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AN ISOMORPHISM PROBLEM FOR ALGEBRAS DEFINED BY SOME QUIVERS AND NONADMISSIBLE IDEALS

BY

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Abstract. Given a quiver Q, a field K and two (not necessarily admissible) ideals I, I' in the path algebra KQ, we study the problem when the factor algebras KQ/I and KQ/I' of KQ are isomorphic. Sufficient conditions are given in case Q is a tree extension of a cycle.

1. Introduction. Let K be an arbitrary field (not necessarily algebraically closed). Assume that $Q = (Q_0, Q_1)$ is a finite quiver and I, I' are two-sided ideals in the path algebra KQ of Q. The aim of this paper is to give a criterion for isomorphism of the factor algebras KQ/I and KQ/I'. We do not assume that the ideals I, I' are admissible and we allow Q to have an oriented cycle, so the structure of the factor algebras can be quite complicated and the general problem is very difficult. Hence we restrict our study to a certain class of quivers Q containing exactly one oriented cycle.

The main results of the paper are Theorem 4.4 and Corollary 4.5 containing sufficient conditions for isomorphism of KQ/I and KQ/I' when Q is a tree extension of a cycle (see Section 4 for the definition). An important part of the proof is a description of the Auslander–Reiten quiver of KQ/I (Theorem 3.6) and the canonical generating set of I (Proposition 3.2) when Q is a single oriented cycle.

A motivation for this work comes from the question, studied in [6], whether the representation-finite algebras over algebraically closed fields form an open $\mathbb{Z}$-scheme. An affirmative answer is given in [7] for the class of triangular algebras by applying van den Dries's test [4]. The key step of the proof of the main result of [7] is to show that given a V-order A over a valuation subring V of K, the K-algebra KA is representation-finite and triangular provided the R-algebra $\overline{A}$, obtained from A by passing to the residue field R of V, is representation-finite and triangular. In a subsequent paper an analogous implication will be proved for V-orders A such that the Gabriel quiver of $\overline{A}$ is a tree extension of a cycle. The criterion given in

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Corollary 4.5 below is one of the main technical tools needed to obtain that result.

Throughout we use the following terminology and notation.

Let $Q = (Q_0, Q_1)$ be a finite quiver with the set of vertices (resp. arrows) Q_0 (resp. Q_1). Given an arrow $\alpha \in Q_1$, $s(\alpha)$ and $t(\alpha)$ is the source and the terminus of α, respectively. By a *path* in Q we mean a sequence $u = \alpha_1 \dots \alpha_m$ of arrows of Q such that $t(\alpha_i) = s(\alpha_{i+1})$ for $i = 1, \dots, m-1$. Then m is the length of u, $s(u) := s(\alpha_1)$ is its source and $t(u) := t(\alpha_m)$ its terminus. Given a vertex x of Q we denote by e_x the stationary path of length 0 associated to x, with $s(e_x) = t(e_x) = x$.

Given a quiver Q, the path algebra of Q is denoted by KQ. By definition, the set of paths in Q is a K-basis of KQ and multiplication is determined by concatenation of paths (see e.g. [1, Chap. II, Def. 1.2]). Denote by KQ_n the two-sided ideal of KQ generated by all paths of length n. A two-sided ideal I of KQ is called *admissible* if $KQ_n \subseteq I \subseteq KQ_2$ for some n.

If u is an arrow or a path in Q then the I-coset of u in KQ/I is denoted also by u.

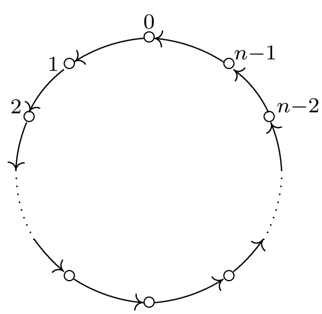
For a ring S, the S-algebra of polynomials in one indeterminate t with coefficients in S is denoted by $S[t]$, and $\mathbb{M}_n(S)$ is the algebra of all $n \times n$-matrices with coefficients in S.

Given a field K and two polynomials $F, G \in K[t]$, we denote by $\gcd(F, G)$ the monic greatest common divisor of F and G.

As usual, we identify the right KQ-modules X with the corresponding representations $(X_i, X_\alpha)_{i \in Q_0, \alpha \in Q_1}$ of Q [1, Chap. III]. Given a K-algebra Λ, we denote by $\operatorname{mod} \Lambda$ the category of right Λ-modules of finite K-dimension.

2. Representations of an oriented cycle. Let $n \geq 1$ and $Q(n)$ be the cyclic quiver with n-vertices

(2.0) $Q(n)$:



that is, $Q(n)_0 = \mathbb{Z}/n\mathbb{Z}$ identified with $\{0, \dots, n-1\}$, $Q(n)_1 = \{\alpha_0, \dots, \alpha_{n-1}\}$ and $s(\alpha_i) = i = t(\alpha_{i-1})$ for $i = 0, \dots, n-1$, where the indices are taken modulo n.

Throughout this section, we fix $n \geq 1$ and we set $Q = Q(n)$.

Given two paths u, v in Q, we say that u is a *subpath* of v if there exist paths w_1, w_2 in Q such that $v = w_1 u w_2$. In that case we write $u \preccurlyeq v$.

We also use the following notation.

For $j \geq 0$ and $i \in Q_0$, $u_{i,j}$ is the (unique) path of length j starting at the vertex i. For simplicity we denote $u_{i,n}$ by u_i.

Let k, l be vertices of Q. We denote by $w_{k,l}$ the shortest path in Q such that $s(w_{k,l}) = k$ and $t(w_{k,l}) = l$.

Given a representation $V = (V_i, V_{\alpha_i})_{i \in Q_0}$ of the cycle Q and a vertex j, we denote by $V_{u_j} : V_j \to V_j$ the composition $V_{\alpha_{j-1}} \circ \cdots \circ V_{\alpha_0} \circ V_{\alpha_{n-1}} \circ \cdots \circ V_{\alpha_j}$.

For a polynomial $F = t^d + a_{d-1}t^{d-1} + \cdots + a_1 t + a_0 \in K[t]$, we denote by M_F the $d \times d$-matrix

$$M_F = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & 0 & \dots & 0 & -a_2 \\ & & & \ddots & & \\ 0 & 0 & 0 & \dots & 0 & -a_{d-2} \\ 0 & 0 & 0 & \dots & 1 & -a_{d-1} \end{bmatrix}.$$

Note that F is the minimal polynomial of M_F.

We define the representation $V(F)$ of the quiver Q as follows: the space K^d is associated to every vertex and the identity map is associated to every arrow of Q except for α_{n-1}; to the arrow α_{n-1} we associate the map defined by M_F, with respect to the standard bases.

We denote by $S(i)$ the simple representation corresponding to the vertex i of Q.

