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EXACT $\mathcal{C}^\infty$ COVERING MAPS OF THE CIRCLE WITHOUT (WEAK) LIMIT MEASURE

BY

ROLAND ZWEIMÜLLER (Salzburg)

Abstract. We construct $\mathcal{C}^\infty$ maps T on the interval and on the circle which are Lebesgue exact preserving an absolutely continuous infinite measure $\mu \ll \lambda$, such that for any probability measure $\nu \ll \lambda$ the sequence $(n^{-1}\sum_{k=0}^{n-1} \nu \circ T^{-k})_{n\geq 1}$ of arithmetical averages of image measures does not converge weakly.

1. Introduction. A measurable map T on some σ-finite measure space $(X, \mathcal{A}, m)$ is called *nonsingular* if $m \circ T^{-1} \ll m$. In this case the image of any absolutely continuous measure $\nu \ll m$ with density $u \in L_1(m)$ again has a density, denoted by $\widehat{T}u := d(\nu \circ T^{-1})/dm$. The positive linear operator $\widehat{T} : L_1(m) \to L_1(m)$ thus defined is the *dual* (or *transfer* or *Perron–Frobenius*) *operator* of T with respect to m. For a probability density $u \in \mathcal{D}(m) := \{v \in L_1(m) : v \geq 0,\ m(v) = 1\}$ on X, $\widehat{T}^n u$ is the density of the distribution of T^n on X. A result of M. Lin (cf. [Li]) shows that T is *exact* with respect to m (meaning that the *tail σ-field* $\mathcal{A}_\infty := \bigcap_{n\geq 0} T^{-n}\mathcal{A}$ only contains sets A for which either A or A^{c} has zero measure) iff for any $u, v \in \mathcal{D}(m)$ we have $\lim_{n\to\infty} \|\widehat{T}^n u - \widehat{T}^n v\|_{L_1(m)} = 0$.

Assume now that X is a compact metrizable space and $\mathcal{A}$ equals $\mathcal{B} = \mathcal{B}_X$, its Borel σ-field. The set $\mathcal{M}_1(X)$ of all probability measures on $\mathcal{B}$ is compact and metrizable in the topology of *weak convergence of measures* (i.e. in the weak*-topology on $\mathcal{C}^*(X)$), where $\nu_n \to \nu$ iff $\lim_{n\to\infty} \nu_n(f) = \nu(f)$ for all $f \in \mathcal{C}(X)$. For any $\nu \in \mathcal{M}_1(X)$ the sequence $(\nu \circ T^{-n})_{n\geq 0}$ of image measures therefore has accumulation points in $\mathcal{M}_1(X)$. If now T is exact with respect to m and there is some $\nu_0 \in \mathcal{M}_1(X)$ with $\nu_0 \ll m$ such that $\nu_0 \circ T^{-n}$ actually converges to some measure $\widetilde{\nu}$ in $\mathcal{M}_1(X)$ (e.g. if there exists an absolutely continuous invariant probability $\widetilde{\nu}$), then Lin's theorem implies that in fact $\lim_{n\to\infty} \nu \circ T^{-n} = \widetilde{\nu}$ for all $\nu \in \mathcal{M}_1(X)$ with $\nu \ll m$. Rudnicki in [Ru] raised the question whether every exact nonsingular map on a compact metric space had such a *weak limit measure*. As pointed out in [Ke], there do exist quadratic maps of the unit interval $X := [0, 1]$ which have no weak

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limit measure (cf. [HK]) but are exact with respect to Lebesgue measure $\lambda =: m$ (cf. [BH]).

The purpose of the present note is to propose a construction which produces simpler exact counterexamples on the interval and on the circle for which (just as for those of [Ke]) even the averaged sequence $(n^{-1}\sum_{k=0}^{n-1}\nu\circ T^{-k})_{n\geq 0}$ does not converge in $\mathcal{M}_1(X)$ if ν is absolutely continuous with respect to Lebesgue measure λ.

2. Construction of examples: $\mathcal{C}^\infty$ covering maps on the interval and the circle. The transformations T considered here will be *piecewise smooth and onto*: there is some finite partition ξ of X into subintervals Z_i, $i\in I$, such that each restriction $T|_{Z^\circ}$, $Z\in\xi$, is a $\mathcal{C}^\infty$ diffeomorphism onto $(0,1)$. They will be almost expanding in that $T'>1$ except at *indifferent fixed points* x_i where $T'x_i=1$, and the mass pushed forward by the map will keep fluctuating between shrinking neighbourhoods of these points.

