COLLOQUIUM MATHEMATICUM

VOL. 93 2002 NO. 2

STANDARDLY STRATIFIED SPLIT AND LOWER TRIANGULAR ALGEBRAS

BY

EDUARDO DO N. MARCOS (São Paulo),
HÉCTOR A. MERKLEN (São Paulo) and CORINA SÁENZ (México, D.F.)

To Idun Reiten for her 60th birthday

Abstract. In the first part, we study algebras A such that $A = R \amalg I$, where R is a subalgebra and I a two-sided nilpotent ideal. Under certain conditions on I, we show that A is standardly stratified if and only if R is standardly stratified. Next, for $A = \left[\begin{smallmatrix} U & 0 \\ M & V \end{smallmatrix}\right]$, we show that A is standardly stratified if and only if the algebra $R = U \times V$ is standardly stratified and ${}_V M$ is a good V-module.

1. Introduction. In this work k will denote an algebraically closed field and "algebra" always means a finite-dimensional basic k-algebra.

We first consider the case in which A is a *split algebra*, that is, an algebra with a subalgebra R and a two-sided ideal I such that there is an R-bimodule decomposition $A = R \amalg I$. The algebra structure in $A = R \amalg I$ is given by

$$(r,i)(r',i') = (rr', ri' + ir' + ii').$$

We prove that, under certain hypotheses, R is standardly stratified if and only if A is standardly stratified. We also study relations between the categories of good modules in both algebras.

In the third section we study the case of a lower triangular matrix algebra of the type

$$A = \begin{bmatrix} U & 0 \\ M & V \end{bmatrix}$$

where U and V are algebras and M is a V-U-bimodule. Algebras of this type can be viewed naturally as split algebras if we let $R = U \times V$ and take M as an R-bimodule in which U acts as zero on the left and V acts as zero on the right. However, the condition **2** assumed in Section 2 is almost never satisfied if we order the idempotents of A in such a way that the idempotents of U

2000 *Mathematics Subject Classification*: 16E10, 16G20, 18G20.

Key words and phrases: split algebra, standardly stratified algebra, good modules.

The first author wants to thank CNPq (Brazil) for financial support. The third author wants to thank Universidad de Guanajuato and Fapesp (Brazil) for financial support.

are smaller than those of V. So our analysis for lower triangular matrices follows a different approach.

Split algebras have been studied recently in various settings. For instance their Hochschild cohomologies have been investigated in [3]. Some relations between their almost split sequences have been found in [1]. Influenced by these works, and others, we study these algebras from the point of view of stratification.

2. Split-by-nilpotent algebras. We start this section by reviewing some definitions and fixing the notations.

Let A be an algebra. Unless otherwise stated, "module" means a finitely generated left module, and A-mod will denote the category of A-modules. Let $\overline{e} = \{e_1, \ldots, e_n\}$ be a complete set of primitive orthogonal idempotents with order given by the indices.

As usual $P_i = Ae_i$ denotes the projective cover of the simple S_i. For each i we define the *standard module* $\Delta_A(i)$ to be the maximal quotient of P_i with composition factors S_j with $j \leq i$. Let Δ be the set of all these standard modules $\Delta_A(i)$. An A-module M will be called a Δ-*good module*, or just a *good module*, if there is a finite chain of submodules

$$0 = M_0 \subset M_1 \subset \ldots \subset M_t = M$$

such that M_i/M_{i-1} is isomorphic to a module in Δ, for all i. The number t does not depend on the filtration. We will call it the Δ-*length* of M and denote it by $l(M)$. The full subcategory of A-mod whose objects are the good modules is denoted by $\mathcal{F}_A(\Delta)$. The algebra A is said to be *left standardly stratified* if A is a good module.

We come back now to our situation of a split algebra $A = R \amalg I$. We assume in addition that I is nilpotent, which is equivalent to the condition that I is contained in the radical of the algebra A, or that there is a complete set of orthogonal primitive idempotents of A which are in the subalgebra R.

Throughout this section, we assume the following conditions on the R-bimodule I:

- (Condition **1**) I_R is a right projective R-module.
- (Condition **2**) For each i the A-module $I \otimes_R S_i$ has composition factors only of the form S_j with $j \leq i$.