Let $X_{i,j}$ be the indecomposable nilpotent representation of Q with top $S(i)$ and of length j [13], that is, $X_{i,j}$ corresponds to the module $e_i KQ/u_{i,j}KQ$. Note that $S(i) \cong X_{i,1}$.

An equivalent description of $X_{i,j}$ can be given in terms of the push-down functor $F_\lambda : \operatorname{mod} K\widetilde{Q} \to \operatorname{mod} KQ$ associated with the universal Galois covering of $F : \widetilde{Q} \to Q$ (see [5], [8]) defined by the infinite linear quiver $\widetilde{Q}$ of type $\mathbb{A}_\infty$, where we identify the vertices of $\widetilde{Q}$ with the integers. The covering map $F : \widetilde{Q} \to Q$ is determined by $F(i) = i + n\mathbb{Z}$.

One can see that $X_{i,j} \cong F_\lambda(Y_{i,j})$, where $Y_{i,j}$ corresponds to the (unique) indecomposable representation of $\widetilde{Q}$ with support $\{i, \dots, i+j-1\}$.

The reader is referred to [1] and [2] for the terminology of Auslander–Reiten theory and to [9] and [12] for basic facts on standard tubes.

For the convenience of the reader we present, with an outline of proof, the following assertion, essentially contained in [13].

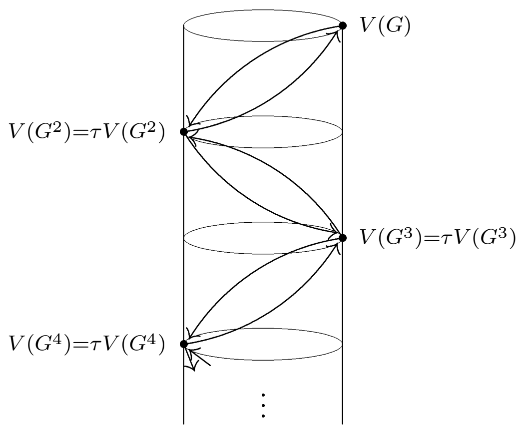
Theorem 2.1. *Let K be a field and $Q = Q(n)$ be the cycle* (2.0).

(a) *Every finite-dimensional indecomposable KQ-module corresponds to one of the following representations*:

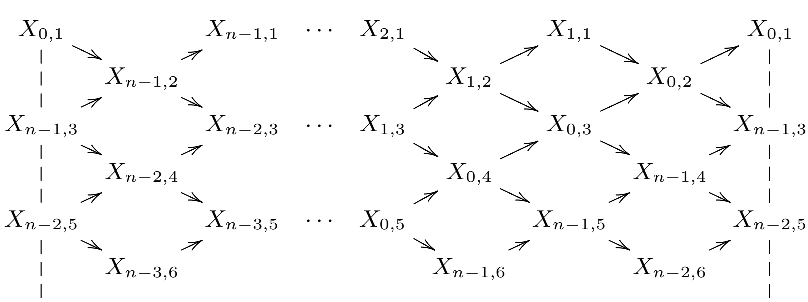
  (1) $X_{i,j}$ *with* $i \in Q_0$ *and* $j \in \mathbb{N}$,
  (2) $V(F)$ *for some F which is a power of an irreducible polynomial in* $K[t]$ *and* $F(0) \neq 0$.

(b) *The Auslander–Reiten quiver* $\Gamma_{KQ} = \Gamma(\operatorname{mod} KQ)$ *of the category* $\operatorname{mod} KQ$ *consists of the family* $\mathcal{T} = \{\mathcal{T}_G\}_G$ *of homogeneous stable tubes indexed by the monic irreducible polynomials* $G \in K[t]$ *such that* $G(0) \neq 0$, *and a stable tube* $\mathcal{T}_t$ *of rank* n. *All the tubes are standard components and they are pairwise orthogonal. Here* $\mathcal{T}_G$ *has the form*

$\mathcal{T}_G$:



and the tube $\mathcal{T}_t$ *has the form*



where we identify the modules along the vertical dashed lines.

Outline of proof. Let $\operatorname{mod}_* KQ$ be the full subcategory of $\operatorname{mod} KQ$ consisting of all KQ-modules (identified with representations $V = (V_i, V_{\alpha_i})_{i \in Q_0})$

such that, for every vertex i, the composite map $V_{u_i} = V_{\alpha_i} \cdots V_{\alpha_{i-1}}$ is invertible. It is easy to check that the map

$$(V_i, V_{\alpha_i})_{i \in Q_0} \mapsto (V_0, V_{u_0})$$

associating to a representation of Q a representation of the one-loop quiver $Q(1)$ determines an equivalence $\operatorname{mod}_* KQ \cong \operatorname{mod} K[t, t^{-1}]$. The structure of the latter category is well known (see e.g. [10, 14.3]).

Now let $\operatorname{mod}_0 KQ$ be the category of all (modules corresponding to) nilpotent representations, that is, representations $V = (V_i, V_{\alpha_i})_{i \in Q_0}$ such that V_{u_0} is nilpotent. This category and its Auslander–Reiten quiver are described in [13]. One can also use Galois covering arguments to prove that the Auslander–Reiten quiver of $\operatorname{mod}_0 KQ$ is just the tube $\mathcal{T}_t$.

It remains to show that every indecomposable representation of Q is either nilpotent or an object of $\operatorname{mod}_* KQ$, and there are no nonzero maps between these two subcategories of $\operatorname{mod} KQ$. We repeat the well-known arguments from the proof of the Jordan theorem. Namely, let $V = (V_i, V_{\alpha_i})_{i \in Q_0}$ be a representation of Q. For $G \in K[t]$ let V^G be the subrepresentation of V such that V_i^G consists of the elements of V_i annihilated by a power of $G(V_{u_i})$, for $i = 0, \ldots, n-1$. (One needs to check that it is really a subrepresentation.) Repeating the well-known arguments, we prove that

$$V \cong \bigoplus_{j=1}^{m} V^{G_j}$$

for some irreducible $G_1, \ldots, G_m$, and there are no nonzero maps between V^{G_i} and V^{G_j} for G_i and G_j relatively prime. ∎

Following the terminology of Galois covering theory [3], the modules (1) and (2) in the theorem are called the *modules of first kind* and *of second kind*, respectively.

We have several direct consequences of Theorem 2.1.

