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ASYMPTOTIC BEHAVIOR OF A SEQUENCE DEFINED BY ITERATION WITH APPLICATIONS

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Abstract. We consider the asymptotic behavior of some classes of sequences defined by a recurrent formula. The main result is the following: Let $f:(0,\infty)^2\to(0,\infty)$ be a continuous function such that (a) $0<f(x,y)<px+(1-p)y$ for some $p\in(0,1)$ and for all $x,y\in(0,\alpha)$, where $\alpha>0$; (b) $f(x,y)=px+(1-p)y-\sum_{s=m}^{\infty}\mathcal{K}_s(x,y)$ uniformly in a neighborhood of the origin, where $m>1$, $\mathcal{K}_s(x,y)=\sum_{i=0}^{s}a_{i,s}x^{s-i}y^i$; (c) $\mathcal{K}_m(1,1)=\sum_{i=0}^{m}a_{i,m}>0$. Let $x_0,x_1\in(0,\alpha)$ and $x_{n+1}=f(x_n,x_{n-1})$, $n\in\mathbb{N}$. Then the sequence (x_n) satisfies the following asymptotic formula:

$$x_n\sim\left(\frac{2-p}{(m-1)\sum_{i=0}^{m}a_{i,m}}\right)^{1/(m-1)}\frac{1}{\sqrt[m-1]{n}}.$$

1. Introduction. In this paper we consider the asymptotic behavior of some classes of sequences. The well known examples are:

(a) $x_n=x_{n-1}-x_{n-1}^2,\ x_0\in(0,1)$;
(b) $x_n=\arctan x_{n-1},\ x_0\in(0,\infty)$;
(c) $x_n=\sin x_{n-1},\ x_0\in(0,\infty)$;
(d) $x_n=\ln(1+x_{n-1}),\ x_0\in(0,\infty)$.

Examining their convergence is a simple task. However a somewhat harder problem is finding the asymptotic behavior for the sequences defined by (a)–(d). Such problems frequently appear in problem section in some journals (see, for example, [5] and [21]).

The following theorem was proved in [15]. The proof appearing there is attributed to Jacobsthal.

THEOREM A. *Let $f:(0,\alpha)\to(0,\alpha)$, where $\alpha>0$, be a continuous function such that $0<f(x)<x$ for every $x\in(0,\alpha)$ and $f(x)=x-ax^k+bx^{k+p}+o(x^{k+p})$ as $x\to+0$, where $k>1$, p, a and b are positive numbers.*

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Let $x_0 \in (0, \alpha)$ *and* $x_n = f(x_{n-1})$, $n \geq 1$. *Then*

$$x_n \sim \frac{1}{((k-1)an)^{1/(k-1)}}.$$

Our proof of Theorem A in [17] is somewhat different from the original one, and the idea and structure of the proof were the starting point and inspiration for our further investigations. For this class of sequences we described, in [17], a method for finding an unlimited number of members in its asymptotic development. Our goal is to generalize Theorem A. We prove the following theorems.

THEOREM 1. *Let* $f : (0,\infty)^2 \to (0,\infty)$ *be a continuous function such that*

(a) $0 < f(x,y) < px + (1-p)y$ *for some* $p \in (0,1)$ *and all* $x, y \in (0,\alpha)$, *where* $\alpha > 0$;

(b) $f(x,y) = px + (1-p)y - \sum_{s=m}^{\infty} \mathcal{K}_s(x,y)$ *uniformly in a neighborhood of the origin, where* $m > 1$, $\mathcal{K}_s(x,y) = \sum_{i=0}^{s} a_{i,s} x^{s-i} y^i$;

(c) $\mathcal{K}_m(1,1) = \sum_{i=0}^{m} a_{i,m} > 0$.

