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POSSIBLY THERE IS NO UNIFORMLY COMPLETELY RAMSEY NULL SET OF SIZE 2^ω

BY

ANDRZEJ NOWIK (Gdańsk)

Abstract. We show that under the axiom $\mathrm{CPA}_{\mathrm{cube}}$ there is no uniformly completely Ramsey null set of size 2^ω. In particular, this holds in the iterated perfect set model. This answers a question of U. Darji.

1. Introduction. The class of uniformly completely Ramsey null sets (UCR_0 sets) was defined and investigated in [Da], and also in [N1] and [N2]. In this paper we continue the investigation of this class and other kinds of small sets defined analogously. The main purpose of this paper is to give a full answer to the problem of U. Darji (see [Da, Question 1]) whether there is always an UCR_0 set of size continuum. We show that under the axiom $\mathrm{CPA}_{\mathrm{cube}}$ there is no such set. In particular, since by Theorem 7.0.4 of [CP], $\mathrm{CPA}_{\mathrm{cube}}$ holds in the iterated perfect set model, there is no such set in this model. Thus the answer to Darji's question is negative.

2. Definitions. We identify $[\omega]^\omega$ with a subset of 2^ω. We often identify $[\omega]^{\le\omega}$ with the space 2^ω via the standard isomorphism. For example, if $x, y \in 2^\omega$, then $x \subseteq y$ means that $\forall_{n\in\omega}\ x(n) \le y(n)$. If $s \in [\omega]^{<\omega}$, $A \in [\omega]^\omega$ and $\max s < \min A$ then we define $[s, A] = \{x \in [\omega]^\omega : s \subseteq x \subseteq s \cup A\}$. These sets are called the *Ellentuck neighborhoods*. Moreover let $[s, A]^{\le\omega} = \{x \subseteq \omega : s \subseteq x \subseteq s \cup A\}$.

We denote by $\psi_{s,A}$ the standard homeomorphism $\psi : 2^\omega \to [s, A]^{\le\omega}$ defined by $\psi(x) = s \cup \{a_n : x(n) = 1\}$, where $A = \{a_0, a_1, a_2, \ldots\}$ with $a_0 < a_1 < a_2 < \ldots$

Recall that $X \subseteq [\omega]^\omega$ is *completely Ramsey null* (for short, X is CR_0) if for every Ellentuck neighborhood $[s, A]$ there exists $B \in [A]^\omega$ such that $[s, B] \cap X = \emptyset$, and X is *completely Ramsey* (X is CR) if for every $[s, A]$ there exists $B \in [A]^\omega$ such that $[s, B] \cap X = \emptyset \vee [s, B] \subseteq X$. A set $X \subseteq 2^\omega$ is *uniformly completely Ramsey null* if for every continuous function

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$F : 2^\omega \to 2^\omega$ and for every $Y \subseteq X$, $F^{-1}[Y]$ is completely Ramsey. We then write $X \in \mathrm{UCR}_0$. We will use the following characterization of UCR_0 sets given in [N1]: A set $X \subseteq 2^\omega$ is UCR_0 if for each continuous function $F : 2^\omega \to 2^\omega$ there exists $A \in [\omega]^\omega$ such that $|F[P(A)] \cap X| \le \omega$.

We will use the following definitions from [CP]:

• A subset C of the product $\prod_{n\in\omega} 2^\omega$ of Cantor sets is said to be a *perfect cube* if $C = \prod_{n\in\omega} C_n$, where $C_n \in \mathrm{Perf}(2^\omega)$ for each n.

• Let $\mathcal{F}_{\mathrm{cube}}$ stand for the family of all continuous injections from a perfect cube C onto a set $P \in \mathrm{Perf}(2^\omega)$. The elements of $\mathcal{F}_{\mathrm{cube}}$ are called *cubes*.

• We say that a family $\mathcal{E} \subseteq \mathrm{Perf}(2^\omega)$ is $\mathcal{F}_{\mathrm{cube}}$-*dense* (or *cube-dense*) in $\mathrm{Perf}(2^\omega)$ provided

$$\forall_{f\in\mathcal{F}_{\mathrm{cube}}}\ \exists_{g\in\mathcal{F}_{\mathrm{cube}}}\ (g \subseteq f \wedge \mathrm{ran}(g) \in \mathcal{E}).$$

• Define

$$s_0^{\mathrm{cube}} = \Big\{2^\omega \setminus \bigcup \mathcal{E} : \mathcal{E} \text{ is } \mathcal{F}_{\mathrm{cube}}\text{-dense in } \mathrm{Perf}(2^\omega)\Big\}.$$