COROLLARY 2.2. *If $V = (V_i, V_{\alpha_i})_{i \in Q_0}$ is an indecomposable representation of Q, then*

$$|\dim_K V_i - \dim_K V_j| \leq 1$$

for any $i, j \in Q_0$. Moreover, $\dim_K V_i = \dim_K V_j$ for all i, j if $V \notin \mathcal{T}_t$. ∎

Now we introduce two partial orders $\preceq_{\mathrm{c}}$ and $\preceq_{\mathrm{r}}$ on the set of vertices of Γ_{KQ}. Define $X \preceq_{\mathrm{c}} Y$ (resp. $X \preceq_{\mathrm{r}} Y$) if X and Y belong to the same tube, lie on the same coray (resp. ray) of this tube and $\dim_K Y \geq \dim_K X$. Clearly, the two orders coincide on the homogeneous tubes. Further, let $\preceq$ be the partial order generated by the union of $\preceq_{\mathrm{c}}$ and $\preceq_{\mathrm{r}}$ on the set of vertices of Γ_{KQ}.

We have another consequence of the description of KQ-modules.

Corollary 2.3. *Every indecomposable KQ-module X is uniserial, that is, the lattice of submodules of X is linear. Moreover, $U \preceq_{\mathrm{r}} X$ for every submodule U of X, and $F \preceq_{\mathrm{c}} X$ for every factor module F of X. If Y is an indecomposable submodule (resp. factor module) of a module X then X has a direct summand U such that $Y \preceq_{\mathrm{r}} U$ (resp. $Y \preceq_{\mathrm{c}} U$).*

Proof. It follows from Theorem 2.1 that the lattice of submodules of $V(G^r)$ is

$$0 \subset V(G) \subset V(G^2) \subset \cdots \subset V(G^r)$$

and the lattice of submodules of $X_{i,j}$ is

$$0 \subset X_{i+j-1,1} \subset \cdots \subset X_{i+1,j-1} \subset X_{i,j}.$$

It is now clear that $U \preceq_{\mathrm{r}} X$ for every submodule U of an indecomposable X. Analogously, we show that $F \preceq_{\mathrm{c}} X$ for every factor module F of X.

For the proof of the remaining statement, assume that $Y \subset X$ and Y is indecomposable. Let $X = X_1 \oplus \cdots \oplus X_m \oplus X'$, where $X_1, \ldots, X_m$ are all indecomposable direct summands of X belonging to the same tube as Y. Then there exists a monomorphism $\mu = [\mu_j] : Y \to X_1 \oplus \cdots \oplus X_m$. Suppose that $Y \not\preceq_{\mathrm{r}} X_j$ for all j. Then, for any j, the kernel of $\mu_j : Y \to X_j$ is nonzero, hence contains the unique simple submodule $\operatorname{soc} Y$ of Y. Therefore $\operatorname{soc} Y \subset \operatorname{Ker} \mu$ and we get a contradiction. Analogously, for an indecomposable factor module Y of X, we prove that X has a direct summand U such that $Y \preceq_{\mathrm{c}} U$. ∎

The proof of the following corollary is routine, and we leave it to the reader.

Corollary 2.4. *Let $V = (V_i, V_{\alpha_i})_{i \in Q_0}$ be an indecomposable representation of Q. Then V is cyclic, that is, it is generated by one element as a KQ-module, and:*

(a) *if V belongs to $\mathcal{T}_t$ and* $\operatorname{top} V = S(i)$, *then the minimal polynomial of the map V_{u_i} is $t^{\dim_K V_i}$,*
(b) *if V belongs to $\mathcal{T}_G$ with $G \neq t$, then the minimal polynomial of V_{u_i} is G^r, where $r \deg G = \dim_K V_i$, for any $i \in Q_0$.* ∎

Corollary 2.5. *Assume that V, W are two indecomposable KQ-modules and $f, g : V \to W$ are two epimorphisms (resp. monomorphisms). Then there exist automorphisms ϕ and ψ of V and W, respectively, such that $g = f\phi$ and $\psi g = f$.*

Proof. We only consider the case when f and g are epimorphisms, because the proof in the other case is analogous. It follows from Theorem 2.1 that either $V \cong V(G^r)$ and $W \cong V(G^s)$ for some irreducible G and $r \geq s$, or $V \cong X_{i,r}$ and $W \cong X_{i,s}$ for some $i \in Q_0$ and $r \geq s$. The assertion follows by simple analysis of homomorphism spaces between KQ-modules, and we leave it to the reader. ■

THEOREM 2.6. *Let $Q = Q(n)$ be the cycle* (2.0) *and let $\Lambda_I = KQ/I$, where I is a nonzero two-sided ideal in KQ. Then*:

(a) $\dim_K \Lambda_I < \infty$.
(b) *Λ_I is representation finite.*
(c) *The Auslander–Reiten quiver Γ_{Λ_I} is a full subquiver of Γ_{KQ} such that*:
  (c1) *if Y is a vertex of Γ_{Λ_I} and $X \preceq Y$ in Γ_{KQ} then X is a vertex of Γ_{Λ_I}*,
  (c2) *every component of Γ_{Λ_I} is finite*,
  (c3) *Γ_{Λ_I} has finitely many components*,
  (c4) *an indecomposable Λ_I-module P is projective* (*resp. injective*) *if and only if P is $\preceq_{\mathrm{c}}$-maximal* (*resp. $\preceq_{\mathrm{r}}$-maximal*) *in Γ_{Λ_I}*,
  (c5) *the Auslander–Reiten translation in Γ_{Λ_I} is the restriction of that in Γ_{KQ} to the set of nonprojective vertices.*

Proof. For simplicity of notation we set $\Lambda = \Lambda_I = KQ/I$. Statement (a) is clear and (b) follows from (c).

(c) Assume that Y is an indecomposable Λ-module. Obviously submodules and factor modules of Y in $\operatorname{mod} KQ$ are Λ-modules. If $X \preceq Y$ then there exist a sequence

$$Y = X_0, X_1, \ldots, X_{m+1} = X$$

of vertices of Γ_{KQ} such that X_{i+1} is either a submodule or a factor module of X_i for $i = 0, \ldots, m-1$. Hence X is a vertex of Γ_Λ and (c1) follows.

Clearly, a homomorphism of Λ-modules which is irreducible in $\operatorname{mod} KQ$ is also irreducible in $\operatorname{mod} \Lambda$. It follows from (c1) and the shape of Γ_{KQ} that every homomorphism between Λ-modules is a composition of morphisms between Λ-modules which are irreducible in $\operatorname{mod} KQ$. Hence Γ_Λ is a full subquiver of Γ_{KQ}.

To prove (c2) assume that ϱ is a relation from I. Then $\varrho = F(u_i)u_{i,j}$ for some $i \in Q_0$, $j < n$ and $F \in K[t]$.

Take an indecomposable representation $V = (V_j, V_{\alpha_j})_{j \in Q_0}$ of Q. Choose $i \in Q_0$ such that $\operatorname{top} V \cong S(i)$ if $V \in \mathcal{T}_t$, and take i arbitrary otherwise. The minimal polynomial of the endomorphism V_{u_i} of V_i divides tF and hence $\dim_K V_i \le \deg F + 1$, by Corollary 2.4. Thanks to Corollary 2.2, we have $\dim_K V = \sum_{j=1}^{n} \dim_K V_j < n(\deg F + 2)$ and hence the components of Γ_Λ are finite.