Observe that the conditions above hold for the R-bimodule I if and only if they hold for the R-bimodule A, since $A = R \amalg I$ as an R-bimodule.

Since $1_A \in R$, we can assume that the ordered set $\overline{e}$ is also a set of orthogonal primitive idempotents of R. Consider the following functors:

$$F : R \otimes_A - : A\text{-mod} \to R\text{-mod} \quad \text{and} \quad G : A \otimes_R - : R\text{-mod} \to A\text{-mod}.$$

We find that the functor $R \otimes_A A \otimes_R -$ is isomorphic to 1_R and also that the functor $F : A \otimes_R -$ preserves projectives and projective covers, and sends, for each i, the R-simples $\mathrm{Top}(Re_i)$ to the A-simples $\mathrm{Top}(Ae_i)$. If there is no danger of misunderstanding, we will call either one S_i. The main result of this section states that, under our conditions **1** and **2**, R is standardly stratified if and only if A is. The following characterization of the family Δ of standard modules, given by Dlab and Ringel, will be used.

THEOREM 1. *For an arbitrary k-algebra Λ, let $D = \{M_1, \dots, M_n\}$ be a family of Λ-modules. Then D is the complete ordered family of standard modules* (*up to isomorphism*) *if and only the following three conditions hold*:

- $\mathrm{Top}(M_i) \simeq S_i$ *for* $1 \le i \le n$.
- *For each i, all composition factors of M_i are of the form S_j with $j \le i$.*
- *For each i,* $\mathrm{Ext}^1_\Lambda(M_i, S_j) = 0$ *for all $j \le i$.*

LEMMA 2. *Under conditions* **1** *and* **2**, *the following statements are valid*:

(1) $\Delta_A(i) \simeq A \otimes_R \Delta_R(i)$.
(2) $\Delta_R(i) \simeq R \otimes_A \Delta_A(i)$.

Proof. (1) We show that the family $\{A \otimes_R \Delta_R(i)\}$ satisfies the conditions of Theorem 1.

• Since the functor $A \otimes_R -$ preserves projective covers, the top of the A-module $A \otimes_R \Delta_R(i)$ is the simple A-module S_i.

• The exact sequence of A-modules $0 \to I \to A \to R \to 0$ gives, for each simple R-simple S_j with $j \le i$, the exact sequence

$$0 = \mathrm{Tor}^1_R(R, S_j) \to I \otimes_R S_j \to A \otimes_R S_j \to R \otimes_R S_j \to 0$$

and so $A \otimes_R S_j$ has composition factors only simples S_k with $k \le i$. From this fact and the exactness of the functor $A \otimes_R -$ we deduce that all composition factors of the A-module $A \otimes_R \Delta_R(i)$ are simple of the form S_j with $j \le i$.

• Finally, we claim that $\mathrm{Ext}^1_A(A \otimes_R \Delta_R(i), S_j) = 0$ for $j \le i$: since A is an R-projective module, $\mathrm{Ext}^1_A(A \otimes_R \Delta_R(i), S_j) \simeq \mathrm{Ext}^1_R(\Delta_R(i), \mathrm{Hom}_A(A, S_j)) \simeq \mathrm{Ext}^1_R(\Delta_R(i), S_j) = 0$ for $j \le i$ (see [8, Exercise 9.21]).

Therefore, from Theorem 1, we conclude that $A \otimes_R \Delta_R(i) \simeq \Delta_A(i)$.

(2) The composition $F \circ G$ is naturally equivalent to $\mathrm{Id}_{R\text{-mod}}$, therefore $\Delta_R(i) \simeq R \otimes_A A \otimes_R \Delta_R(i) \simeq R \otimes_A \Delta_A(i)$.

LEMMA 3. *For any good A-module M we have* $\mathrm{Tor}^1_A(R, M) = 0$.