Throughout this note we will use one fixed bijection $\langle\cdot,\cdot\rangle : \omega\times\omega \to \omega$. For each $A \in [\omega]^\omega$ and $n < \omega$ we define $(A)_n = \{a_{\langle k,n\rangle} : k \in \omega\}$ where $A = \{a_0, a_1, a_2 \ldots\}$ and $a_0 < a_1 < a_2 < \ldots$

Finally, for a finite set $A \in [\omega]^{<\omega}$ we define $(A)_n = \{a_{\langle k,n\rangle} : k \in \omega \wedge \langle k,n\rangle \le r\}$ where $A = \{a_0, a_1, \ldots, a_r\}$ and $a_0 < a_1 < \ldots < a_r$.

For $X \subseteq (2^\omega)^2$ we denote by $X_{(x)}$ and $X^{(y)}$ the *x-section* and *y-section* of X, respectively (i.e. $X_{(x)} = \{y : \langle x,y\rangle \in X\}$, $X^{(y)} = \{x : \langle x,y\rangle \in X\}$).

For completeness we briefly outline the definition of Σ and $\mathrm{w}\Sigma\mathrm{QN}$ sets (see [BRR]).

We say that a sequence of functions $f_k : X \to \mathbb{R}_+$ *converges quasinormally* to 0 ($f_k \overset{\mathrm{QN}}{\to} 0$) if there is a sequence $\varepsilon_n \to 0$ such that $\forall_{x\in X}\ \forall_k^\infty\ f_k(x) < \varepsilon_k$.

We say that a sequence of functions $f_k : X \to \mathbb{R}_+$ (Σ) *converges* to 0 if $\forall_{x\in X}\ \sum_{k=1}^\infty f_k(x) < \infty$.

A topological space X is a Σ *space* if for each sequence of continuous functions $f_k : X \to \mathbb{R}_+$, if $f_k \to 0$ pointwise then there is a subsequence k_l such that $f_{k_l} \overset{(\Sigma)}{\to} 0$.

Analogously, X is a $\mathrm{w}\Sigma\mathrm{QN}$ *space* if for each sequence of continuous functions $f_k : X \to \mathbb{R}_+$, if $f_k \overset{(\Sigma)}{\to} 0$ then there is a subsequence k_l such that $f_{k_l} \overset{\mathrm{QN}}{\to} 0$.

3. $\mathcal{F}$- UCR_0 sets. The following terminology will be useful in our proof.

Definition 3.1. Suppose that $\mathcal{F}$ is an arbitrary family of Borel functions from 2^ω to 2^ω. We say that $X \subseteq 2^\omega$ is an $\mathcal{F}$-UCR_0 *set* if for every $F \in \mathcal{F}$ and every $Y \subseteq X$, $F^{-1}[Y]$ is completely Ramsey.

First note that $\mathcal{F}$-UCR_0 is a σ-ideal and if $\mathcal{F} \subseteq \mathcal{G}$ then $\mathcal{G}$-$\mathrm{UCR}_0 \subseteq \mathcal{F}$-$\mathrm{UCR}_0$. In what follows we shall consider this general definition for the following families of functions from 2^ω to 2^ω:

- 1-1 = all one-to-one continuous functions.
- Count = all continuous functions with countable preimages of points.
- 1-1-Borel = all one-to-one Borel functions.
- Count-Borel = all Borel functions with countable preimages of points.

Note that in [N2], a special case of this definition was considered for $\mathcal{F}$ being the family of all Borel functions from 2^ω to 2^ω.

FACT 3.2. 1-1-Borel-UCR_0 = Count-Borel-UCR_0.

Proof. The inclusion $\supseteq$ is obvious. Conversely, let $X \in$ 1-1-Borel-UCR_0. Let $Y \subseteq X$ and let $[s,E]$ be an Ellentuck neighborhood. Let $B : 2^\omega \to 2^\omega$ be a Borel mapping with countable preimages of points.

