis there are preprojective components $\mathcal{P}_{S_x}(1),\ldots,\mathcal{P}_{S_x}(m)$ of Γ_{S_x} such that there exists a dimension-preserving isomorphism between $\mathcal{P}_{R_x}(i)$ and $\mathcal{P}_{S_x}(i)$ for $i=1,\ldots,m$. Since the combinatorial data of (K,R) and (L,S) coincide the components $\mathcal{P}_{S_x}(i)$, $i\le m$, do not contain summands of $\operatorname{rad} e_xS$ and therefore they are preprojective components in the Auslander–Reiten quiver Γ_S of S.

Assume that e_xR is not preprojective. Then $\mathcal{P}_{R_x}(1),\ldots,\mathcal{P}_{R_x}(m)$ are all preprojective components of Γ_R by [4, Theorem 1.3]. Moreover this happens if and only if one of the following conditions holds.

1. There exists a direct summand of $\operatorname{rad} e_xR$ which is not preprojective as an R_x-module.

2. There exist direct summands M_1 and M_2 of $\operatorname{rad} e_xR$ and a path of irreducible morphisms from M_1 to M_2 containing a path of the form $\tau Y\to X\to Y$ for some indecomposable R_x-modules Y,X, where τ is the Auslander–Reiten translate in $\operatorname{mod}(R_x)$.

It is easy to observe that if 1 or 2 holds then the analogous condition with respect to e_xS is also satisfied and therefore e_xS is not preprojective and $\mathcal{P}_{S_x}(1),\ldots,\mathcal{P}_{S_x}(m)$ are all preprojective components of Γ_S.

Otherwise the components $\mathcal{P}_{R_x}(m+1),\ldots,\mathcal{P}_{R_x}(n)$ together with e_xR form a part of the preprojective component $\mathcal{P}_R(x)$ containing e_xR and it is easy to construct inductively a dimension-preserving isomorphism between

$\mathcal{P}_R(x)$ and the corresponding component $\mathcal{P}_S(x)$ of Γ_S. In this case the components $\mathcal{P}_{R_x}(1),\ldots,\mathcal{P}_{R_x}(m),\mathcal{P}_R(x)$ (resp. $\mathcal{P}_{S_x}(1),\ldots,\mathcal{P}_{S_x}(m),\mathcal{P}_S(x)$) are all preprojective components of Γ_R (resp. Γ_S). ■

REMARK 2.8. With a suitably modified notion of combinatorial data the theorem can be extended to the non-schurian case.

Proof of Theorem 1.3. The theorem follows from Lemma 2.6 and Theorem 2.7. ■

3. Application of van den Dries's test. Let $\mathbb{L}$ be the language of the first order theory of fields or rings having countably many variables, two two-argument function symbols $+$ and $\cdot$ and two constants $0, 1$. Let $\mathbb{A}$ be the *two-sorted first order language of algebras over fields* (see [7]), that is, the disjoint union $\mathbb{L}_1 \amalg \mathbb{L}_2$ of two copies of $\mathbb{L}$ equipped with another function symbol $\cdot$ which associates to a pair of variables from $\mathbb{L}_1$ and $\mathbb{L}_2$ a variable from $\mathbb{L}_2$. The language $\mathbb{A}$ has the usual logical connectives: $\wedge$, $\vee$, $\neg$, $\rightarrow$ and allows quantification on both sorts of variables.

By a model for this language we mean a pair (K, R), where K and R are models for $\mathbb{L}_1$ and $\mathbb{L}_2$ respectively and the new function symbol is interpreted as a function

$$\cdot : K \times R \to R.$$

It is clear that if K is a field and R is a K-algebra with identity then the obvious interpretation of the symbols of the language $\mathbb{A}$ allows us to treat the pair (K, R) as a model for $\mathbb{A}$.

Fix a natural number d and denote by $\mathbf{Alg}(d)$ the class of models (K, R) for $\mathbb{A}$ such that K is an algebraically closed field and R is a d-dimensional associative K-algebra with identity.

Let Σ be the first order theory of algebraically closed fields. First order formulas $\phi(x_1,\ldots,x_r)$, $\psi(x_1,\ldots,x_r)$ are called *Σ-equivalent* provided $K \models \phi \Leftrightarrow \psi$ for every algebraically closed field K. The following theorem is a consequence of Tarski's quantifier elimination theorem for algebraically closed fields and van den Dries's test [7, Theorem 12.7, Corollary 12.8].