We begin with the globally simplest prototypical family of interval maps with two branches and two indifferent fixed points $x_0=0$ and $x_1=1$. A slight variation of this will then result in equally smooth covering maps of the circle.

Starting from a map T_1 we shall give an inductive scheme producing a sequence $(T_j)_{j\geq 1}$ of maps by changing T_j on the set $(0,\beta_j)\cup(1-\beta_j,1)$ (where $\beta_j\searrow 0$) to obtain T_{j+1}. One suitable choice for T_1 is as follows. Let $H(t):=t+1_{(0,\infty)}(t)\cdot 2^{-1}\exp(2-1/t)$, so that $H\in\mathcal{C}^\infty(\mathbb{R})$, and let S denote its restriction to $[0,1/2]$. Then take $T_1(x):=S(x)$ for $x\in[0,1/2]$ and $T_1(x):=1-S^*(1-x)$ for $x\in(1/2,1]$, where $S^*:=S$.

We want the modification procedure to preserve a few convenient properties of the branches of T_j, clearly shared by the preceding example, which we collect in the following definition: We let $\mathcal{S}$ denote the collection of all $\mathcal{C}^\infty$ diffeomorphisms $S:[0,1/2]\to[0,1]$ of the form $Sx=x+Dx$ with $D:[0,1/2]\to[0,1/2]$ increasing, $D'\leq\kappa_D\cdot D''$ on $[0,1/4]$ for some $\kappa_D\in(0,\infty)$, and $D^{(n)}(0^+)=0$ for all $n\geq 0$. Observe that in this case both S and D are convex (but D may well vanish on some interval $[0,\varepsilon]$), and D can be extended to a $\mathcal{C}^\infty$ function on $(-\infty,1/2]$ by letting $Dx:=0$ for $x<0$.

Given $S,S^*\in\mathcal{S}$ we let $[S,S^*]$ denote the piecewise $\mathcal{C}^\infty$ map $T:[0,1]\to[0,1]$ with $Tx=Sx$ for $x\in Z_0:=[0,1/2]$ and $Tx=1-S^*(1-x)$ for $x\in Z_1:=(1/2,1]$. If both S and S^* are strictly convex, then $T=[S,S^*]$ belongs to the class $\mathcal{T}$ of endomorphisms studied in [T2], and we write $T\in\mathcal{T}_{\mathcal{S}}$. Thus T is conservative ergodic with respect to Lebesgue measure λ and preserves an infinite measure $\mu\ll\lambda$ which has a continuous positive density h with singularities of the type specified in [T1] at the indifferent fixed points $x_i=0,1$. According to Theorem 1 of [T2], T is Lebesgue exact.

As a consequence of these properties, $\lim_{n\to\infty}\int_{(\varepsilon,1-\varepsilon)}\widehat{T}^n u\,d\lambda = 0$ for any $\varepsilon > 0$ and $u \in L_1(\lambda)$ (cf. [T3]), showing that the mass of the iterated densities $\widehat{T}^n u$ accumulates near the fixed points x_i. We are going to construct a transformation of this type for which this mass fluctuates between the two points (δ_x denotes unit point mass at x):

THEOREM 1 (Existence of $T \in \mathcal{T}_{\mathcal{S}}$ without weak limit measures). *The class $\mathcal{T}_{\mathcal{S}}$ contains maps $T = [S, S^*]$ which are exact and have no weak limit measures on* $[0,1]$, *as for any $u \in \mathcal{D}(\lambda)$ the set of weak accumulation points of the measures* $(n^{-1}\sum_{k=0}^{n-1}\widehat{T}^k u\,d\lambda)_{n\geq 1}$ *equals* $\{s\delta_0 + (1-s)\delta_1 : s \in [0,1]\}$.