Since $\forall_{y\in 2^\omega}\ |(\mathrm{Graph}(B))^{(y)}| \le \omega$, by the Luzin–Novikov Theorem (see for instance [Ke, Theorem 18.10]) there are Borel sets $b_n \subseteq (2^\omega)^2$ such that $\bigcup_{n\in\omega} b_n = \mathrm{Graph}(B)$ and $\forall_{y\in 2^\omega}\ |b_n^{(y)}| \le 1$. Define A_n for $n \in \omega$ by letting $A_n = \pi_x[b_n]$. Then $\bigcup_{n\in\omega} A_n = 2^\omega$, since $2^\omega = \pi_x[\mathrm{Graph}(B)] = \bigcup_{n\in\omega} \pi_x[b_n] = \bigcup_{n\in\omega} A_n$. Since $A_n \in \Sigma^1_1$, A_n is completely Ramsey, hence there exists an Ellentuck neighborhood $[s_1,E_1] \subseteq [s,E]$ and $n_0 \in \omega$ such that $[s_1,E_1] \subseteq A_{n_0}$.

Define $B_1 = B\restriction[s_1,E_1]$. It is easy to see that B_1 is one-to-one. There is an $i \in 2$ and an Ellentuck neighborhood $[s_2,E_2] \subseteq [s_1,E_1]$ such that $[s_2,E_2] \subseteq B_1^{-1}[\mathcal{C}_{\langle i\rangle}]$, where $\mathcal{C}_s = \{x \in 2^\omega : s \subseteq x\}$. Next, let $B_2 : [s_1,E_1]^{\le\omega} \to 2^\omega$ be an extension of $B_1\restriction[s_2,E_2]$ to a one-to-one Borel function. Since $X \in$ 1-1-Borel-UCR_0, there exists $E_3 \in [E_2]^\omega$ such that

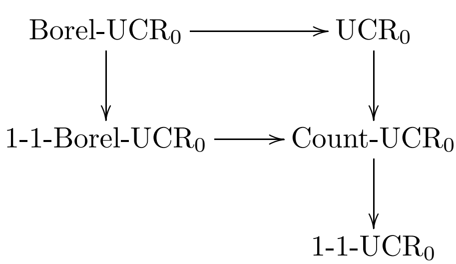
$$[s_2,E_3] \subseteq B_2^{-1}[Y] \vee [s_2,E_3] \cap B_2^{-1}[Y] = \emptyset.$$

Hence

$$[s_2,E_3] \subseteq B^{-1}[Y] \vee [s_2,E_3] \cap B^{-1}[Y] = \emptyset.$$

This proves that $X \in$ Count-Borel-UCR_0. ∎

Thus we have the following diagram of inclusions:



Next, we reformulate the characterization of UCR_0 sets from Theorem 1 of [N1] in our more general language:

THEOREM 3.3. *Suppose that $\mathcal{F}$ is a family of Borel functions from 2^ω to 2^ω. Assume that*:

1. *For each $s \in [\omega]^{<\omega}$, $E \in [\omega]^\omega$ such that* $\max(s) < \min(E)$ *and $F \in \mathcal{F}$ we have $F \circ \psi_{s,E} \in \mathcal{F}$.*
2. *For every perfect set $P \subseteq 2^\omega$ there exist $F \in \mathcal{F}$ and $X \subseteq P$ such that $f^{-1}[X] \notin \mathrm{CR}$.*

Then for every $X \subseteq 2^\omega$ the following statements are equivalent:

$$X \in \mathcal{F}\text{-}\mathrm{UCR}_0. \tag{1}$$

$$\forall_{F\in\mathcal{F}}\ \exists_{A\in[\omega]^\omega}\ |F[P(A)] \cap X| \le \omega. \tag{2}$$

Proof. The proof is essentially the same as the proof of Theorem 1 in [N1]. We use assumption 2 to assure that there is no perfect set in $\mathcal{F}$-UCR_0. Assumption 1 is necessary in the proof of the implication (2)⇒(1). ■

Unfortunately, we do not know of any examples of sets distinguishing the properties Borel-UCR_0, UCR_0, Count-UCR_0, 1-1-UCR_0, 1-1-Borel-UCR_0.

4. Main result

THEOREM 4.1. *Every* 1-1-UCR_0 *set is an s_0^{cube} set.*

Proof. By Fact 1.0.3 from [CP] it is enough to consider functions defined on the entire space $(2^\omega)^\omega$. So, suppose that $f : \prod_{n\in\omega} 2^\omega \to 2^\omega$ is a cube. Let

$$F : 2^\omega \to (2^\omega)^\omega, \qquad F(Z) = ((Z)_n)_{n\in\omega}.$$

Note that a variant of this function has been used in [N2] to prove that every UCR_0 set is an (s_0^2) set.