THEOREM 3.1. *A first order formula $\phi(x_1,\ldots,x_r)$ is Σ-equivalent to a positive quantifier-free formula if and only if the following is satisfied*:

For any algebraically closed fields K, L and every homomorphism $f : V \to L$ from a valuation subring V of K to L and for any tuple $(a_1,\ldots,a_r) \in V^r$ that satisfies ϕ in K the tuple $(f(a_1),\ldots,f(a_r))$ satisfies ϕ in L. ■

Recall that by a *positive quantifier-free formula* we mean a formula built without quantifiers and negation.

For the proof of the following fact we refer to [7, 12.56].

LEMMA 3.2. *The class of representation finite algebras is finitely axiomatizable as a subclass of* $\mathbf{Alg}(d)$. *There exist functions* $\nu, \mu : \mathbb{N} \to \mathbb{N}$ *such that for every* $(L, R) \in \mathbf{Alg}(d)$ *if* R *is representation finite then the number of isomorphism classes and the dimensions of indecomposable* R*-modules are bounded by* $\nu(d)$ *and* $\mu(d)$ *respectively.* ■

We say that a class $\mathcal{C} \subseteq \mathbf{Alg}(d)$ is *finitely axiomatizable as a subclass of* $\mathbf{Alg}(d)$ if there exists a sentence α such that $(K, R) \in \mathcal{C}$ if and only if $(K, R) \models \alpha$ for every $(K, R) \in \mathbf{Alg}(d)$.

LEMMA 3.3. *The class of representation-directed* d*-dimensional algebras over algebraically closed fields is finitely axiomatizable as a subclass of* $\mathbf{Alg}(d)$.

Proof. Let ψ be an axiom for the class of representation finite algebras as a subclass of $\mathbf{Alg}(d)$. Now let ξ be the first order sequence in the language $\mathbb{A}$ expressing:

"For every sequence $M_1, \ldots, M_n, M_{n+1} = M_1$, $n \leq \nu(d)$, *of indecomposable* R*-modules of dimension less than or equal to* $\mu(d)$ *and any collection* $(f_i : M_i \to M_{i+1})$, $i = 1, \ldots, n$, *of non-isomorphisms one of the homomorphisms* f_i *is zero."*

It is easy to observe that $\eta = \psi \wedge \xi$ is an axiom for the class of representation-directed algebras as a subclass of $\mathbf{Alg}(d)$. ■

3.4. *Proof of Corollary 1.4.* Let ϕ be a first order formula expressing the axiom for the class of algebras which are not representation-directed in terms of their structure constants, as in [7, Corollary 12.57]. Let V be a valuation subring of a field K and $f : V \to L$ be a homomorphism into an algebraically closed field L. After replacing V by its localization with respect to the ideal $\operatorname{Ker} f$ we can assume that L is an extension of the residue field k of V.

Let $a = (a_{ijl}) \in V^{d^3}$ be a system of structure constants of a V-order A and assume that $\phi(a)$ is satisfied in K. Then $A^{(K)}$ is not representation-directed. Assume that $\phi(f(a_{ijl}))$ is not satisfied in L, that is, the algebra $A \otimes_V L$ is representation-directed (L has a V-module structure defined by f). Then, by Lemma 2.1, the algebra $\bar{A}$ is representation-directed and thus $A^{(K)}$ is representation-directed by Theorem 1.3, a contradiction.

It follows by Theorem 3.1 that ϕ is equivalent to a positive quantifier-free formula. Thus it defines a closed $\mathbb{Z}$-scheme. ■

4. Remarks and comments. The following example shows that the algebra $\bar{A}$ may not be representation-directed although $A^{(K)}$ is.

EXAMPLE 4.1. Let

$$A = \begin{pmatrix} V & \mathfrak{p} & \mathfrak{p} \\ 0 & V & \mathfrak{p} \\ 0 & 0 & V \end{pmatrix}.$$

Then $A^{(K)}$ is isomorphic to the path algebra of the quiver

$$\circ \longrightarrow \circ \longrightarrow \circ$$

over K whereas $\bar{A}$ is isomorphic to the path k-algebra of the quiver

$$\begin{array}{ccc} \circ & \xrightarrow{\alpha} & \circ \\ & \searrow & \Big\downarrow \beta \\ & & \circ \end{array}$$

bounded by the relation $\alpha\beta = 0$.