Proof. Using the exact sequence of right A-modules $0 \to I \to A \to R \to 0$, we get, for each $\Delta_A(i)$, an exact sequence

$$0 \to \mathrm{Tor}^1_A(R, \Delta_A(i)) \to I \otimes_A \Delta_A(i) \to \Delta_A(i) \to \Delta_R(i) \to 0.$$

Since $\Delta_A(i) = A \otimes_R \Delta_R(i) \simeq R \otimes_R \Delta_R(i) \amalg I \otimes_R \Delta_R(i)$ (as R-modules) and since $I \otimes_A \Delta_A(i) \simeq I \otimes_A A \otimes_R \Delta_R(i) \simeq I \otimes_R \Delta_R(i)$, we see by counting dimensions that $\mathrm{Tor}^1_A(R, \Delta_A(i)) = 0$. Then it follows by induction that $\mathrm{Tor}^1_A(R, M) = 0$ for all $M \in \mathcal{F}_A(\Delta)$. ■

PROPOSITION 4. *With the general assumptions made for this section, we have*:

(1) $N \in \mathcal{F}_R(\Delta)$ *implies* $A \otimes_R N \in \mathcal{F}_A(\Delta)$.
(2) $N \in \mathcal{F}_A(\Delta)$ *implies* $R \otimes_A N \in \mathcal{F}_R(\Delta)$.

Proof. (1) We use induction on $l(N)$. By the previous lemma, the result is true if N is one of the standard modules $\Delta_R(i)$. So, assume that N contains properly one of the $\Delta_R(i)$ for some $i = 1, \dots, n$. Then there is a short exact sequence

$$0 \to \Delta_R(i) \to N \to N/\Delta_R(i) \to 0$$

in $\mathcal{F}_R(\Delta)$. Applying the functor $G = A \otimes_R -$ we get

$$0 \to \Delta_A(i) \to A \otimes_R N \to A \otimes_R (N/\Delta_R(i)) \to 0.$$

By induction and the fact that $\mathcal{F}_A(\Delta)$ is closed under extensions, it follows that $A \otimes_R N$ is in $\mathcal{F}_A(\Delta)$.

(2) Again, we use induction on $l(N)$. Here, we get a long exact sequence

$$\ldots \to \mathrm{Tor}^1_A(R, N/\Delta_A(i)) \to R \otimes_A \Delta_A(i) \to R \otimes_A N \to R \otimes_A N/\Delta_A(i) \to 0.$$

Since by the previous lemma $\mathrm{Tor}^1_A(R, M) = 0$ for all $M \in \mathcal{F}_A(\Delta)$, the result follows. ■

COROLLARY 5. *A is a standardly stratified algebra if and only if R is.* ■

COROLLARY 6. *If the category $\mathcal{F}_A(\Delta)$ is of finite representation type, then so is $\mathcal{F}_R(\Delta)$.*

Proof. Let $\{M_1, \dots, M_m\}$ be a complete set of isomorphism classes of indecomposable A-modules in $\mathcal{F}_A(\Delta)$ and let $M = M_1 \amalg \ldots \amalg M_m$. We claim that $\mathcal{F}_R(\Delta) = \mathrm{add}(R \otimes_A M)$. Let $L \in \mathcal{F}_R(\Delta)$. Then $A \otimes_R L \simeq M_1^{t_1} \amalg M_2^{t_2} \amalg \ldots \amalg M_m^{t_m}$ so

$$M \simeq (R \otimes_A M_1)^{t_1} \amalg \ldots \amalg (R \otimes_A M_m)^{t_m}, \quad \text{which is in } \mathrm{add}(R \otimes_A M). \ ■$$

In [6] Reiten and Platzeck observed that the subcategory of good modules is always contained in the subcategory of modules of finite projective dimension. They also gave conditions for these subcategories to be equal. We have the following proposition which also relates to these concepts. For the statement we use the notation of [6].