(c3) If Γ_Λ had infinitely many components, then Λ would be a product of infinitely many algebras by Auslander's result (see e.g. [1, Chap. IV, 5.4]). This would contradict (a).

For the proof of (c4) note that if P is $\preceq_c$-maximal then, by Corollary 2.3, P is not an image of a nonsplit epimorphism from a Λ-module. This means that P is projective.

If P is not $\preceq_c$-maximal then there exists a Λ-module Y such that $P \prec_c Y$, and we conclude by (c1) that there is a nonsplit exact sequence of Λ-modules ending at P, thus P is not projective.

The proof of (c5) is easy. ∎

COROLLARY 2.7. *Assume that K is algebraically closed and let I, I' be ideals of KQ. The algebras $\Lambda = KQ/I$ and $\Lambda' = KQ/I'$ are Morita equivalent if and only if the translation quivers Γ_Λ and $\Gamma_{\Lambda'}$ are isomorphic.*

Proof. It is obvious that if Λ is Morita equivalent to Λ' then their Auslander–Reiten quivers are isomorphic. For the converse, note that if Γ_Λ and $\Gamma_{\Lambda'}$ are isomorphic then the configuration of the projective vertices in Γ_Λ is the same as in $\Gamma_{\Lambda'}$. Since the tubes in Γ_{KQ} are standard and orthogonal we then have isomorphism of the basic algebras associated to Λ and Λ' (see [1, I.6.3]). Thus Λ and Λ' are Morita equivalent. ∎

3. The canonical generating set of an ideal of $KQ(n)$. Throughout this section, we fix $n \geq 1$ and denote by $Q = Q(n)$ the cycle (2.0).

Let I be a two-sided ideal of $KQ(n)$ generated by a set $\mathcal{G}$. The aim of this section is to describe the Auslander–Reiten quiver of $\Lambda_I = KQ(n)/I$ in terms of $\mathcal{G}$. To do this, we first reduce $\mathcal{G}$ to a "canonical" set of generators. Let us explain the idea on an example.

EXAMPLE 3.1. Assume that $\operatorname{char} K \neq 2, 3$ and consider the ideal I of $KQ(3)$ generated by the following elements:

$$F_0(u_0)\alpha_0\alpha_1 = u_0^2\alpha_0\alpha_1 - 3u_0\alpha_0\alpha_1 + 2\alpha_0\alpha_1,$$
$$F_1(u_0)\alpha_0 = u_0^3\alpha_0 + u_1^2\alpha_0 - u_0\alpha_0 - \alpha_0,$$
$$F_2(u_1)\alpha_1\alpha_2 = u_1^3\alpha_1\alpha_2 - u_1^2\alpha_1\alpha_2 - u_1\alpha_1\alpha_2 + \alpha_1\alpha_2,$$

where

$$F_0(t) = (t-1)(t-2), \quad F_1(t) = (t-1)(t+1)^2, \quad F_2(t) = (t+1)(t-1)^2.$$

Observe that I contains the elements $F_0(u_0)u_0\alpha_0 = F_0(u_0)\alpha_0\alpha_1\alpha_2\alpha_0$, $F_1(u_0)\alpha_0\alpha_1$ and $F_2(u_1)\alpha_1\alpha_2$.

Now, using the fact that $G = t-1$ is the greatest common divisor of tF_0 and F_1, we see that the ideal I is generated by

$$G(u_0)\alpha_0, \qquad F_2(u_1)\alpha_1\alpha_2.$$

Note that

$$G_1(u_1)\alpha_1\alpha_2 = \alpha_1\alpha_2 G(u_0)\alpha_0\alpha_1\alpha_2 \in I,$$

where $G_1 = tG$. Since G is the greatest common divisor of F_2 and G_1, it follows that I is generated by

$$G(u_0)\alpha_0, \quad G(u_1)\alpha_1\alpha_2.$$

By generalizing this procedure we reduce any set of generators of any ideal in KQ to the form described in the proposition below.

PROPOSITION 3.2. *Let I be a two-sided ideal of $KQ(n)$. There exist:*

(i) *a monic polynomial $G \in K[t]$ such that $G(0) \neq 0$,*
(ii) *a nonnegative integer g,*
(iii) *vertices $i_1, \dots, i_r$ of $Q(n)$ and integers $0 \leq m_a < 2n$, $a = 1, \dots, r$, such that*
   (a) *the paths u_{i_a,m_a}, $a=1, \dots, r$, are pairwise incomparable with respect to the subpath order $\preceq$ (in particular, the numbers $i_1, \dots, i_r$, as well as $i_1+m_1, \dots, i_r+m_r$, are pairwise different modulo n),*
   (b) *the elements*

$$u_{i_1}^g G(u_{i_1})u_{i_1,m_1}, \ \dots, \ u_{i_r}^g G(u_{i_r})u_{i_r,m_r}$$

generate I as a two-sided ideal.

A set of generators of this form is called a *canonical set of generators* of I.

To present the proof we need some preparation. Clearly, I has a finite set of generators of the form

$$\mathcal{G} = \{G_a(u_{i_a})u_{i_a,j_a} : a = 1, \dots, p\}$$

for some $i_a \in \{0, \dots, n-1\}$, $G_a \in K[t]$, $j_a \geq 0$.

Without loss of generality, we can assume that $j_a < n$ for any a. Indeed, if $j_a = kn + j'_a$ for some $k \in \mathbb{N}$, then

$$G_a(u_{i_a})u_{i_a,j_a} = G'_a(u_a)u_{i_a,j'_a}, \quad \text{where} \quad G'_a = t^k G_a.$$

Let n_a be the multiplicity of t as a factor of G_a for $a = 1, \dots, p$. We keep the notation introduced above.

LEMMA 3.3. *Assume that $\mathcal{G}$ satisfies the following condition:*

(R1) *There are $a \neq b$ such that $u_{i_a,j_a} \preccurlyeq u_{i_b,j_b}$ and ($i_a = i_b$ or $i_a + j_a = i_b + j_b$) in $Q_0 = \mathbb{Z}/n\mathbb{Z}$.*

Let

$$\mathcal{G}_1 = \begin{cases} (\mathcal{G} \setminus \{G_a(u_{i_a})u_{i_a,j_a}, G_b(u_{i_b})u_{i_b,j_b}\}) \cup \{H(u_{i_a})u_{i_a,j_a}\} & \text{if } n_b \geq n_a, \\ (\mathcal{G} \setminus \{G_a(u_{i_a})u_{i_a,j_a}, G_b(u_{i_b})u_{i_b,j_b}\}) \cup \{H(u_{i_b})u_{i_b,j_b}\} & \text{if } n_b < n_a, \end{cases}$$

where $H = \gcd(G_a, G_b)$. Then $(\mathcal{G}) = (\mathcal{G}_1)$.