Note that since we do not assume that the algebra $A^{(K)}$ is separable (see [14], [16]) the category $\mathrm{latt}(A)$ usually does not have Auslander–Reiten sequences, even if V is a complete discrete valuation domain, as the following simple example shows.

EXAMPLE 4.2. Let $A = \left(\begin{smallmatrix} V & V \\ 0 & V \end{smallmatrix}\right)$ where V is a discrete valuation ring with $\mathfrak{p} = (p)$. We treat A as the path algebra of the quiver $\circ \to \circ$ with coefficients in V and identify A-lattices with V-representations of this quiver. Observe that there does not exist a minimal right almost split map [1, Chapter V] ending at the lattice $I = (V \to 0)$ although I is not projective. To see this assume that

$$\phi : X \to I$$

is a minimal right almost split map in $\mathrm{latt}(A)$ and let $g : X_1 \to X_2$ be the representation corresponding to X. Given a number m denote by Y_m the lattice defined by $p^m : V \to V$.

Every projection from Y_m to I factorizes through ϕ for every m. We are going to show that this is impossible. Since there is no non-zero homomorphism from Y_m to the projective lattice $P = (0 \to V)$, with no loss of generality we can assume that X has no direct summands isomorphic to P.

Let $X \cong X' \oplus U$, where $U = I^n$ for some n and X' has no direct summands I. Since ϕ is not splitting it follows that $\phi(U) \subseteq pI$. It follows that for every $m \geq 0$ any epimorphism $\pi : Y_m \to I$ factorizes through X'. Let X' correspond to a representation $X'_1 \xrightarrow{g'} X'_2$; under our assumptions $\operatorname{Ker} g' \subseteq pX'_1$ and $X'_2/\operatorname{Im} g'$ is a finitely generated torsion V-module. Assume that $p^l X'_2 \subseteq \operatorname{Im} g'$.

Let $u = (u_1, u_2) : Y_{l+1} \to X'$ be a homomorphism such that $\phi \circ u$ is an epimorphism onto I. It follows that $\phi(u_1(1)) \notin pI$ and hence $u_1(1) \notin pX_1$. Observe that $g'(u_1(1)) = p^{l+1}u_2(1) \in p \operatorname{Im} g'$. Let $g'(u_1(1)) = pg'(x)$. It follows that $u_1(1) - px \in \operatorname{Ker} g'$, contrary to the assumption that $\operatorname{Ker} g' \subseteq pX'_1$.

However in the case when $A = VQ/I$ for some admissible ideal I of VQ and $\bar{A}$ is representation-directed the Auslander–Reiten quivers of $\bar{A}$ and $A^{(K)}$

are isomorphic and induced by a common system of A-lattices. Let us precede the precise formulation of this fact (Proposition 4.4) by some lemmas. From now on by an irreducible morphism we always mean an irreducible morphism between indecomposable modules.

LEMMA 4.3. *Assume that A is a V-lattice of the form VQ/I and $\bar{A}$ is representation-directed. There exist A-lattices $X_1, \ldots, X_r$ such that $\{X_i^{(K)} : i = 1, \ldots, r\}$ (resp. $\{\overline{X}_i : i = 1, \ldots, r\}$) is a full set of representatives of isomorphism classes of indecomposable $A^{(K)}$-modules (resp. $\bar{A}$-modules). Moreover, there exists a set $W \subseteq \{1, \ldots, r\}^2$ and a system $(f_{ij} : X_i \to X_j)_{(i,j)\in W}$ of A-homomorphisms such that $\{f_{ij}^{(K)} : X_i^{(K)} \to X_j^{(K)}; (i,j) \in W\}$ and $\{\bar{f}_{ij} : \overline{X}_i \to \overline{X}_j; (i,j) \in W\}$ are the sets of all (up to composition with isomorphisms) irreducible morphisms in* $\mathrm{mod}(A^{(K)})$ *and* $\mathrm{mod}(\bar{A})$ *respectively.*

Proof. Given an indecomposable $A^{(K)}$-module X' let $s(X')$ denote the number of proper successors of X' in the Auslander–Reiten quiver of $A^{(K)}$.