COROLLARY 7. *If $\mathcal{F}_A(\Delta) = P^{<\infty}(A)$, then $\mathcal{F}_R(\Delta) = P^{<\infty}(R)$.*

Proof. If $M \in P^{<\infty}(R)$, then it has a finite projective resolution of R-modules, $0 \to P^n \to \ldots \to P^1 \to P^0 \to M \to 0$. Applying the functor

$G = A \otimes_R -$ we get the following projective resolution of the A-module $A \otimes_R M$:

$$0 \to A \otimes_R P^n \to \ldots \to A \otimes_R P^1 \to A \otimes_R P^0 \to A \otimes_R M \to 0.$$

Hence, $A \otimes_R M$, having finite A-projective dimension, is in $\mathcal{F}_A(\Delta)$. Therefore, by Proposition 4, $M \simeq R \otimes_A (A \otimes_R M) \in \mathcal{F}_R(\Delta)$. ∎

REMARKS 8. (1) Since the functor $(R \otimes_A (A \otimes_R -))$ is isomorphic to the functor $\mathrm{Id}_{R\text{-mod}}$, for M and $N \in R$-mod, we find that $M \simeq N$ if and only if $A \otimes_R M \simeq A \otimes_R N$.

(2) Since I is contained in the radical of A, $R \otimes_A M = 0$ if and only if $M = 0$. ∎

COROLLARY 9. *The R-module M is decomposable* (*resp. indecomposable*) *if and only if the A-module $A \otimes_R M$ is decomposable* (*resp. indecomposable*).

One particular case of a split algebra satisfying our assumptions is the algebra $A = R[x]/(x^2)$, which is isomorphic to $R \amalg R$ with multiplication given by $(r, s)(r', s') = (rr', rs' + sr')$. The isomorphism is given by $r + sx \mapsto (r, s)$.

It is clear that the quiver Q_R is a subquiver of Q_A. Moreover, Q_A is obtained from Q_R by adding one loop, denoted by l_i, at each vertex v_i. This is a complete description of Q_A.

Now if $J = \{r_1, \ldots, r_n\}$ are relations defining a presentation of R, then a set of relations for a presentation of A is obtained by adding to J all loops l_i^2 and all differences

$$\{l_i^2 \text{ for each } i, \alpha l_{o(\alpha)} - l_{t(\alpha)}\alpha \text{ for each arrow } \alpha\}.$$

We now give some examples.

EXAMPLE 10. This example shows that it can happen that $\mathcal{F}_R(\Delta)$ is of finite representation type but $\mathcal{F}_A(\Delta)$ is not.

Let $A = k(1 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 2)$ be the Kronecker algebra, and $R = k(1 \xrightarrow{\alpha} 2)$. In this case $I = \langle\beta\rangle$ is the two-sided ideal generated by β (which is just the one-dimensional vector space $k\beta$). As R-bimodule, $\langle\beta\rangle$ is the simple $S_2 \otimes_k S_1$ which, as a right R-module, is isomorphic to S_1 which is a projective right R-module. In this case our algebras are hereditary and $\mathcal{F}_A(\Delta) = A$-mod and $\mathcal{F}_R(\Delta) = R$-mod. The category R-mod is of finite representation type but A-mod is not.

EXAMPLE 11. We have shown that there is an embedding on the indecomposable objects from the category $\mathcal{F}_R(\Delta)$ into $\mathcal{F}_A(\Delta)$, given by $A \otimes_R -$. We consider the algebra $A = R[x]/(x^2)$ in which $\mathcal{F}_R(\Delta)$ has 3 indecomposables but $\mathcal{F}_A(\Delta)$ has 4 indecomposables (see [4]). In our example, the algebras will be IIP, so that their good modules are the modules of finite projective dimension.

Let $R \simeq k(1 \xrightarrow{\alpha} 2)$ and $A \simeq kQ_A/I$ where Q_A is obtained from the quiver of Q_R by adding one loop l_i at each vertex i and the relations $\{l_1^2, l_2^2, \alpha l_1 - l_2\alpha\}$ (as described before). We see that $\Delta_R(i) = S_i$ and the indecomposables of the category R-mod $= \mathcal{F}_R(\Delta)$ are S_1, S_2 and P_1. So we have $A \otimes_R S_1 \simeq \Delta_A(1)$, $A \otimes_R S_2 \simeq \Delta_A(2)$ and $A \otimes_R P_1 \simeq P_A(1)$ are indecomposable A-modules in $\mathcal{F}_A(\Delta)$, but in addition we have the following indecomposable A-module:

$$\begin{array}{ccc} k & & k \\ \| & \searrow & \| \\ k & & k \end{array}$$

which belongs to $\mathcal{F}_A(\Delta)$ and is not of the form $A \otimes_R M$ for any M in R-mod.