REMARK 1. The maps $T \in \mathcal{T}$ are not only exact, but share another strong ergodic property: They are *pointwise dual ergodic*, i.e. for each $T \in \mathcal{T}$ there are constants $a_n = a_n(T) \in (0,\infty)$, $n \geq 1$, such that $a_n^{-1}\sum_{k=0}^{n-1}\widehat{T}^k u \to \lambda(u)h$ a.e. on $[0,1]$ for any $u \in L_1(\lambda)$; see [A1], or Sections 3.7 and 4.8 of [A0]. In fact, if $u \in \mathcal{C}^1([0,1])$ and $u > 0$, this convergence is even uniform on each $(\varepsilon, 1-\varepsilon)$, $\varepsilon \in (0,1/2)$; cf. [T3] and [Zw]. Still, this regular asymptotic behaviour of the $\sum_{k=0}^{n-1}\widehat{T}^k u$ on any center interval cannot prevent the mass fluctuations between the endpoints.

Proof of Theorem 1. By exactness we need only consider one specific $u \in \mathcal{D}(\lambda)$ which we choose to be (uniformly) continuous. Since for any $\varepsilon > 0$ and $T \in \mathcal{T}_{\mathcal{S}}$, $\lim_{n\to\infty}\int_{(\varepsilon,1-\varepsilon)} n^{-1}\sum_{k=0}^{n-1}\widehat{T}^k u\,d\lambda = 0$, it is enough to construct some $T_\infty \in \mathcal{T}_{\mathcal{S}}$ and a subsequence $n_j \nearrow \infty$ of $\mathbb{N}$ such that

$$\int_{I_j}\left(\frac{1}{n_j}\sum_{k=0}^{n_j-1}\widehat{T}_\infty^k u\right)d\lambda \leq \frac{1}{j}$$

for all $j \geq 1$, where $I_j = Z_0$ if j is even and $I_j = Z_1$ if j is odd. The map T_∞ we are going to construct will be the limit of a sequence $T_j = [S_j, S_j^*]$, $j \geq 1$, in $\mathcal{T}_{\mathcal{S}}$ with all $S_j^{(*)} = \mathrm{Id} + D_j^{(*)}$ strictly convex, where $S^{(*)} \in \{S, S^*\}$ (the same convention for $D^{(*)}$ respectively), and $T_i = T_j$ on $(\beta_j, 1-\beta_j)$ for all $i \geq j \geq 1$, where $(\beta_j)_{j\geq 1}$ is a suitable sequence in $(0,1/2)$ with $\beta_j \searrow 0$.

Clearly, $S_\infty^{(*)} := \lim_{j\to\infty} S_j^{(*)} =: \mathrm{Id} + D_\infty^{(*)}$ are then continuous and strictly increasing on $[0,1/2]$. They are strictly convex and $\mathcal{C}^\infty$ on each $(\delta, 1/2]$, $\delta \in (0,1/2)$, which immediately proves strict convexity on all of $[0,1/2]$. To show that these functions are in fact $\mathcal{C}^\infty$ on $[0,1/2]$ we need to check that $\lim_{x\to 0^+}(D_\infty^{(*)})^{(k)}(x) = 0$ and $(D_\infty^{(*)})^{(k)}(0^+) = 0$ for all $k \geq 1$. This follows by a simple induction from the fact that we shall have $|(D_{j+1}^{(*)})^{(k)}| \leq (1+\varepsilon_j)|(D_j^{(*)})^{(k)}|$ for $1 \leq k \leq j+1$, where $\varepsilon_j := 2^{-j}$. Finally, T_∞ will belong to $\mathcal{T}_{\mathcal{S}}$, as $S_\infty^{(*)} \in \mathcal{S}$ results from the estimate $(D_{j+1}^{(*)})' \leq (\kappa_j + \varepsilon_j)(D_{j+1}^{(*)})''$ provided below.

To start the inductive construction at step $j = 1$, we choose any $T_1 = [S_1, S_1^*] \in \mathcal{T}_{\mathcal{S}}$ for which each derivative $T_1^{(k)}$, $k \geq 1$, is strictly monotone in a suitable neighbourhood N_k of the fixed points, e.g. $S_1^{(*)} = H|_{[0,1/2]}$ with H as above. Let $n_1 := 1$, $\beta_1 := 1/4$. Since $\int_{[0,1]} n^{-1} \sum_{k=0}^{n-1} \widehat{T}^k u \, d\lambda = 1$ in any case, we have

$$\int\limits_{I_j} \left(\frac{1}{n_j} \sum_{k=0}^{n_j-1} \widehat{T}^k u \right) d\lambda \leq \frac{1}{j} \quad \text{for } j = 1 \text{ and any } T \in \mathcal{T}_{\mathcal{S}}.$$