Let us enumerate the representatives of isomorphism classes of indecomposable $A^{(K)}$-modules: $X_1', \ldots, X_r'$, in such a way that $i \geq j$ implies $s(X_i') \geq s(X_j')$ for $i, j \in \{1, \ldots, r\}$. By induction on m we construct A-lattices X_m and homomorphisms f_{mj} satisfying the conditions in the lemma and such that $X_m^{(K)} \cong X_m'$.

Let us agree that "all irreducible morphisms" means "all (up to composition with isomorphisms) irreducible morphisms".

The construction is obvious when $m = 1$ (then X_m' is simple injective). Assume that $m > 1$ and we have constructed X_i and f_{ij} with the required properties for $i < m$.

First consider the case when X_m' is injective. It is easy to construct a lattice X_m and A-homomorphisms $f_{mj} : X_m \to Y_j$ in $\mathrm{latt}(A)$ such that $Y_j^{(K)}$, $\overline{Y}_j$ are indecomposable for $j = 1, \ldots, s$ and $f_{mj}^{(K)}$ (resp. $\bar{f}_{mj}$) for $j = 1, \ldots, s$ are all irreducible morphisms starting at $X_m^{(K)}$ (resp. $\overline{X}_m$).

Assume that $Y_j^{(K)} \cong X_{i_j}'$, $j = 1, \ldots, s$. Then $i_j < m$ and $\overline{Y}_j \cong \overline{X}_{i_j}$ for $j = 1, \ldots, s$. Here we use Lemma 2.6 asserting that the combinatorial data of $A^{(K)}$ and $\bar{A}$ coincide. Observe that since the dimensions of homomorphism spaces between indecomposable modules over a representation-directed algebra are determined by their positions in the Auslander–Reiten quiver we have $\dim_K \mathrm{Hom}_{A^{(K)}}(X_{i_j}^{(K)}, Y_j^{(K)}) = \dim_k \mathrm{Hom}_{\bar{A}}(\overline{X}_{i_j}, \overline{Y}_j) = 1$ for each $j \leq s$. It follows by Lemma 4.5 below (see also Corollary 4.6) that the V-modules $\mathrm{Ext}_A^1(X_{i_j}, Y_j)$ are torsion-free and $Y_j \cong X_{i_j}$ for $j = 1, \ldots, s$ thanks to [7, Corollary 12.26]. We can finish the construction in this case.

Now assume that X'_m is not injective and let

$$0 \to X'_m \to \bigoplus_{i=1}^{q} X_{j_i}^{(K)} \xrightarrow{f^{(K)}} X_s^{(K)} \to 0,$$

where $f^{(K)} = (f_{j_i s}^{(K)})_{i=1,\ldots,q}$, be the Auslander–Reiten sequence in $\operatorname{mod}(A^{(K)})$. Such a sequence exists by the inductive hypothesis, we also use the fact that all arrows in the Auslander–Reiten quiver of $A^{(K)}$ have trivial valuation [1, VII.2.3]. Let $X_m = \operatorname{Ker} f$. Since X_s is free as a V-module, X_m is an A-lattice and both induced sequences

$$0 \to X_m^{(K)} \to \bigoplus_{i=1}^{q} X_{j_i}^{(K)} \xrightarrow{f^{(K)}} X_s^{(K)} \to 0,$$

$$0 \to \overline{X}_m \to \bigoplus_{i=1}^{q} \overline{X}_{j_i} \xrightarrow{\bar{f}} \overline{X}_s \to 0$$

are exact. It follows that $X_m^{(K)} \cong X'_m$. By Theorem 2.7 we know that the dimensions of the middle terms of the Auslander–Reiten sequences ending at $X_s^{(K)}$ and $\overline{X}_s$ are equal, hence $\bar{f}_{j_i s}$, $i = 1, \ldots, q$, are all the irreducible morphisms ending at $\overline{X}_s$. It follows that the latter sequence is an Auslander–Reiten sequence in $\operatorname{mod}(\bar{A})$. Then $\overline{X}_m$ is indecomposable as the Auslander–Reiten translate of $\overline{X}_s$.