Example 12. We now describe an example where I is not zero, but nevertheless the functors F and G induce bijections between the indecomposable *good* modules.

We take as R the hereditary algebra $k(1 \xrightarrow{\alpha} 2)$. Then $A \simeq kQ_A/(\mathrm{rad})^2$ has radical square zero, where Q_A is obtained from Q_R by adding one loop l at vertex 1. It is easy to see that in this case both categories have 3 indecomposable objects.

3. Algebras in lower triangular form. In this section we study algebras, which are given as matrix algebras in lower triangular form, with respect to being standardly stratified. We choose the idempotents conveniently. Let us observe again that these are always split algebras. But, with our choice of idempotents, they almost never satisfy all the hypotheses of the former section. So the point of view here is another one and the results that we obtain are of a different nature.

We fix the following notations. U and V denote finite-dimensional k-algebras, M a V-U-bimodule and A the finite-dimensional k-algebra $A = \left[\begin{smallmatrix} U & 0 \\ M & V \end{smallmatrix}\right]$. Also, we take the ordered set $\overline{g} = \{e_1, \ldots, e_t, f_{t+1}, \ldots, f_{t+r}\}$ as the complete ordered set of orthogonal primitive idempotents of A, where $\overline{e} = \{e_1, \ldots, e_t\} \subset U$ and $\overline{f} = \{f_{t+1}, \ldots, f_{t+r}\} \subset V$ are the fixed complete ordered sets of orthogonal idempotents of U, V, respectively. (Here, of course, we identify e_i with $\left[\begin{smallmatrix} e_i & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and f_j with $\left[\begin{smallmatrix} 0 & 0 \\ 0 & f_i \end{smallmatrix}\right]$.)

Let us begin by quoting two well known results that will be useful.

Given Λ, a standardly stratified algebra with respect to $\overline{h} = (h_1, \ldots, h_n)$, let j be such that $1 \leq j \leq n$ and denote by ε_j the sum $\varepsilon = h_j + \ldots + h_n$. We now state two well known results of Dlab and Ringel.

THEOREM 13. (1) *The algebra $A/A\varepsilon_j A$ is standardly stratified and the good $A/A\varepsilon_j A$-modules are the good A-modules annihilated by $A\varepsilon_j A$.*

(2) *The algebra $\varepsilon_j A\varepsilon_j$ is standardly stratified with respect to $\{h_j, \ldots, h_n\}$.*

REMARK 14. It follows from Theorem 13 that the set of standard A-modules is the union of the set of standard U-modules and the set of standard V-modules.

We recall the well known fact that there is an equivalence between the category of A-modules and the category $\mathcal{C}$ whose objects are triples (X, Y, f) where X is in U-mod, Y in V-mod and $f : M \otimes_U X \to Y$ is a V-module homomorphism. In what follows, by abuse of language, we identify the X of U (resp. of V) with the corresponding triple $(X, 0, 0)$ (resp. $(0, X, 0)$).

The sequence $(A, B, f) \xrightarrow{(\alpha,\beta)} (A', B', f') \xrightarrow{(\alpha'\beta')} (A'', B'', f'')$ is *exact* if and only if the sequences $A \xrightarrow{\alpha} A' \xrightarrow{\alpha'} A''$ and $B \xrightarrow{\beta} B' \xrightarrow{\beta'} B''$ are exact. Moreover the indecomposable A-projective modules are of the form $(P, M \otimes_V P, \mathrm{Id})$ where ${}_U P$ is projective, or of the form $(0, Q, 0)$, where ${}_V Q$ is projective. The indecomposable injective objects in $\mathcal{C}$ are objects of the form $(I, 0, 0)$ where I is an indecomposable injective U-module, and objects isomorphic to objects of the form $(\mathrm{Hom}_V(M, J), J, \phi)$ where J is an indecomposable injective V-module and $\phi : M \otimes_U \mathrm{Hom}_V(M, J) \to J$ is given by $\phi(m \otimes f) = f(m)$ for $m \in M$ and $f \in \mathrm{Hom}_V(M, J)$.