Proof. Observe that the elements $u_{i_a} G_b(u_{i_a}) u_{i_a,j_a}, G_a(u_{i_b}) u_{i_b,j_b}$ belong to

$$J_1 = (G_a(u_{i_a}) u_{i_a,j_a}, G_b(u_{i_b}) u_{i_b,j_b}).$$

For instance, if $i_a = i_b$ then

$$u_{i_a} G_b(u_{i_a}) u_{i_a,j_a} = G_b(u_{i_b}) u_{i_b,j_b} w_{i_b+j_b, i_a+j_a}.$$

The remaining case is similar and we leave it to the reader.

Let $H_1 = \gcd(tG_b, G_a)$. Then the ideal J_1 is generated by $H(u_{i_b}) u_{i_b,j_b}$ and $H_1(u_{i_a}) u_{i_a,j_a}$. To finish the proof it is enough to observe that if $n_b \geq n_a$ then $H = H_1$ and $H(u_{i_a}) u_{i_a,j_a}$ generates J_1, whereas if $n_b < n_a$ then $H(u_{i_b}) u_{i_b,j_b}$ does. ∎

Observe that after applying the operation $\mathcal{G} \mapsto \mathcal{G}_1$ finitely many times we can assume that our generating set $\mathcal{G}$ satisfies $i_a \neq i_b$ and $i_a + j_a \neq i_b + j_b$ modulo n, for any $a \neq b$, that is, (R1) is not satisfied.

LEMMA 3.4. *Assume that $\mathcal{G}$ satisfies the following condition*:

(R2) *there are $a \neq b$ such that $u_{i_a,j_a} \preccurlyeq u_{i_b,j_b}$ and $G_a \neq tG_b$.*

Let

$$\mathcal{G}_2 = (\mathcal{G} \setminus \{G_a(u_{i_a}) u_{i_a,j_a}, G_b(u_{i_b}) u_{i_b,j_b}\}) \cup S,$$

where

$$S = \begin{cases} \{H(u_{i_a}) u_{i_a,j_a}\} & \text{if } n_a \leq n_b, \\ \{H(u_{i_b}) u_{i_b,j_b}, u_{i_a} H(u_{i_a}) u_{i_a,j_a}\} & \text{if } n_a = n_b + 1, \\ \{H(u_{i_b}) u_{i_b,j_b}\} & \text{if } n_a > n_b + 1, \end{cases}$$
$$H = \gcd(G_a, G_b).$$

Then $(\mathcal{G}) = (\mathcal{G}_2)$.

Proof. Observe that the elements $u_{i_a}^2 G_b(u_{i_a}) u_{i_a,j_a}, G_a(u_{i_b}) u_{i_b,j_b}$ belong to

$$J_2 = (G_a(u_{i_a}) u_{i_a,j_a}, G_b(u_{i_b}) u_{i_b,j_b}).$$

Let $H_2 = \gcd(t^2 G_b, G_a)$. Then J_2 is generated by the elements

$$H(u_{i_b}) u_{i_b,j_b} \quad \text{and} \quad H_2(u_{i_a}) u_{i_a,j_a}.$$

To finish the proof it is enough to observe that if $n_a \leq n_b$ then $H(u_{i_a}) u_{i_a,j_a}$ generates J_2, and if $n_a > n_b + 1$ then $H(u_{i_b}) u_{i_b,j_b}$ does. If $n_a = n_b + 1$ then $H_2 = tH$. ∎

Observe that if $\mathcal{G}$ does not satisfy (R1) then neither does the set $\mathcal{G}_2$ obtained from $\mathcal{G}$ as in Lemma 3.4.

LEMMA 3.5. *Assume that $\mathcal{G}$ satisfies the following condition*:

(R3) *there are $a \neq b$ such that $u_{i_b,j_b} \not\preccurlyeq u_{i_a,j_a}$, $u_{i_a,j_a} \not\preccurlyeq u_{i_b,j_b}$ and $G_a \neq G_b$.*

Let

$$\mathcal{G}_3 = (\mathcal{G} \setminus \{G_a(u_{i_a})u_{i_a,j_a}, G_b(u_{i_b})u_{i_b,j_b}\}) \cup S,$$

where

$$S = \begin{cases} \{H(u_{i_a})u_{i_a,j_a}, H(u_{i_b})u_{i_b,j_b}\} & \text{if } n_b = n_a, \\ \{H(u_{i_a})u_{i_a,j_a}\} & \text{if } n_a < n_b, \\ \{H(u_{i_b})u_{i_b,j_b}\} & \text{if } n_a > n_b, \end{cases}$$
$$H = \gcd(G_a, G_b).$$

Then $(\mathcal{G}) = (\mathcal{G}_3)$.

Proof. As in the proofs of Lemmas 3.3 and 3.4 we observe that the elements $u_{i_b}G_a(u_{i_b})u_{i_b,j_b}, u_{i_a}G_b(u_{i_a})u_{i_a,j_a}$ belong to

$$J_3 = (G_a(u_{i_a})u_{i_a,j_a}, G_b(u_{i_b})u_{i_b,j_b})$$

and hence $J_3 = (H_3(u_{i_a})u_{i_a,j_a}, H_1(u_{i_b})u_{i_b,j_b})$, where we put $H_3 = \gcd(G_a, tG_b)$ and $H_1 = \gcd(tG_a, G_b)$. If $n_a < n_b$ then $H_3 = H$, $H_1 = tH$ and $H_1(u_{i_b})u_{i_b,j_b} \in (H(u_{i_a})u_{i_a,j_a})$. The case $n_a > n_b$ is analogous. If $n_a = n_b$ then $H_1 = H_3 = H$.