For the inductive step assume that for some $j \geq 1$ we have constructed $T_j = [S_j, S_j^*] \in \mathcal{T}_{\mathcal{S}}$ with all derivatives monotone near the fixed points, and found $n_j \geq 1$, $\beta_j \in (0, 1/4)$ such that

$$\int\limits_{I_j} \left(\frac{1}{n_j} \sum_{k=0}^{n_j-1} \widehat{T}^k u \right) d\lambda \leq \frac{1}{j} \quad \text{for any } T \in \mathcal{T}_{\mathcal{S}} \text{ with } T = T_j \text{ on } [\beta_j, 1 - \beta_j].$$

We show that we can do likewise for $j + 1$ with some $\beta_{j+1} \in (0, \beta_j/2)$, thereby respecting the estimates on derivatives mentioned above. Suppose without loss of generality that j is even, so that $I_{j+1} = Z_1$. (In case j is odd, apply the argument to follow to $[S_j^*, S_j]$ obtaining $[S_{j+1}^*, S_{j+1}]$ and take $T_{j+1} := [S_{j+1}, S_{j+1}^*]$.) We shall isolate the main steps of the construction in the form of four lemmas, whose proofs are deferred to the next section.

The first crucial observation is that no matter how high the degree of tangency of some strictly convex $S \in \mathcal{S}$ to the identity may be, we can still do much better without leaving $\mathcal{S}$:

Lemma 1 (Locally deforming $S \in \mathcal{S}$ towards the identity). *For any strictly convex $S = \mathrm{Id} + D \in \mathcal{S}$ with all derivatives $D^{(k)}$, $k \geq 1$, strictly monotone on neighbourhoods of 0^+, any $j \geq 1$, $\varepsilon > 0$ and $\beta \in (0, 1/2)$ there is a decreasing family $(\Phi_t)_{t \in [0,1]} = (\mathrm{Id} + \Psi_t)_{t \in [0,1]}$ in $\mathcal{S}$, $\mathcal{C}^r$ for any $r \geq 1$, with the following properties: $\Phi_1 = S$, $\Phi_t|_{[\beta,1/2]} = S|_{[\beta,1/2]}$ for all $t \in [0,1]$, for each $t \in (0,1]$ the function Ψ_t is strictly convex with derivatives strictly monotone around 0^+, and there is some $\eta \in (0, \beta)$ such that $\{x : \Phi_0 x = x\} = [0, \eta]$. Moreover, we can ensure that for all $t \in [0,1]$ we have $\Psi_t' \leq (\kappa_D + \varepsilon) \Psi_t''$ on $[0, 1/4]$, and $|\Psi_t^{(k)}| \leq (1 + \varepsilon) |D^{(k)}|$ for $1 \leq k \leq j + 1$.*

We apply Lemma 1 with $S := S_j$, $\varepsilon := \varepsilon_j$, and $\beta := \beta_j$, to obtain $\eta > 0$ and a family $(\Phi_t)_{t \in [0,1]}$ in $\mathcal{S}$ as specified there. We can thus locally modify $T_j = [S_j, S_j^*]$ near $x = 0$ to obtain maps $[\Phi_t, S_j^*] \in \mathcal{T}_{\mathcal{S}}$, $t \in (0,1]$, which are close to the identity on $[0, \eta]$. This is perfect for our purpose since the limit map $[\Phi_0, S_j^*]$ traps all the mass into this now absorbing set:

Lemma 2 (Mass accumulates in the absorbing set). *Let $T = [S, S^*]$ with $S^* \in \mathcal{S}$ strictly convex and $S \in \mathcal{S}$ with $\{Sx = x\} = [0, \eta]$ for some $\eta > 0$. Then $\lim_{n \to \infty} \int_{[\eta,1]} \widehat{T}^n u \, d\lambda = 0$ for any $u \in L_1(\lambda)$.*

Let $P_{(t)}$ denote the dual operator of $[\Phi_t, S_j^*]$, $t \in [0,1]$. According to the Lemma 2, there is some $n_{j+1} > n_j$ such that

$$\int_{Z_1} \left(\frac{1}{n_{j+1}} \sum_{k=0}^{n_{j+1}-1} P_{(0)}^k u \right) d\lambda \le \frac{1}{3(j+1)}.$$

We cannot take T_{j+1} to be this limit map, as we need strictly convex branches. However, all the Φ_t with $t > 0$ are strictly convex and approximate Φ_0 in the $\mathcal{C}^1$-norm, which by the next lemma is enough to let us conclude that $P_{(t)}^k u \to P_{(0)}^k u$ in $\mathcal{C}^0$.