Let $x_0, x_1 \in (0,\alpha)$ *and*

$$x_{n+1} = f(x_n, x_{n-1}), \qquad n \in \mathbb{N}. \tag{1}$$

Then the sequence (x_n) *satisfies the following asymptotic formula*:

$$x_n \sim \left(\frac{2-p}{(m-1)\sum_{i=0}^{m} a_{i,m}}\right)^{1/(m-1)} \frac{1}{\sqrt[m-1]{n}}. \tag{2}$$

THEOREM 2. *Let* $f : (0,\infty)^{k+1} \to (0,\infty)$ *be a continuous function such that*

(a) $0 < f(z_1, \dots, z_{k+1}) < 1$ *for all* $z_1, \dots, z_{k+1} \in (0,\alpha)$, *where* $\alpha > 0$;

(b) $f(z_1, \dots, z_{k+1}) = 1 - \sum_{s=m}^{\infty} \mathcal{K}_s(z_1, \dots, z_{k+1})$ *uniformly in a neighborhood of the origin, where* $m \geq 1$, *and* $\mathcal{K}_s(z_1, \dots, z_{k+1})$ *is a homogeneous polynomial of order* s;

(c) $\mathcal{K}_m(1, \dots, 1) > 0$.

Let $x_0, x_1, \dots, x_k \in (0,\alpha)$ *and*

$$x_{n+1} = x_n f(x_n, x_{n-1}, \dots, x_{n-k}), \qquad n \geq k. \tag{3}$$

Then the sequence defined by (3) *satisfies the following asymptotic formula*:

$$x_n \sim \left(\frac{1}{m\mathcal{K}_m(1,\dots,1)}\right)^{1/m} \frac{1}{\sqrt[m]{n}}. \tag{4}$$

In [6] (see also [1] and [8]) the generalized *Beddington–Holt stock recruitment model* was considered, i.e.

$$x_{n+1} = ax_n + \frac{bx_{n-1}}{1 + cx_{n-1} + dx_n}, \qquad x_0, x_1 > 0, \qquad n = 1, 2, \ldots, \tag{5}$$

where $a \in (0,1)$, $b \in \mathbb{R}_+$ and $c, d \in \mathbb{R}_+ \cup \{0\}$, with $c + d > 0$.

In [18] it was proved that in the case $a + b < 1$, the zero equilibrium is a geometrically global attractor of all positive solutions of equation (5).

In the case $a + b = 1$ and $c > 0$, it was proved that $x_n = \mathcal{O}(1/n)$. We also proved for a special choice of parameters that $x_n \sim c/n$ as $n \to \infty$ (see Example 1 in [18]). This example was the starting point for this paper and it hints that a similar asymptotic formula holds for every sequence defined by (5). Theorem 1 confirms this conjecture.

We can apply the main result of this paper to the following two equations:

$$x_{n+1} = (ax_n + bx_{n-1}e^{-x_{n-1}})e^{-x_n}, \qquad x_0, x_1 > 0, \qquad n = 1, 2, \ldots, \tag{6}$$

where $a \in (0,1)$, $b \in [0, \infty)$, in the case when $a + b = 1$ and

$$x_{n+1} = (\alpha x_n + \beta x_{n-1})e^{-x_n}, \qquad x_0, x_1 > 0, \qquad n = 1, 2, \ldots, \tag{7}$$

where $\alpha \in (0,1)$, $\beta \in (0, \infty)$, in the case when $\alpha + \beta = 1$.

Equations (6) and (7) may be viewed as describing some population models. Equation (6) describes the growth of a *mosquito population*. Equation (7) is derived from a two life stage model where the young mature into adults, and adults produce young. The global stability of these equations was studied in [9].

We can also apply the main result of this paper to the *perennial grass model*

$$x_{n+1} = ax_n + \frac{bx_{n-1} + c}{e^{x_n}}, \qquad x_0, x_1 > 0, \qquad n = 1, 2, \ldots,$$

where $a \in (0,1)$, $b, c \geq 0$, $b + c > 0$, in the case when $c = 0$ and $a + b = 1$.

The stability and oscillatory character of solutions of this equation have been studied in [6] and [7].

Theorem 2 can be applied to the following discrete analogue of the delay *logistic equation* (see, for example, [13] and [14]):

$$N_{t+1} = \frac{\alpha N_t}{1 + \beta N_{t-k}}, \qquad \alpha, \beta > 0, \qquad k \in \mathbb{N}. \tag{8}$$

In [10] the authors obtain conditions for the oscillation and asymptotic stability of all positive solutions of (8) about its positive equilibrium $(\alpha - 1)/\beta$. If $\alpha = 1$ we can easily see that $N_t \to 0$ as $t \to \infty$. The rate of convergence is determined by Theorem 2.