LEMMA 15. *The A-module (X, Y, f) is in $\mathcal{F}_A(\Delta)$ if and only if $X \in \mathcal{F}_U(\Delta)$ and $Y \in \mathcal{F}_V(\Delta)$.*

Proof. Let $L = (X, Y, f)$. Then we have the following filtration: $L = A\varepsilon_1 L \supseteq A\varepsilon_2 L \supseteq \ldots \supseteq A\varepsilon_{t-1} L \supseteq (0, Y, 0) \supseteq A\varepsilon_{t+2} L \supseteq \ldots \supseteq A\varepsilon_{t+r+1} L = 0$. Assuming that L is A-good we deduce that $L/(0, Y, 0) \simeq (X, 0, 0)$ is A-good, and it is annihilated by $\left[\begin{smallmatrix} 0 & 0 \\ M & V \end{smallmatrix}\right]$. It follows that $X \in \mathcal{F}_U(\Delta)$ and $Y \in \mathcal{F}_V(\Delta)$. The converse is analogous. ∎

PROPOSITION 16. *The algebra A is standardly stratified with respect to $\overline{g}$ if and only if the following conditions are satisfied.*

(a) *U is standardly stratified with respect to $\overline{e}$.*
(b) *V is standardly stratified with respect to $\overline{f}$.*
(c) *${}_V M \in \mathcal{F}_V(\Delta)$.*

Proof. First, assume that A is standardly stratified. Since ${}_A A \in \mathcal{F}_A(\Delta)$ and $A = (U, M, 1) \amalg (0, V, 0)$, Lemma 15 shows that ${}_U U \in \mathcal{F}_U(\Delta)$ and $M \amalg V \in \mathcal{F}_V(\Delta)$. The converse follows analogously using the other implication of Lemma 15. ∎

We recall that a left standardly stratified algebra A is called *quasi-hereditary* if $\mathrm{End}_A(\Delta_A(i))$ is a division ring for each i belonging to the index set of the simple A-modules.

COROLLARY 17. *The algebra A is quasi-hereditary if and only if U and V are quasi-hereditary and $M \in \mathcal{F}_V(\Delta)$.*

Proof. This follows easily from Remark 14 and the previous proposition. ∎

We now want to investigate conditions, for lower triangular matrix algebras, which imply that the category of good modules is the category of modules of finite projective dimension. We write $A = \left[\begin{smallmatrix} U & 0 \\ M & V \end{smallmatrix}\right]$ and keep the notations above. Then we know that there is an exact, full and faithful functor V-mod $\to$ A-mod, given by $Y \mapsto (0, Y, 0)$, which takes projectives to projectives, and also $\mathcal{F}_V(\Delta)$ into $\mathcal{F}_A(\Delta)$.

THEOREM 18 [5]. *Let $A = \left[\begin{smallmatrix} U & 0 \\ M & V \end{smallmatrix}\right]$ be such that ${}_VM$ has finite projective dimension. If $L = (X, Y, f)$ and $\operatorname{pd} L < \infty$, then ${}_VY$ and ${}_UX$ both have finite projective dimension.*

Proof. It is always true that if L has finite projective dimension then so does ${}_UX$. (The resolution of L induces a resolution of X.)

We now show that ${}_VY$ also has finite projective dimension.

In fact, if L is A-projective then ${}_VY$ is in $\operatorname{add}(M \amalg V)$ so it has finite projective dimension.

Let L be any A-module with finite projective resolution of the form

$$0 \to P_n \to P_{n-1} \to \ldots \to P_0 \to L \to 0.$$

This induces an exact sequence

$$0 \to Y_n \to Y_{n-1} \to \ldots \to Y_0 \to Y \to 0$$

where all ${}_VY_i$ are in $\operatorname{add}(M \amalg V)$ and so have finite projective dimension. It follows that Y has finite projective dimension. ∎

THEOREM 19. *$\mathcal{F}_A(\Delta) = P^{<\infty}(A)$ if and only if $\mathcal{F}_V(\Delta) = P^{<\infty}(V)$, $\mathcal{F}_U(\Delta) = P^{<\infty}(U)$ and $M \in P^{<\infty}(V)$.*