We show that F is continuous. Let $Z \in 2^\omega$ and $(W_n)_{n<n_0}$ be a finite sequence of open subsets of 2^ω and suppose that $F(Z) \in \prod_{n<n_0} W_n \times \prod_{n\ge n_0} 2^\omega$. Then we can find $K_0 \in \omega$ such that for every $B \in 2^\omega$, if $B \cap K_0 = Z \cap K_0$ and $n < n_0$ then $(B)_n \in W_n$. Therefore F is continuous. It is also easy to see that $f \circ F$ is one-to-one, because f is a cube and F is one-to-one.

Let X be a 1-1-UCR_0 set. Then there is a set $A \in [\omega]^\omega$ such that $|(f \circ F)[P(A)] \cap X| \le \omega$. Let $\{a_0, a_1, a_2, \ldots\}$ be an increasing enumeration of the elements of A. We define $\Xi : (2^\omega)^\omega \to P(A)$ by

$$a_{2\langle k,n\rangle} \in \Xi((x_n)_{n\in\omega}) \iff x_n(k) = 0, \quad a_{2\langle k,n\rangle+1} \in \Xi((x_n)_{n\in\omega}) \iff x_n(k) = 1.$$

CLAIM 4.2. *The image $F[\Xi[(2^\omega)^\omega]]$ is a perfect cube in $(2^\omega)^\omega$, denoted by C.*

Proof. Define $E_{k,n} = \{a_{2\langle k,n\rangle}, a_{2\langle k,n\rangle+1}\}$. It is clear that $(E_{k,n})_{k,n\in\omega}$ is a partition of A into 2-element subsets.

Define

$$D_n = \Big\{x \in 2^\omega : x \subseteq \bigcup_{k\in\omega} E_{k,n} \wedge \forall_{k\in\omega}\ |E_{k,n}\cap x| = 1\Big\}.$$

These sets are perfect. We will show that

$$F[\varXi[(2^\omega)^\omega]] = \prod_{n\in\omega} D_n.$$

"$\subseteq$": Let $(x_n) \in F[\varXi[(2^\omega)^\omega]]$ and fix $n_0 \in \omega$. Let $B \in \varXi[(2^\omega)^\omega]$ be such that $(x_n)_{n\in\omega} = F(B)$. Obviously, $B \in P(A)$. From the definition of F we conclude that

$$\forall_{n\in\omega}\ x_n = (B)_n.$$

It easily follows from the definition of $\varXi$ that $\operatorname{ran}(\varXi) \subseteq [\omega]^\omega$. Therefore, let $\{b_0, b_1, \ldots\}$ be an increasing enumeration of the elements of B.

Since from the definition of $\varXi$ we easily see that $\forall_{k,n\in\omega}\ |B\cap E_{k,n}| = 1$, we have $\forall_{k,n\in\omega}\ b_{\langle k,n\rangle} \in E_{k,n}$. Hence, $(B)_{n_0} = \{b_{\langle k,n_0\rangle} : k\in\omega\}$ and therefore $x_{n_0} = (B)_{n_0} \subseteq \bigcup_{k\in\omega} E_{k,n_0}$ and $\forall_{k\in\omega}\ |E_{k,n_0}\cap x_{n_0}| = 1$. Finally, $x_{n_0} \in D_{n_0}$.

"$\supseteq$": Let $(x_n)_{n\in\omega} \in \prod_{n\in\omega} D_n$. Define $x = \bigcup_{n\in\omega} x_n$. Since $x_n \in D_n$, we conclude that $(x_n)_{n\in\omega}$ are pairwise disjoint and $\forall_{k,n\in\omega}\ |E_{k,n}\cap x| = 1$. Let $\{b_0, b_1, \ldots\}$ be an increasing enumeration of the elements of x. Then $\forall_{k,n\in\omega}\ b_{\langle k,n\rangle} \in E_{k,n}$. Hence $F(x) = (x_n)_{n\in\omega}$.

Now, define $(z_n)_{n\in\omega} \in (2^\omega)^\omega$ by

$$z_n(k) = 0 \ \Leftrightarrow\ a_{2\langle k,n\rangle} \in x, \qquad z_n(k) = 1 \ \Leftrightarrow\ a_{2\langle k,n\rangle+1} \in x.$$

Since $\varXi((z_n)_{n\in\omega}) = x$, we finally obtain $F(\varXi((z_n)_{n\in\omega})) = (x_n)_{n\in\omega}$, therefore $(x_n)_{n\in\omega} \in F[\varXi[(2^\omega)^\omega]]$. This finishes the proof of the Claim.