Lemma 3 (Continuous dependence of $\widehat{T}^k u$ on $\mathcal{C}^1$-branches). *Let $u \in \mathcal{C}([0,1])$, $S^* \in \mathcal{S}$ and for $S \in \mathcal{S}$ let P_S denote the dual operator of $[S, S^*]$ with respect to λ. For any $k \ge 1$, $S \mapsto P_S^k u$ is then continuous as a map from $(\mathcal{S}, \|\cdot\|_{\mathcal{C}^1})$ into $(\mathcal{C}([0,1]), \|\cdot\|_{\mathcal{C}^0})$.*

Therefore $\int_{Z_1} n_{j+1}^{-1} \sum_{k=0}^{n_{j+1}-1} P_{(t)}^k u \, d\lambda$ is continuous in t, so that if we define $S_{j+1}^* := S_j^*$ and $S_{j+1} := \Phi_t$ for some sufficiently small $t > 0$, $T_{j+1} := [S_{j+1}, S_{j+1}^*] \in \mathcal{T}_{\mathcal{S}}$ still satisfies

$$\int_{Z_1} \left(\frac{1}{n_{j+1}} \sum_{k=0}^{n_{j+1}-1} \widehat{T}_{j+1}^k u \right) d\lambda \le \frac{1}{2(j+1)}.$$

We finally need to provide some space for the modifications to be done in the subsequent steps of the construction. Let us point out that this does not depend on the particular class of maps, but works for any nonsingular system:

Lemma 4 (Small modifications of nonsingular transformations). *Let T be a nonsingular map on the σ-finite measure space $(X, \mathcal{A}, m)$, and consider a decreasing sequence $(B_l)_{l\ge1}$ in $\mathcal{A}$ with $\lim_{l\to\infty} m(B_l) = 0$. Then for any $n \ge 1$, $u \in L_1(m)$, and $\varepsilon > 0$ there is some $l = l(n, u, \varepsilon) \ge 1$ such that*

$$\|\widehat{T}^k u - \widehat{T}_*^k u\|_{L_1(m)} < \varepsilon \quad \textit{for } k \in \{0, 1, \dots, n-1\}$$

whenever T_* is a nonsingular map on $(X, \mathcal{A}, m)$ with $T_* = T$ on B_l^{c}.

As a consequence, there is some $\beta_{j+1} \in (0, \beta_j/2)$ such that

$$\int_{Z_1} \left(\frac{1}{n_{j+1}} \sum_{k=0}^{n_{j+1}-1} \widehat{T}^k u \right) d\lambda \le \frac{1}{j+1}$$

for any $T \in \mathcal{T}_{\mathcal{S}}$ with $T = T_{j+1}$ on $[\beta_{j+1}, 1 - \beta_{j+1}]$. This completes the inductive step and hence the proof of Theorem 1. ∎

Observe that if we start our construction with a map $T_1 \in \mathcal{T}_{\mathcal{S}}$ which can be regarded as a $\mathcal{C}^\infty$ covering map of the circle by identifying the endpoints of the interval (i.e. if $T_1^{(k)}((1/2)^-) = T_1^{(k)}((1/2)^+)$ for $k \geq 1$), then the same is true for the limit map T, as $T_j = T_1$ for all $j \geq 1$ around the critical point $x = 1/2$ and the fixed point is flat on either side. However, T then has a weak limit measure on the circle, as the two accumulation points of the measures coincide. Still, a slight modification of the construction yields

THEOREM 2 (Covering maps of the circle). *There exist Lebesgue exact $\mathcal{C}^\infty$ covering maps of the circle without weak limit measure.*