Let $u : X_m \to \bigoplus_{i=1}^{q} X_{j_i}$, $u = (u_i)_{i=1,\ldots,q}$, be the natural embedding. Then we see that $u_i^{(K)}$ and $\overline{u}_i$, $i = 1, \ldots, q$, are all the irreducible morphisms starting at $X_m^{(K)}$ and $\overline{X}_m$ respectively. ∎

The following proposition is a direct consequence of Lemma 4.3.

PROPOSITION 4.4. *Assume that $A = VQ/I$ for some quiver Q and a two-sided ideal I of VQ, and $\bar{A} = A \otimes_V k$ is representation-directed. Then there is an isomorphism of the Auslander–Reiten quivers of $\bar{A}$ and $A^{(K)}$ preserving dimension vectors.*

Proof. Let $X_1, \ldots, X_r$ be A-lattices as in Lemma 4.3. The required isomorphism of quivers is uniquely determined by $\overline{X}_i \mapsto X_i^{(K)}$ for $i = 1, \ldots, r$. ∎

LEMMA 4.5. (a) *For all A-lattices X, Y,*

$$\operatorname{Hom}_{A^{(K)}}(X^{(K)}, Y^{(K)}) = K \operatorname{Hom}_A(X, Y).$$

(b) *The following conditions are equivalent*:

(1) *the group* $\operatorname{Ext}_A^1(X, Y)$ *is torsion-free*,
(2) *the canonical map* $\operatorname{Hom}_A(X, Y) \to \operatorname{Hom}_{\bar{A}}(\overline{X}, \overline{Y})$ *given by* $f \mapsto f \otimes k$ *is surjective*,
(3) *the V-rank of the module* $\operatorname{Hom}_A(X, Y)$ *equals the k-dimension of* $\operatorname{Hom}_{\bar{A}}(\overline{X}, \overline{Y})$.

Proof. Assertion (a) is Lemma 12.21 of [7]. A simple refinement of the proof of Proposition 12.25 in [7] proves (b). For the convenience of the reader we sketch the argument. Let

$$P_1 \xrightarrow{d} P_0 \to X \to 0$$

be a projective presentation of X in $\mathrm{mod}(A)$. Application of $\mathrm{Hom}_A(-,Y)$ yields two exact sequences

$$0 \to \mathrm{Hom}_A(X,Y) \to \mathrm{Hom}_A(P_0,Y) \xrightarrow{q} N \to 0$$

and

$$0 \to N \xrightarrow{u} \mathrm{Hom}_A(P_1,Y) \to \mathrm{Ext}^1_A(X,Y) \to 0$$

where $uq = d^*$, $d^* = \mathrm{Hom}_A(d,Y)$, $N = \mathrm{Im}\, d^*$ and u is the identity embedding. Now apply the functor $\overline{(-)}$ to get the exact sequences

$$0 \to \overline{\mathrm{Hom}_A(X,Y)} \to \overline{\mathrm{Hom}_A(P_0,Y)} \xrightarrow{\overline{q}} \overline{N} \to 0$$

(N is torsion-free) and

$$0 \to \mathrm{Tor}^V_1(\mathrm{Ext}^1_A(X,Y),k) \xrightarrow{\partial} \overline{N} \xrightarrow{\overline{u}} \overline{\mathrm{Hom}_A(P_1,Y)} \to \overline{\mathrm{Ext}^1_A(X,Y)} \to 0$$

Let $M = \overline{q}^{-1}(\partial(\mathrm{Tor}^V_1(\mathrm{Ext}^1_A(X,Y),k)))$. There is a commutative diagram of k-spaces with exact rows:

$$\begin{array}{ccccccc} 0 \to & M & \to & \overline{\mathrm{Hom}_A(P_0,Y)} & \xrightarrow{\overline{d^*}} & \overline{\mathrm{Hom}_A(P_1,Y)} \\ & & & \downarrow & & \downarrow \\ 0 \to & \mathrm{Hom}_{\bar{A}}(\overline{X},\overline{Y}) & \to & \mathrm{Hom}_{\bar{A}}(\overline{P}_0,\overline{Y}) & \xrightarrow{(\bar{d})^*} & \mathrm{Hom}_{\bar{A}}(\overline{P}_1,\overline{Y}) \end{array}$$

where the vertical homomorphisms are the canonical isomorphisms (P_0 and P_1 are projective). Observe that M and $\overline{\mathrm{Hom}_A(X,Y)}$ are isomorphic if and only if $\mathrm{Tor}^V_1(\mathrm{Ext}^1_A(X,Y),k) = 0$. Now the equivalence of the conditions in (b) follows. ∎