Proof. Assume that $\mathcal{F}_A(\Delta) = P^{<\infty}(A)$. Then ${}_AA$ is standardly stratified and it follows that $\mathcal{F}_V(\Delta) = P^{<\infty}(V)$ and, by Proposition 16, ${}_VM$ has finite projective dimension. To see that $\mathcal{F}_U(\Delta) = P^{<\infty}(U)$, take a U-module X in $P^{<\infty}(U)$. We show by induction on the projective dimension of X that X is A-good and therefore U-good. The hypothesis implies that U is standardly stratified and therefore the projective U-modules are good. Assume now that X has projective dimension n and that all U-modules of projective dimension $n-1$ are U-good. We have an exact sequence

$$0 \to (\Omega_U(X), M \otimes_U P(X), f) \to (P(X), M \otimes_U P(X), \mathrm{Id}) \to (X, 0, 0) \to 0,$$

where $P(X)$ denotes the projective cover of X, and $\Omega_U(X)$ the first syzygy of X, which has projective dimension $n-1$. We also have the following exact sequence:

$$0 \to (0, M \otimes_U P(X), 0) \to (\Omega, M \otimes_U P(X), f) \to (\Omega_U X, 0, 0) \to 0.$$

By induction $(\Omega_U X, 0, 0)$ has finite projective dimension and since $(0, M \otimes_U P(X), 0)$ is projective, $(\Omega, M \otimes_U P(X), f)$ also has finite projective dimension.

Now using the first exact sequence we conclude that $(X, 0, 0)$ has finite projective dimension and therefore it is A-good.

Assume now that $\mathcal{F}_V(\Delta) = P^{<\infty}(V)$, $\mathcal{F}_U(\Delta) = P^{<\infty}(U)$ and $M \in \mathcal{F}_V(\Delta) = P^{<\infty}(V)$. Then, by Proposition 16, A is standardly stratified. Take any A-module (X, Y, f) of finite projective dimension. Using Theorem 18 and the fact that $(0, Y, 0)$ is good, we infer that X has finite projective dimension and therefore it is U-good.

It follows that (X, Y, f) is A-good. ∎

REFERENCES

[1] I. Assem and D. Zacharia, *Full embeddings of almost split sequences over split-by-nilpotent extensions*, Colloq. Math. 81 (1999), 21–31.

[2] M. Auslander, I. Reiten and S. Smalø, *Representation Theory of Artin Algebras*, Cambridge Stud. Adv. Math. 36, Cambridge Univ. Press, 1995.

[3] C. Cibils, E. N. Marcos, M. J. Redondo and A. Solotar, *On the Hochschild cohomology of split algebras and trivial extensions*, preprint.

[4] F. U. Coelho, E. N. Marcos, H. Merklen and M. I. Platzeck, *Modules of infinite projective dimension over algebras whose idempotent ideals are projective*, Tsukuba J. Math. 21 (1997), 345–359.

[5] K. Fuller and M. Saorin, *On the finitistic dimension conjecture for artinian rings*, Manuscripta Math. 74 (1992), 117–132.

[6] M. I. Platzeck and I. Reiten, *Modules of finite projective dimension for standardly stratified algebras*, Comm. Algebra 29 (2001), 973–986.

[7] C. M. Ringel, *The category of good modules over a quasi-hereditary algebra has almost split sequences*, Math. Z. 208 (1991), 209–223.

[8] J. Rotman, *An Introduction to Homological Algebra*, Academic Press, 1979.

Departamento de Matemática
Universidade de São Paulo
Caixa Postal 66.281
São Paulo–SP, 05315-970, Brasil
E-mail: enmarcos@ime.usp.br
merklen@ime.usp.br

Departamento de Matemáticas
Facultad de Ciencias
Universidad Nacional Autónoma de México
Circuito Exterior, Ciudad Universitaria
C.P. 04510, México, D.F., Mexico
E-mail: ecsv@lya.fciencias.unam.mx

Received 20 November 2001;
revised 13 March 2002 (4136)