2. Auxiliary results. Before we prove the main result we need three auxiliary results.

LEMMA 1. *Let the real sequence* (a_n) *of nonnegative numbers satisfy the inequality*

$$a_{n+2} \le (1-p)a_{n+1} + pa_n, \qquad n \in \mathbb{N}, \tag{9}$$

where $p \in [0,1)$. *Then* (a_n) *converges.*

Proof. Let $b_n = a_{n+1} + pa_n$. Then (b_n) is nonnegative and nonincreasing by (9). Hence it converges. Let $\lim_{n\to\infty} b_n = b$.

Since

$$a_n = b_{n-1} + b_{n-2}(-p) + \ldots + b_0(-p)^{n-1} + a_0(-p)^n,$$

in the standard manner, we obtain

$$\lim_{n\to\infty} a_n = \frac{b}{1+p},$$

as desired. ∎

For generalizations of this useful lemma and closely related results, see [2]–[4], [16], [19] and [20].

Note that Lemma 1 solves problems 5.2.3(i) and 5.2.4(i) in [11]. This is incorporated in the following corollary.

COROLLARY 1. *Consider the difference equation*

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-1}}{A + \delta x_{n-1} + \alpha x_n}, \tag{10}$$

where $\beta, \gamma, A \in (0,\infty)$, $\alpha, \delta \in [0,\infty)$ *and* $\alpha + \delta > 0$. *Then every positive solution of* (10) *converges to zero if and only if* $\beta + \gamma \le A$.

Proof. Let $\beta + \gamma = A$. From (10) we have

$$x_{n+1} = \frac{\beta x_n + \gamma x_{n-1}}{A + \delta x_{n-1} + \alpha x_n} \le \frac{\beta}{A} x_n + \frac{\gamma}{A} x_{n-1}.$$

Thus by Lemma 1 the sequence (x_n) converges, say to $x \ge 0$. Letting $n \to \infty$ in (10) we obtain $x = \frac{\beta x + \gamma x}{A + (\delta + \alpha)x}$. Hence $x = 0$.

If $\beta + \gamma < A$, then from (10) it follows that

$$x_{n+1} \le \frac{\beta + \gamma}{A} \max\{x_n, x_{n-1}\}.$$

By Corollary 1 of [18] we find that (x_n) geometrically converges to zero.

Let $\beta + \gamma > A$. Suppose that there exists a positive solution (x_n) of (10) such that $x_n \to 0$ as $n \to \infty$. Then for every $\varepsilon > 0$ there exists n_0 such that

$|\delta x_{n-1}+\alpha x_n|<\varepsilon$ for all $n\ge n_0$. Let $0<\varepsilon<\beta+\gamma-A$. From (10) we have

$$x_{n+1}\ge \frac{\beta}{A+\varepsilon}\,x_n+\frac{\gamma}{A+\varepsilon}\,x_{n-1}\qquad \text{for } n\ge n_0.$$

Let the sequence (z_n) satisfy the difference equation

$$z_{n+1}=\frac{\beta}{A+\varepsilon}\,z_n+\frac{\gamma}{A+\varepsilon}\,z_{n-1},$$

where $z_0=x_0$ and $z_1=x_1$. It is easy to show by induction that $z_n\le x_n$ for all $n\in\mathbb{N}$. The positive characteristic root of the characteristic equation of this equation is

$$\lambda_1=\frac{\beta+\sqrt{\beta^2+4\gamma(A+\varepsilon)}}{2(A+\varepsilon)}>1,$$

since $\beta+\gamma>A+\varepsilon$.

The other root λ_2 is negative and $|\lambda_2|\le|\lambda_1|$. Since the sequence (z_n) is positive we have

$$z_n=c_1\lambda_1^n+c_2\lambda_2^n$$

where $c_1>0$. Hence $z_n\to\infty$ as $n\to\infty$, and consequently $x_n\to\infty$ as $n\to\infty$, which is a contradiction.

LEMMA 2. *Let $p\in(0,1)$, and let the sequence (y_n) satisfy the difference equation*

$$y_n=\frac{(1-p+p^2)y_{n-1}+p(1-p)}{py_{n-1}+1-p},\qquad y_0>0. \tag{11}$$

Then $y_n\to 1$ as $n\to\infty$.