Thus we obtain $|f[C]\cap X| \le \omega$. Since next we can find a perfect cube $D \subseteq C$ such that $f[D]\cap X = \emptyset$, we finally conclude that X is an s_0^{cube} set. ∎

Corollary 4.3. *Assume* $\mathrm{CPA}_{\mathrm{cube}}$. *Then every* 1-1-$\mathrm{UCR}_0$ *set has size* $\le \omega_1$ *and* $2^\omega = \omega_2$ *holds.*

Proof. This follows immediately from Proposition 1.0.4 of [CP] (stating that under $\mathrm{CPA}_{\mathrm{cube}}$, $s_0^{\mathrm{cube}} \subseteq [2^\omega]^{\le\omega_1}$) and from the previous theorem. ∎

Corollary 4.4. *Under the axiom* $\mathrm{CPA}_{\mathrm{cube}}$ *there is no* UCR_0 *set of size* 2^ω. *In particular, this holds in the iterated perfect set model.*

This answers a question of Darji from [Da]. ∎

Recall the following notion of smallness considered in [Sc]. We say that $X \subseteq 2^\omega\times 2^\omega$ has *property* (s_0^2) if every set $P\times Q \in \mathrm{Perf}\times\mathrm{Perf}$ has a subset $P_1\times Q_1 \in \mathrm{Perf}\times\mathrm{Perf}$ disjoint from X. It was proven in [Sc] that the class of (s_0^2) sets is a σ-ideal on $2^\omega\times 2^\omega$. It was also proven in [N2] that every UCR_0 set is an (s_0^2) set. Note that Theorem 4.1 is a strengthening of this result, since we have the following easy fact:

Observation 4.5. *Every s_0^{cube} set is an (s_0^2) set.*

Proof. Let $X \in s_0^{\mathrm{cube}}$. Suppose that $P, Q \subseteq 2^\omega$ are perfect sets. We may assume that $P \approx 2^\omega$ and $Q \approx 2^\omega$. Let h_P, h_Q be arbitrary homeomorphisms between $\prod_{n\in\omega} 2^\omega$ and P, Q, respectively. Define the following cube:

$$f : \prod_{n\in\omega} 2^\omega \to 2^\omega \times 2^\omega, \qquad f((x_n)_{n\in\omega}) = \langle h_P((x_{2n})_{n\in\omega}), h_Q((x_{2n+1})_{n\in\omega})\rangle.$$

Since $X \in s_0^{\mathrm{cube}}$ there exists a subcube $g \subseteq f$, $g : \prod_{n\in\omega} C_n \to 2^\omega \times 2^\omega$ ($C_n \in \mathrm{Perf}$), such that $\mathrm{ran}(g) \cap X = \emptyset$. We define two perfect sets, P_1 and Q_1, by putting

$$P_1 = h_P\Big[\prod_{n\in\omega} C_{2n}\Big] \quad \text{and} \quad Q_1 = h_Q\Big[\prod_{n\in\omega} C_{2n+1}\Big].$$

It is easy to see that P_1, Q_1 are perfect sets and $P_1 \times Q_1 \subseteq \mathrm{ran}(g)$. Therefore $(P_1 \times Q_1) \cap X = \emptyset$ and $P_1 \times Q_1 \subseteq P \times Q$. This proves that $X \in (s_0^2)$. ∎

5. Thin sets related to trigonometric series. In this section we briefly discuss the relation between thin sets related to trigonometric series and $\mathcal{F}$-UCR$_0$ sets.

Theorem 5.1. *Let $X \subseteq 2^\omega$ be a Σ set and let $h : 2^\omega \to 2^\omega$ be a continuous one-to-one function. Then $h^{-1}[X] \in \mathrm{CR}_0$. In particular, every Σ set is a* 1-1-UCR$_0$ *set.*

Proof. Since every continuous image of a Σ set is a Σ set, it is sufficient to show that $X \cap [\omega]^\omega \in \mathrm{CR}_0$. We notice that for every Ellentuck neighborhood $[s, A]$ the set $[s, A]^{\le\omega}$ is homeomorphic to 2^ω, hence finally it is enough to show that there exists $A \in [\omega]^\omega$ such that $[A]^\omega \cap X = \emptyset$.