Proof of Proposition 3.2. We define the degree of a relation $G(u_i)u_{i,j}$ to be $n \deg G + j$. Observe that if $\mathcal{G}$ satisfies one of the conditions (RN), $N = 1, 2, 3$, then applying the corresponding operation $\mathcal{G} \mapsto \mathcal{G}_N$ decreases the sum of the degrees of the relations in the generating set. For instance, if (R2) is satisfied and $n_a = n_b + 1$ then $2 \deg H + 1 < \deg G_a + \deg G_b$ because $G_a \neq tG_b$. Therefore after finitely many reductions, we obtain a generating set $\mathcal{G} = \{G_a(u_{i_a})u_{i_a,j_a} : a = 1, \ldots, r\}$ that does not satisfy any of the conditions (R1)–(R3). Observe that since (R1) is not satisfied the numbers i_a for $a = 1, \ldots, r$ are pairwise different modulo n, as also are $i_a + j_a$, $a = 1, \ldots, r$. Since (R2) is not satisfied, $u_{i_a,j_a} \preccurlyeq u_{i_b,j_b}$ yields $G_a = tG_b$. Analogously, since (R3) is not satisfied, $u_{i_b,j_b} \not\preccurlyeq u_{i_a,j_a}$, and $u_{i_a,j_a} \not\preccurlyeq u_{i_b,j_b}$ yields $G_a = G_b$. Now let $t^g G$, where $G(0) \neq 0$, be the greatest common divisor of all G_a, $a = 1, \ldots, r$. Moreover, put $m_a = j_a + 1$ whenever u_{i_a,j_a} is a subpath of another u_{i_b,j_b}, and let $m_a = j_a$ otherwise. It is easy to check that the conditions of the proposition are satisfied. ∎

Remark. Note that if

$$\mathcal{G} = \{G_a(u_{i_a})u_{i_a,j_a} : a = 1, \ldots, p,\ j_a < n\}$$

generates I then the polynomial G in Proposition 3.2 equals $(1/t^g)\gcd\{G_a : a = 1, \ldots, p\}$ for some $g \in \mathbb{N}$. This is a direct consequence of the proof of Proposition 3.2.

Now we describe the Auslander–Reiten quiver of $KQ(n)/I$ assuming that we know a canonical set of generators of I:

$$u_{i_1}^g G(u_{i_1})u_{i_1,m_1}, \ldots, u_{i_r}^g G(u_{i_r})u_{i_r,m_r}.$$

Throughout the rest of this section, we assume that I is as above and we set $\Lambda = KQ(n)/I$.

Assume moreover that the polynomial G has the following decomposition in $K[t]$:

$$G = F_1^{r_1} \dots F_s^{r_s},$$

where $F_1, \dots, F_s$ are pairwise relatively prime irreducible polynomials.

Given $i \in Q_0$ and $m \in \mathbb{N}$, we denote by $\mathcal{K}_{i,m}$ the set of paths from the set $\{u_{i_1,m_1}, \dots, u_{i_r,m_r}\}$ which pass through α_{i-1} exactly m times, that is,

$$\mathcal{K}_{i,m} = \{u_{i_s,m_s} : u_{i-1}^{m-1}\alpha_{i-1} \preccurlyeq u_{i_s,m_s},\ u_{i-1}^{m}\alpha_{i-1} \not\preccurlyeq u_{i_s,m_s}\} \quad \text{for } m \geq 1,$$
$$\mathcal{K}_{i,0} = \{u_{i_s,m_s} : \alpha_{i-1} \not\preccurlyeq u_{i_s,m_s}\}.$$

Note that, under our assumptions, if $\mathcal{K}_{i,0}$ is empty then $\mathcal{K}_{i,1}$ is not empty, for every i.

Assume that $\mathcal{K}_{i,m} \neq \emptyset$. Suppose we are walking from vertex i following the direction of arrows; denote by $q_{i,m}$ the terminus of a path from $\mathcal{K}_{i,m}$ which we meet first. Let $\ell_{i,m}$ be the length of the shortest path from i to $q_{i,m}$.

For example let $\Lambda = KQ(3)/I$, where a canonical set of generators of I is $\{G(u_0)\alpha_0\alpha_1, G(u_1)\alpha_1\alpha_2\}$ for some $G \in K[X]$, $G(0) \neq 0$. Then

$$\begin{aligned} &\mathcal{K}_{2,0} = \emptyset, \\ &\mathcal{K}_{1,0} = \{\alpha_1\alpha_2\}, && q_{1,0} = 0, \quad \ell_{1,0} = 2, \\ &\mathcal{K}_{2,1} = \{\alpha_0\alpha_1, \alpha_1\alpha_2\}, && q_{2,1} = 2, \quad \ell_{2,1} = 0. \end{aligned}$$

Let $\mathcal{S}(\Lambda)$ be set of vertices $i \in Q_0$ such that the simple KQ-module $S(i)$ is a Λ-module. It is easy to see that $i \in \mathcal{S}(\Lambda)$ if and only if $i \notin \{i_1, \dots, i_r\}$ or ($i = i_k$ and $g + j_k \geq 1$, for some $k \in \{1, \dots, r\}$).

THEOREM 3.6. *Suppose that $\Lambda = KQ/I$ and I has a canonical set of generators*

$$\{u_{i_1}^g G(u_{i_1}) u_{i_1,m_1}, \dots, u_{i_1}^g G(u_{i_r}) u_{i_r,m_r}\},$$

where $G = F_1^{r_1} \dots F_s^{r_s}$ and $F_1, \dots, F_s$ are pairwise relatively prime irreducible polynomials. The Auslander–Reiten quiver of the category $\operatorname{mod} \Lambda$ *is a disjoint union*

$$\mathcal{T}_{F_1}(\Lambda) \cup \dots \cup \mathcal{T}_{F_s}(\Lambda) \cup \mathcal{T}_t(\Lambda),$$

where

(a) *$\mathcal{T}_{F_j}(\Lambda)$ is a connected component and is the part of the tube $\mathcal{T}_{F_j}$ consisting of modules of regular length less than or equal to r_j.*
(b) *$\mathcal{T}_t(\Lambda)$ consists of all $\preceq$-predecessors of one of the modules Π_i, $i \in \mathcal{S}(\Lambda)$, where*

$$\Pi_i = \begin{cases} X_{i,gn+\ell_{i,0}} & \text{if } \mathcal{K}_{i,0} \neq \emptyset, \\ X_{i,(g+1)n+\ell_{i,1}} & \text{if } \mathcal{K}_{i,0} = \emptyset. \end{cases}$$

$\mathcal{T}_t(\Lambda)$ is not connected in general.

(c) *The modules* $V(F_j^{r_j})$ *for* $j = 1, \ldots, s$ *and* Π_i, $i \in \mathcal{S}(\Lambda)$, *described above, form a complete set of indecomposable projective* Λ*-modules (up to isomorphism).*

Proof. Statements (a) and (b) follow from (c) and Theorem 2.6.

(c) One checks directly that $V(F_j^{r_j})$ and Π_i are Λ-modules. By Theorem 2.6 it is enough to see that that they are $\preceq_c$-maximal. Observe that if $V = (V_i, V_{\alpha_i})_{i\in Q_0}$ is a representation satisfying the relations $u_{i_1}^g G(u_{i_1})u_{i_1,m_1}, \ldots, u_{i_r}^g G(u_{i_r})u_{i_r,m_r}$ then the minimal polynomial of V_{u_i} divides $t^{2+g}G$ for any i. It follows that Γ_Λ is contained in $\bigcup_{j=1}^s \mathcal{T}_{F_j} \cup \mathcal{T}_t$.