Proof. We solve this difference equation. We look for a solution of (11) in the form $y_n=u_n/v_n$, where the sequences u_n and v_n satisfy the system of difference equations

$$\begin{aligned} u_{n+1}&=(1-p(1-p))u_n+p(1-p)v_n,\\ v_{n+1}&=pu_n+(1-p)v_n, \end{aligned} \tag{12}$$

where $u_0=y_0$ and $v_1=1$. This is a standard method for solving equations of this type (see, for example, [12]).

From (12) we obtain

$$v_{n+2}-((1-p)^2+1)v_{n+1}+(1-p)^2v_n=0\qquad \text{for } n=0,1,\dots,$$

and consequently

$$v_n=c_1+c_2(1-p)^{2n}\qquad \text{for } n=0,1,\dots \tag{13}$$

for some $c_1,c_2\in\mathbb{R}$. From (12) and (13) we obtain

$$u_n=\frac{1}{p}\,(v_{n+1}-(1-p)v_n)=c_1-c_2(1-p)^{2n+1}.$$

Hence

$$y_n = \frac{c_1 + c_2(1-p)^{2n}}{c_1 - c_2(1-p)^{2n+1}} \quad \text{for } n = 0, 1, \ldots \tag{14}$$

Since $y_n > 0$ for all $n \geq 0$ we have $c_1 \neq 0$. Letting $n \to \infty$ in (14) we obtain the result. ■

LEMMA 3. *Let $f : (0,\infty)^2 \to (0,\infty)$ be a continuous function such that*

(a) $0 < f(x,y) < px + (1-p)y$ *for some* $p \in (0,1)$ *and all* $x, y \in (0,\alpha)$, *where* $\alpha > 0$;

(b) $f(x,y) = px + (1-p)y + xo(1) + yo(1)$ *as* $x^2 + y^2 \to 0$.

Let $x_0, x_1 \in (0,\alpha)$ and (x_n) be defined by (1). *Then* $\lim_{n\to\infty} x_n = 0$ *and*

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 1. \tag{15}$$

Proof. Step 1. It is clear that (x_n) is a positive sequence. Then from (1) and by (a) we have

$$x_{n+1} = f(x_n, x_{n-1}) < px_n + (1-p)x_{n-1}.$$

Thus, Lemma 1 shows that the limit $x = \lim_{n\to\infty} x_n$ exists. Assume that $x > 0$; letting $n \to \infty$ in (1), we get $x = f(x,x)$. On the other hand, from (a) it follows that $f(x,x) < px + (1-p)x = x$ for $x > 0$. Therefore x must be zero.

Step 2. Let us show (15). It is obvious that (1) can be written in the form

$$y_n = p - \varepsilon_n + \frac{1 - p - \delta_n}{y_{n-1}}$$

where

$$y_n = \frac{x_{n+1}}{x_n}, \qquad \varepsilon_n = o(1), \qquad d_n = o(1).$$

Hence

$$y_n = \frac{[(p-\varepsilon_n)(p-\varepsilon_{n-1}) + 1 - p - \delta_n]y_{n-2} + (p-\varepsilon_n)(1-p-\delta_{n-1})}{(p-\varepsilon_{n-1})y_{n-2} + 1 - p - \delta_{n-1}}.$$

Let $z_n = y_{2n}$ and $\gamma_n = y_{2n+1}$. From the conditions of the lemma for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ we have $|\varepsilon_n| < \varepsilon$ and $|\delta_n| < \varepsilon$. We may suppose $n_0 = 1$. Thus we have

$$\frac{[(p-\varepsilon)^2 + 1 - p - \varepsilon]z_{n-1} + (p-\varepsilon)(1-p-\varepsilon)}{(p+\varepsilon)z_{n-1} + 1 - p + \varepsilon} \leq z_n \leq \frac{(1 - p + \varepsilon + (p+\varepsilon)^2)z_{n-1} + (p+\varepsilon)(1-p+\varepsilon)}{(p-\varepsilon)z_{n-1} + 1 - p - \varepsilon}.$$

Let the sequences (f_n) and (g_n) satisfy the equations

$$f_n = \frac{[(p-\varepsilon)^2 + 1 - p - \varepsilon]f_{n-1} + (p-\varepsilon)(1-p-\varepsilon)}{(p+\varepsilon)f_{n-1} + 1 - p + \varepsilon},$$

$$g_n = \frac{(1-p+\varepsilon+(p+\varepsilon)^2)g_{n-1} + (p+\varepsilon)(1-p+\varepsilon)}{(p-\varepsilon)g_{n-1} + 1 - p - \varepsilon},$$

where $f_1 = z_1 = g_1$.