COROLLARY 4.6. *Let $X_1,\ldots,X_r$ be A-lattices as in Lemma* 4.3, *and assume that $\bar{A}$ is representation-directed. Then the group* $\mathrm{Ext}^1_A(X_i,X_j)$ *is torsion-free as a V-module and there is a natural isomorphism*

$$\overline{\mathrm{Hom}_A(X_i,X_j)} \cong \mathrm{Hom}_{\bar{A}}(\overline{X}_i,\overline{X}_j) \quad \textit{for all } i,j = 1,\ldots,r.$$

Proof. Note that the dimension of the homomorphism space between indecomposable modules over a representation-directed algebra depends only on their position in the Auslander–Reiten quiver. It follows that

$$\dim_K \mathrm{Hom}_{A^{(K)}}(X_i^{(K)},X_j^{(K)}) = \dim_k \mathrm{Hom}_{\bar{A}}(\overline{X}_i,\overline{X}_j) \quad \text{for } i,j = 1,\ldots,r.$$

Thus the corollary follows from Lemma 4.5. ∎

REFERENCES

[1] M. Auslander, I. Reiten and S. Smalø, *Representation Theory of Artin Algebras*, Cambridge Stud. in Adv. Math. 36, Cambridge Univ. Press, 1995.

[2] S. Balcerzyk and T. Józefiak, *Pierścienie przemienne*, PWN, Warszawa, 1985 (in Polish); English transl. of Chapters I-IV: *Commutative Noetherian and Krull rings*, PWN, Warszawa, Ellis Harwood, Chichester, and Halsted Press, New York, 1989.

[3] C. C. Chang and H. J. Keisler, *Model Theory*, Stud. Logic Found. Math. 73, North-Holland, 1973.

[4] P. Dräxler and J. A. de la Peña, *On the existence of postprojective components in the Auslander–Reiten quiver of an algebra*, Tsukuba J. Math. 20 (1996), 457–469.

[5] P. Gabriel, *Finite representation type is open*, in: Representations of Algebras, Lecture Notes in Math. 488, Springer, Berlin, 1975, 132–155.

[6] C. Jensen and H. Lenzing, *Homological dimension and representation type of algebras under base field extension*, Manuscripta Math. 39 (1982), 1–13.

[7] —, —, *Model Theoretic Algebra: with particular emphasis on fields, rings, modules*, Algebra Logic Appl. 2, Gordon and Breach, New York, 1989.

[8] I. Kaplansky, *Maximal fields with valuations*, Duke Math. J. 9 (1942), 313–321.

[9] S. Kasjan and J. A. de la Peña, *Constructing the preprojective components of an algebra*, J. Algebra 179 (1996), 793–807.

[10] H. Kraft, *Geometric methods in representation theory*, in: Representations of Algebras (Puebla, 1980), Lecture Notes in Math. 944, Springer, 1982, 180–258.

[11] J. D. Monk, *Mathematical Logic*, Grad. Texts in Math. 37, Springer, New York, 1976.

[12] B. Poonen, *Maximally complete fields*, Enseign. Math. 39 (1993), 87–106.

[13] A. Raggi-Cárdenas and L. Salmerón, *Quivers for algebras over commutative Noetherian complete local rings*, Comm. Algebra 15 (1987), 2617–2623.

[14] I. Reiner, *Topics in Integral Representation Theory*, Lecture Notes in Math. 744, Springer, Berlin, 1979.

[15] C. M. Ringel, *Tame Algebras and Integral Quadratic Forms*, Lecture Notes in Math. 1099, Springer, Berlin, 1984.

[16] K. W. Roggenkamp, *The construction of almost split sequences for integral group rings and orders*, Comm. Algebra 5 (1977), 1363–1373.

[17] A. Skowroński, *On nonexistence of oriented cycles in Auslander–Reiten quivers*, Acta Univ. Carolin. Math. Phys. 25 (1984), 45–52.

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