Assume, to the contrary, that $V(F_j^{r_j})$ is not $\preceq_c$-maximal. Then, by Corollary 2.3, $W = V(F_j^{r_j+1})$ is a Λ-module. If we view W as a representation $(W_i, W_{\alpha_i})_{i\in Q_0}$, then the minimal polynomial of W_{u_i} is $F_j^{r_j+1}$, a contradiction.

Let $\mathcal{K}_{i,0} \neq \emptyset$ and suppose that $u_{k,l} \in \mathcal{K}_{i,0}$, $q_{i,0} = t(u_{k,l})$. If $\Pi_i = X_{i,gn+\ell_{i,0}}$ is not $\preceq_c$-maximal then $X_{i,gn+\ell_{i,0}+1}$ is a Λ-module. Moreover $u_k^g G(u_k)u_{k,l} \in I$ and then $u_i^g G(u_i)u_{i,l} \in I$. But $X_{i,gn+\ell_{i,0}+1}$ does not satisfy that relation, a contradiction.

The assumption $\mathcal{K}_{i,0} = \emptyset$ yields $\mathcal{K}_{i,1} \neq \emptyset$ and we proceed analogously.

It is easy to observe that there are no other $\preceq_c$-maximal Λ-modules. ∎

Let Λ be $KQ(3)/I$ as in Example 3.1. We have four projective Λ-modules $X_{0,1}$, $X_{2,2}$, $X_{1,2}$, $V_0(G)$, and Γ_Λ consists of two connected components:

$$\begin{array}{lccccc}
\mathcal{T}_{t-1}(\Lambda): & V_0(M_G) & & & & \\
\mathcal{T}_t(\Lambda): & X_{0,1} & & X_{2,1} & & X_{1,1} \\
 & & \searrow \quad \nearrow & & \searrow \quad \nearrow & \\
 & & X_{2,2} & & X_{1,2} &
\end{array}$$

The basic algebra Λ^b associated with Λ (see [1, I.6.3]) is isomorphic to the algebra $KD/(\alpha\beta)$, where

$$D: \qquad \bullet \xrightarrow{\alpha} \bullet \xrightarrow{\beta} \bullet \qquad\qquad \bullet$$

Proposition 3.7. *Under the assumptions and notation of Theorem* 3.6, *there is an isomorphism*

$$e_i\Lambda \cong \bigoplus_{j=1}^{s} V(F_j^{r_j}) \oplus \Pi_i$$

for $i \in \{0, \ldots, n-1\}$, *where we set* $\Pi_i = 0$ *if* $i \notin \mathcal{S}(\Lambda)$.

Proof. There are surjective maps from $e_i\Lambda$ to each of $V(F_j^{r_j})$ and Π_i. Moreover, these modules are indecomposable projective and pairwise nonisomorphic, thus $\bigoplus_{j=1}^s V(F_j^{r_j}) \oplus \Pi_i$ is a direct summand of $e_i\Lambda$. It suffices to prove that

$$\dim_K e_i\Lambda \leq \dim_K \Big(\bigoplus_{j=1}^{s} V(F_j^{r_j}) \oplus \Pi_i\Big).$$

First assume that $\mathcal{K}_{i,0} \neq \emptyset$. It is easy to check that $\dim_K e_i\Lambda e_{i+k} \leq d+1$ for any $k < n$, where $d = g + \deg G$. Moreover, $\dim_K e_i\Lambda e_{i+k} \leq d$ if $\ell_{i,0} \leq k < n$. Hence $\dim_K e_i\Lambda \leq nd + \ell_{i,0}$. On the other hand, $ng + \ell_{i,0} = \dim_K \Pi_i$ and $\sum_{j=1}^{s} \dim_K V(F_j^{r_j}) = n \deg G$.

Similarly if $\mathcal{K}_{i,0} = \emptyset$ then we prove that $\dim_K e_i\Lambda \leq n(d+1) + \ell_{i,1}$ and the assertion follows, since $\dim_K \Pi_i = n(g+1) + \ell_{i,1}$. ■

An isomorphism of algebras $f : \Lambda \to \Lambda'$ induces an isomorphism of categories

$$\Phi_f : \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda'$$

defined by associating to any Λ-module M the vector space $\Phi_f(M) = M$ equipped with multiplication given by the formula $m \cdot_{\Lambda'} \lambda' = m \cdot_{\Lambda} f^{-1}(\lambda')$. Moreover, $\Phi_f(\gamma) = \gamma$ for every homomorphism γ. It is clear that $\Phi_f(e\Lambda) \cong f(e)\Lambda'$ for any $e \in \Lambda$.

COROLLARY 3.8. *Assume that $F_1, \ldots, F_M$ (resp. $F'_1, \ldots, F'_M$) are pairwise relatively prime polynomials of degree 1 in $K[X]$, $F_s(0) \neq 0$, $F'_s(0) \neq 0$, $0 \leq j_p < n$, $r_p, n_{p,s} \in \mathbb{N}\cup\{0\}$ for $s = 1, \ldots, M$, $p = 1, \ldots, N$, $i_1, \ldots, i_r \in Q_0$. Let $\Lambda = KQ/I$, $\Lambda' = KQ/I'$ where*

$$I = (u_{i_p}^{r_p} F_1^{n_{p,1}}(u_{i_p}) \cdot \ldots \cdot F_M^{n_{p,M}}(u_{i_p}) u_{i_p,j_p};\ p = 1, \ldots, N),$$
$$I' = (u_{i_p}^{r_p} (F'_1)^{n_{p,1}}(u_{i_p}) \cdot \ldots \cdot (F'_M)^{n_{p,M}}(u_{i_p}) u_{i_p,j_p};\ p = 1, \ldots, N).$$

There is an isomorphism of algebras $f : \Lambda \to \Lambda'$ such that $\Phi_f(X_{i,j}) \cong X_{i,j}$ and $\Phi_f(V(F_i^r)) \cong V((F'_i)^r)$ whenever $X_{i,j}$, $V(F_i^r)$ are Λ-modules.

Under the assumptions of the corollary we say that I and I' *have generating sets of the same shape.*

Proof. Observe that the set of generators of I satisfies one of the conditions (RN), $N = 1, 2, 3$, if and only if the set of generators of I' does. Then an application of a suitable operation $\mathcal{G} \mapsto \mathcal{G}_N$ on both of them gives us two new generating sets of I and I' of the same shape. By induction we can assume that I has the canonical set of generators

$$\{u_{i_1}^{g} G(u_{i_1}) u_{i_1,m_1}, \ldots, u_{i_r}^{g} G(u_{i_r}) u_{i_r,m_r}\},$$

where $0 \leq m_a < 2n$ for $a = 1, \ldots, r$, $G = F_1^{r_1} \ldots F_L^{r_L}$ for some $L \leq M$, and I' has a canonical set of generators of the same shape

$$\{u_{i_1}^{g} G'(u_{i_1}) u_{i_1,m_1}, \ldots, u_{i_r}^{g} G'(u_{i_r}) u_{i_r,m_r}\},$$

where $G' = (F'_1)^{r_1} \ldots (F'_L)^{r_L}$.