It is easy to see that for $\varepsilon \in (0, \min\{p, 1-p\})$ we have $0 \le f_n \le z_n \le g_n$. As in Lemma 2 we can find explicit expressions for (f_n) and (g_n).

Let

$$f_n = f_n(\varepsilon) = \frac{u_n(\varepsilon)}{v_n(\varepsilon)} = \frac{u_n}{v_n}.$$

Then as in Lemma 2 we have

$$v_n = c_1(\varepsilon, p)\lambda_1^n(\varepsilon, p) + c_2(\varepsilon, p)\lambda_2^n(\varepsilon, p),$$

where, for sufficiently small $\varepsilon > 0$, $\lambda_1(\varepsilon, p)$ is close to 1 and $\lambda_2(\varepsilon, p)$ is close to $(1-p)^2$. Since

$$v_{n+1} = (p+\varepsilon)u_n + (1-p+\varepsilon)v_n$$

we have

$$\begin{aligned} u_n &= \frac{1}{p+\varepsilon}(v_{n+1} - (1-p+\varepsilon)v_n) \\ &= [\lambda_1 - (1-p+\varepsilon)]\,\frac{c_1\lambda_1^n}{p+\varepsilon} + [\lambda_2 - (1-p+\varepsilon)]\,\frac{c_2\lambda_2^n}{p+\varepsilon}. \end{aligned}$$

By the same argument as in Lemma 2 we have $c_1(\varepsilon, p) > 0$ for such ε.

From that we obtain

$$\lim_{n\to\infty} f_n = \lim_{n\to\infty} \frac{u_n(\varepsilon)}{v_n(\varepsilon)} = \frac{\lambda_1 - (1-p+\varepsilon)}{p+\varepsilon} \to 1 \quad \text{as } \varepsilon \to 0.$$

Similarly we obtain

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} g_n(\varepsilon) = 1,$$

which implies $\lim_{n\to\infty} z_n = 1$. In the same manner we obtain $\lim_{n\to\infty} \gamma_n = 1$, finishing the proof. ∎

3. Proof of the main results. We are now in a position to prove the main results.

Proof of Theorem 1. Step 1. $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} x_{n+1}/x_n = 1$ are simple consequences of Lemma 3.

Step 2. We may suppose that (x_n, x_{n-1}) belong to the set where (b) holds for every $n \in \mathbb{N}$. We transform (1) to a system of difference equations.

Let $x_n = y_{n-1}$. Then

$$\begin{aligned} y_n &= py_{n-1} + (1-p)x_{n-1} - \mathcal{K}_m(y_{n-1}, x_{n-1}) + o((y_{n-1}^2 + x_{n-1}^2)^{m/2}),\\ x_n &= y_{n-1}. \end{aligned} \tag{16}$$

System (16) can be written in the following matrix form:

$$\begin{bmatrix} y_n \\ x_n \end{bmatrix} = \begin{bmatrix} p & 1-p \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y_{n-1} \\ x_{n-1} \end{bmatrix} - \begin{bmatrix} \mathcal{K}_m(y_{n-1}, x_{n-1}) + o((y_{n-1}^2 + x_{n-1}^2)^{m/2}) \\ 0 \end{bmatrix}.$$

Let

$$A = \begin{bmatrix} p & 1-p \\ 1 & 0 \end{bmatrix}.$$

The characteristic polynomial of A is $P_2(\lambda) = (\lambda - 1)(\lambda - (p-1))$. For $\lambda = 1$ and $\lambda = p-1$ we obtain the corresponding characteristic vectors

$$\vec{w}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{w}_2 = \begin{bmatrix} 1-p \\ -1 \end{bmatrix}.$$

Let

$$\begin{bmatrix} y_n \\ x_n \end{bmatrix} = \begin{bmatrix} 1 & 1-p \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_n \\ u_n \end{bmatrix}, \qquad n \in \mathbb{N}.$$