Let us briefly sketch how to construct an orientation-preserving map of degree 3 with the required properties. (To get a degree 2 map, we need to reverse orientation.) Take some 3-to-1 map T_1 from Thaler's class $\mathcal{T}$ with flat indifferent fixed points at $x_i \in \{0^+, 1/2, 1^-\}$ (i.e. $T_1'(x_i) = 1$ and $T_1^{(k)}(x_i) = 0$ for $k \geq 2$) which is $\mathcal{C}^\infty$ on the circle and satisfies $(T_1 - \mathrm{Id})' \leq \kappa(T_1 - \mathrm{Id})''$ and monotonicity of derivatives near the fixed points. Then use the same inductive scheme as before, modifying T_j near 0^+ and 1^- if j is even, or near $(1/2)^\pm$ if j is odd, to obtain T_{j+1}. The straightforward formal modifications are left to the reader.

3. Proofs of the lemmas. We conclude with the technical proofs of the lemmas announced before.

Proof of Lemma 1. Assume without loss of generality that β is so small that each $D^{(k)}$, $1 \leq k \leq j+1$, is strictly monotone on $[0, \beta]$. It is enough to construct $\widetilde{S} \leq S$ satisfying the requirements for Φ_0 and take $\Phi_t := t \cdot S + (1-t) \cdot \widetilde{S}$, $t \in [0,1]$. To this end we let $\widetilde{S}x := x + \widetilde{D}x$ with $\widetilde{D} := D \circ \varphi_\alpha$ and φ_α chosen as follows.

Take $a, b \in (0, \beta)$, $a < b$; then we can choose some concave $\mathcal{C}^\infty$ function F on $\mathbb{R}$ with $F = 0$ on $[b, \infty)$ and $F' = 1$ on $(-\infty, a]$. For $\alpha \in (0,1)$ we define $\varphi_\alpha(x) := x + \alpha F(x)$, $x \in \mathbb{R}$. Then $\varphi_\alpha \in \mathcal{C}^\infty(\mathbb{R})$, $\varphi_\alpha = \mathrm{Id}$ on $[\beta, \infty)$, and φ_α is strictly increasing and concave, and $\varphi_\alpha(0) < 0$, so that there is a unique zero $\eta_\alpha \in (0, \beta)$. Notice that for any $r \geq 1$, $\varphi_\alpha \to \mathrm{Id}$ in $\mathcal{C}^r([0, 1/2])$ as $\alpha \searrow 0$, in particular we have $\eta_\alpha \to 0$.

The function $\widetilde{D}$ is $\mathcal{C}^\infty$ and increasing with $\widetilde{D} = 0$ on $[0, \eta_\alpha]$, $\widetilde{D} = D$ on $[b, 1/2]$, and satisfies $\widetilde{D} \leq D$. The first of these properties immediately implies that for $t \in (0,1]$ each derivative $\Psi_t^{(k)}$, $k \geq 1$, has the same monotonicity behaviour around 0^+ as $D^{(k)}$. As φ_α is affine on $[0, a]$, we have $\widetilde{D}^{(k)} = (1+\alpha)^k D^{(k)} \circ \varphi_\alpha$ on this set, which in view of the monotonicity of $D^{(k)}$ there shows that $|\widetilde{D}^{(k)}| \leq (1+\varepsilon)|D^{(k)}|$ on $[0, a]$ for $1 \leq k \leq j+1$ provided α is small enough.

To deal with the $\widetilde{D}^{(k)}$ on $[a,b]$, we notice that each is a finite sum of terms of the form $\mathrm{const}\cdot(D^{(i)}\circ\varphi_\alpha)\cdot\prod_{1\le l\le k}(\varphi_\alpha^{(l)})^{m_l}$ with $1\le i\le k$ and $m_l\ge 0$, and that the only one containing no factor $\varphi_\alpha^{(l)}$ with $l\ge 2$ (and hence not necessarily tending to zero as $\alpha\to 0$) is $(D^{(k)}\circ\varphi_\alpha)(\varphi'_\alpha)^k$. By strict monotonicity of the derivatives, $D^{(k)}$ has no zero in $[a,b]$, and we can conclude that $|\widetilde{D}^{(k)}|\le(1+\varepsilon)|D^{(k)}|$ on $[a,b]$ for $1\le k\le j+1$ for α sufficiently small.