We will make use of the following sequence of functions $f_n : 2^\omega \to \mathbb{R}$ defined in [BRR] and used there to prove that $2^\omega \notin \mathrm{wQN}$:

$$f_n(x) = \begin{cases} \dfrac{1}{|\{i \le n : x(i) = 1\}|} & \text{if } x(n) = 1,\\[2ex] \dfrac{1}{n+1} & \text{if } x(n) = 0. \end{cases}$$

Obviously $f_n \in C(2^\omega)$. One easily checks that $f_n(x) \to 0$ for each $x \in 2^\omega$. By assumption, there exists a subsequence $(f_{n_k})_{k\in\omega}$ such that $\sum_{k\in\omega} f_{n_k}(x) < \infty$ for each $x \in X$. Let $A = \{n_{k_0}, n_{k_1}, \ldots\}$. Suppose that $B \in [A]^\omega$. Clearly $B = \{n_{k_{r_0}}, n_{k_{r_1}}, \ldots\}$ where $(r_l)_{l\in\omega}$ is an increasing sequence of natural numbers. Moreover,

$$f_{n_{k_{r_l}}}(B) = \frac{1}{|\{s \le n_{k_{r_l}} : s \in B\}|} = \frac{1}{l+1}.$$

Thus $\sum_{k\in\omega} f_{n_k}(B)=\infty$ so $B\notin\{x:\sum_{k\in\omega} f_{n_k}(x)<\infty\}$. Therefore

$$[A]^\omega\cap\Big\{x:\sum_{k\in\omega} f_{n_k}(x)<\infty\Big\}=\emptyset.$$

This proves that $[A]^\omega\cap X=\emptyset$. ■

THEOREM 5.2. *Let* $X\subseteq 2^\omega$ *be a* wΣQN *set and let* $h:2^\omega\to 2^\omega$ *be a continuous one-to-one function. Then* $h^{-1}[X]\in \mathrm{CR}_0$. *In particular*, *every* wΣQN *set is a* 1-1-UCR$_0$ *set.*

Proof. As in the proof of Theorem 5.1, it is enough to show that there exists $A\in[\omega]^\omega$ such that $X\cap[A]^\omega=\emptyset$.

We will slightly modify the functions from the previous proof. Namely, set

$$f_n(x)=\begin{cases}2^{-|\{i\le n:x(i)=1\}|} & \text{if } x(n)=1,\\ 2^{-(n+1)} & \text{if } x(n)=0.\end{cases}$$

Obviously $f_n\in C(2^\omega)$. Let $x\in 2^\omega$. Then

$$\sum_{n\in\omega} f_n(x)\le\sum_{n\in\omega}2^{-(n+1)}+\sum_{n\in\omega}2^{-(n+1)}<\infty.$$

Since $X\in$ wΣQN there exists a subsequence (n_k) such that $f_{n_k}\restriction X\overset{\mathrm{QN}}{\to}0$. Then there exists a sequence $(\varepsilon_k)_{k\in\omega}$ of positive numbers converging to zero such that

$$X\subseteq\{x\in 2^\omega:\forall_k^\infty\ f_{n_k}(x)<\varepsilon_k\}.$$

Next, choose an increasing sequence $(i_j)_{j\in\omega}$ of natural numbers such that $1/2^{j+1}>\varepsilon_{i_j}$. Let $A=\{n_{i_0},n_{i_1},n_{i_2},\ldots\}$. Suppose that $B\in[A]^\omega$. Clearly $B=\{n_{i_{r_0}},n_{i_{r_1}},n_{i_{r_2}},\ldots\}$, where $(r_l)_{l\in\omega}$ is an increasing sequence of natural numbers. Moreover,

$$f_{n_{i_{r_l}}}(B)=2^{-|\{s\le n_{i_{r_l}}:s\in B\}|}=2^{-(l+1)}\ge 2^{-(r_l+1)}>\varepsilon_{i_{r_l}}.$$

Thus $\exists_k^\infty\ f_{n_k}(x)>\varepsilon_k$. Therefore $[A]^\omega\cap\{x:\forall_k^\infty\ f_{n_k}(x)<\varepsilon_k\}=\emptyset$. This proves that $[A]^\omega\cap X=\emptyset$. ■

Unfortunately, we are still unable to prove that every Σ set is a UCR$_0$ set. Even, we do not know whether every γ-set is a UCR$_0$ set.

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Permanent address:
Institute of Mathematics
University of Gdańsk
Wita Stwosza 57
80-952 Gdańsk, Poland
E-mail: matan@julia.univ.gda.pl

Current address:
Institute of Mathematics
Polish Academy of Sciences
Abrahama 18
81-825 Sopot, Poland
E-mail: nowik@impan.gda.pl

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