Then

$$\begin{aligned} \begin{bmatrix} v_n \\ u_n \end{bmatrix} = &\begin{bmatrix} 1 & 0 \\ 0 & p-1 \end{bmatrix} \begin{bmatrix} v_{n-1} \\ u_{n-1} \end{bmatrix} \\ &- \frac{1}{2-p} \begin{bmatrix} 1 & 1-p \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \mathcal{K}_m(v_{n-1} + (1-p)u_{n-1}, v_{n-1} - u_{n-1}) + \dots \\ 0 \end{bmatrix}. \end{aligned}$$

Thus we have

$$v_n = v_{n-1} - \frac{1}{2-p} \widehat{\mathcal{K}}_m(v_{n-1}, u_{n-1}) + o((v_{n-1}^2 + u_{n-1}^2)^{m/2}) \tag{17}$$

and

$$u_n = (p-1)u_{n-1} - \frac{1}{2-p} \widehat{\mathcal{K}}_m(v_{n-1}, u_{n-1}) + o((v_{n-1}^2 + u_{n-1}^2)^{m/2}),$$

where

$$\widehat{\mathcal{K}}_m(v_{n-1}, u_{n-1}) = \mathcal{K}_m(v_{n-1} + (1-p)u_{n-1}, v_{n-1} - u_{n-1}).$$

Applying Step 1 we obtain

$$0 \le \lim_{n\to\infty} \left| \frac{u_n}{v_n} \right| = \lim_{n\to\infty} \frac{|x_{n+1} - x_n|}{x_{n+1} + (1-p)x_n} \le \lim_{n\to\infty} \frac{|x_{n+1} - x_n|}{(1-p)x_n} = 0.$$

Hence, from (17),

$$v_n = v_{n-1} - \frac{\sum_{i=0}^m a_{i,m}}{2-p} v_{n-1}^m + o(v_{n-1}^m).$$

Finally, applying Theorem A and using the fact that $x_n = v_n - u_n = v_n + o(v_n)$ we obtain the result. ■

EXAMPLE 1. Consider (5), where $a + b = 1$. Since

$$f(x,y) = ax + \frac{(1-a)y}{1+cy+dx} = ax + (1-a)y + (1-a)y\sum_{n=1}^{\infty}(-1)^n(cy+dx)^n$$

uniformly in $|cy + dx| < 1$, by Theorem 1 we obtain

$$x_n \sim \frac{2-a}{(1-a)(c+d)}\frac{1}{n}.$$

EXAMPLE 2. Consider (6), where $a, b \in (0,1)$ and $a + b = 1$. Since

$$\begin{aligned} f(x,y) &= (ax + (1-a)ye^{-y})e^{-x} \\ &= ax + (1-a)y + ax\sum_{n=1}^{\infty}(-x)^n + (1-a)y\sum_{n=1}^{\infty}(-1)^n(x+y)^n \end{aligned}$$

uniformly in $x^2 + y^2 \le r, r > 0$, by Theorem 1 we obtain $x_n \sim 1/n$.

EXAMPLE 3. Consider (7), where $\alpha, \beta \in (0,1)$ and $\alpha + \beta = 1$. Since

$$\begin{aligned} f(x,y) &= (\alpha x + (1-\alpha)y)e^{-x} \\ &= \alpha x + (1-\alpha)y + x\sum_{n=1}^{\infty}(-x)^n + (1-\alpha)y\sum_{n=1}^{\infty}(-1)^n x^n \end{aligned}$$

uniformly in $x^2 + y^2 \le r, r > 0$, by Theorem 1 we obtain $x_n \sim (2-\alpha)/n$.

Proof of Theorem 2. From the conditions of the theorem we have $x_{n+1} < x_n$ for all $n \ge k+1$. As in Theorem 1, we have $x_n \to 0$ and $x_{n+1}/x_n \to 1$ as $n \to \infty$.

From this and (3) we obtain

$$x_{n+1} = x_n - \mathcal{K}_m(1,\ldots,1)x_n^{m+1} + o(x_n^{m+1}).$$

By Theorem A we obtain the result. ■

EXAMPLE 4. Consider (8), when $\alpha = 1$ and $x_0, x_1, \ldots, x_k \in \mathbb{R}$. Since

$$f(z_1,\ldots,z_{k+1}) = \frac{z_1}{1+\beta z_{k+1}} = z_1\sum_{n=0}^{\infty}(-\beta z_{k+1})^n$$

uniformly on $|\beta z_{k+1}| < 1$, by Theorem 2 we obtain $x_n \sim 1/(\beta n)$.

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