Straightforward calculation finally shows that $\widetilde{D}'\le(\kappa_D+\varepsilon)\widetilde{D}''$ for α small enough if we recall that $\varphi'_\alpha\to 1$ and $\varphi''_\alpha\to 0$ uniformly on $[0,1/2]$ as $\alpha\searrow 0$. ∎

Proof of Lemma 2. We are going to show that $M:=[0,\eta)$ is a sweep-out set, i.e. $\bigcup_{n\ge1}T^{-n}M=[0,1]\bmod\lambda$, implying $\lim_{n\to\infty}\lambda(\bigcap_{k=0}^n T^{-k}M^{\mathrm{c}})=0$. The assertion then follows immediately: M being an absorbing set (i.e. $TM\subseteq M$), we have $\int_{M^{\mathrm{c}}}\widehat{T}^n u\,d\lambda=\int_{\bigcap_{k=0}^n T^{-k}M^{\mathrm{c}}}u\,d\lambda$.

Let $M':=Z_1\cap T^{-1}M$; then $T^{-1}M\backslash M'=M$, showing that the sweep-out property for M follows once we prove that M' is a sweep-out set for $T|_{M^{\mathrm{c}}}$. This however is easily seen, as the map $T_0:M^{\mathrm{c}}\to M^{\mathrm{c}}$ with $T_0=T$ on $M^{\mathrm{c}}\backslash M'$ which maps M' affinely onto M^{c} is of type $\mathcal{T}$ and hence conservative ergodic on M^{c} (so that any set of positive Lebesgue measure is a sweep-out set for T_0). ∎

Proof of Lemma 3. For T piecewise smooth and onto and $(i_0,\ldots,i_{n-1})\in I^n$ we let $Z_{i_0,\ldots,i_{n-1}}:=\bigcap_{k=0}^{n-1}T^{-k}Z_{i_k}$ denote the cylinders of order n, and write $f_{i_0,\ldots,i_{n-1}}:=(T^n|_{Z_{i_0,\ldots,i_{n-1}}})^{-1}:(0,1)\to Z_{i_0,\ldots,i_{n-1}}$. The dual operator $\widehat{T}$ with respect to Lebesgue measure λ then has a version which admits a simple explicit representation as

$$\widehat{T}^n u=\sum_{(i_0,\ldots,i_{n-1})\in I^n}u\circ f_{i_0,\ldots,i_{n-1}}\cdot|f'_{i_0,\ldots,i_{n-1}}|.$$

It is therefore enough to show that each $f_{i_0,\ldots,i_{n-1}}$ depends $\mathcal{C}^1$-continuously on the $\mathcal{C}^1$-branch S, which is immediate from the following observations whose elementary proofs are omitted: Let J and J' be compact intervals; then the operation of inverting $\mathcal{C}^1$ diffeomorphisms of J onto J', $g\mapsto g^{-1}$, is $\mathcal{C}^1$-$\mathcal{C}^1$-continuous. Moreover, for $\mathcal{C}^1$ maps $g:J\to J'$ and $f:J'\to\mathbb{R}$ the operation of composition $(f,g)\mapsto f\circ g$ is continuous as a map $\mathcal{C}^1\times\mathcal{C}^1\to\mathcal{C}^1$. ∎

Proof of Lemma 4. Fix u and n. Writing $C_l:=\bigcup_{k=0}^{n-1}T^{-k}B_l$ we have $\widehat{T}_*^k(1_{C_l^{\mathrm{c}}}\cdot u)=\widehat{T}^k(1_{C_l^{\mathrm{c}}}\cdot u)$ for $0\le k\le n-1$ provided $T_*=T$ on B_l^{c}. Consequently, $\|\widehat{T}^k u-\widehat{T}_*^k u\|_{L_1(m)}=\|\widehat{T}^k(1_{C_l}\cdot u)-\widehat{T}_*^k(1_{C_l}\cdot u)\|_{L_1(m)}\le 2\|1_{C_l}\cdot u\|_{L_1(m)}$, and this bound decreases to zero as $l\to\infty$: since $m\circ T^{-k}\ll m$ for each k, we have $\lim_{l\to\infty}m(T^{-k}B_l)=0$ so that $C_l\searrow\emptyset \bmod m$. ∎

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Institut für Mathematik
Universität Salzburg
Hellbrunnerstraße 34
A-5020 Salzburg, Austria
E-mail: roland.zweimueller@sbg.ac.at

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