Projective Λ-modules and projective Λ'-modules in the tube $\mathcal{T}_t$ are the same. By Proposition 3.7, we have

$$e_i\Lambda \cong \bigoplus_{j=1}^{L} V(F_j^{r_j}) \oplus \Pi_i, \qquad e_i\Lambda \cong \bigoplus_{j=1}^{L} V((F'_j)^{r_j}) \oplus \Pi_i.$$

Hence there are decompositions into projective indecomposables:

$$\Lambda_\Lambda \cong \bigoplus_{j=1}^{L} V(F_j^{r_j})^n \oplus \bigoplus_{i\in\mathcal{S}(\Lambda)} \Pi_i,$$

$$\Lambda'_{\Lambda'} \cong \bigoplus_{j=1}^{L} V((F'_j)^{r_j})^n \oplus \bigoplus_{i\in\mathcal{S}(\Lambda')} \Pi_i.$$

Since the algebras $\mathrm{End}_\Lambda(V(F_j^{r_j}))$ and $\mathrm{End}_{\Lambda'}(V((F'_j)^{r_j}))$ are isomorphic, and by Theorem 3.6 the components in Γ_Λ (resp. in $\Gamma_{\Lambda'}$) are orthogonal, there exists an isomorphism

$$\Lambda \cong \mathrm{End}_\Lambda(\Lambda_\Lambda) \cong \mathrm{End}_{\Lambda'}(\Lambda'_{\Lambda'}) \cong \Lambda'$$

satisfying the required condition. ∎

Let us formulate separately a special case of our results.

Corollary 3.9. *Assume that F has the following decomposition in $K[t]$:*

$$G = t^g F_1^{r_1} \ldots F_s^{r_s},$$

where $F_1, \ldots, F_s$ are pairwise relatively prime irreducible polynomials not divisible by t. Then there is an isomorphism of algebras

$$KQ/(G(u_i), i = 0, \ldots, n-1) \cong A_{n,g} \times \prod_{j=1}^{s} \mathbb{M}_n(D_j[t]/(t^{r_j})),$$

where $A_{n,g}$ is the Nakayama algebra $KQ/(u_{i,g}, i = 0, \ldots, n-1)$ (see [1, V.3]), *and $D_j = K[t]/(F_j)$.*

Proof. Let $\Lambda = KQ/(G(u_i), i = 0, \ldots, n-1)$. It follows from Theorem 3.6 that the components in Γ_Λ are orthogonal. Moreover, Proposition 3.7 yields

$$e_i\Lambda \cong \bigoplus_{j=1}^{s} V(F_j^{r_j}) \oplus \Pi_i$$

for $i = 0, \ldots, n-1$. Then there is an isomorphism of algebras

$$\Lambda \cong \mathrm{End}_\Lambda(\Lambda_\Lambda) \cong \mathrm{End}_\Lambda(V(F_1^{r_1})^n) \times \cdots \times \mathrm{End}_\Lambda(V(F_s^{r_s})^n) \times \mathrm{End}_\Lambda\Big(\bigoplus_{i=0}^{n-1} \Pi_i\Big).$$

Moreover

$$\mathrm{End}_\Lambda(V(F_j^{r_j})^n) \cong \mathbb{M}_n(\mathrm{End}(V(F_j^{r_j})), \quad \mathrm{End}(V(F_j^{r_j})) \cong D_j[t]/(t^{r_j}).$$

The projective Λ-modules $\Pi_0, \ldots, \Pi_{n-1}$ are pairwise nonisomorphic and the algebra $\mathrm{End}_\Lambda(\bigoplus_{i=0}^{n-1} \Pi_i)$ is basic. As the additive hull of $\{\Pi_0, \ldots, \Pi_{n-1}\}$ is isomorphic to the category of projective $A_{n,g}$-modules, it follows that $\mathrm{End}_\Lambda(\bigoplus_{i=0}^{n-1} \Pi_i)$ is isomorphic to $A_{n,g}$. ∎

4. The main result. Recall that an isomorphism of algebras $f : \Lambda \to \Lambda'$ induces an isomorphism of categories $\Phi_f : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda'$. If M is an A-Λ-bimodule for some algebra A then $\Phi_f(M)$ has a natural A-Λ'-bimodule structure.

PROPOSITION 4.2. *Let*

$$B = \begin{bmatrix} A & M \\ 0 & \Lambda \end{bmatrix}, \quad B' = \begin{bmatrix} A & M' \\ 0 & \Lambda' \end{bmatrix}$$

be generalized matrix algebras, where M is an A-Λ-bimodule and M' is an A'-Λ'-bimodule (see [1, I.2.10]). *If there exist an isomorphism $f : \Lambda \to \Lambda'$ of algebras and an isomorphism $\sigma : \Phi_f(M) \to M'$ of A-Λ'-bimodules, then $B \cong B'$.*

Proof. The map $g : B \to B'$ given by

$$\begin{pmatrix} a & m \\ 0 & \lambda \end{pmatrix} \mapsto \begin{pmatrix} a & \sigma(m) \\ 0 & f(\lambda) \end{pmatrix}$$

for $a \in A, m \in M, \lambda \in \Lambda$ is an isomorphism. ∎

We consider the algebras of the form KQ/I, where Q is obtained from the cycle $Q(n)$ of (2.0) by attaching trees. More precisely, suppose we have pairwise disjoint finite connected trees $T_1, \ldots, T_m$ with distinguished vertices $\omega_1, \ldots, \omega_m$, $\omega_i \in (T_i)_0$, for $i = 1, \ldots, m$, and vertices $z_1, \ldots, z_m$ of $Q(n)$. Let $\underline{T} = (T_1, \ldots, T_m)$ and let $\underline{\omega} = (\omega_1, \ldots, \omega_m)$ and $\underline{z} = (z_1, \ldots, z_m)$. Define a new quiver $Q = Q(n, \underline{T}, \underline{\omega}, \underline{z})$ by setting

$$Q_0 = Q(n)_0 \cup \bigcup_{i=1}^{m} (T_i)_0, \quad Q_1 = Q(n)_1 \cup \bigcup_{i=1}^{m} (T_i)_1 \cup \{\beta_1, \ldots, \beta_m\},$$

where $\beta_1, \ldots, \beta_m$ are new arrows with $s(\beta_i) = \omega_i$ and $t(\beta_i) = z_i$ for $i = 1, \ldots, m$. We call such a quiver a *tree extension of the cycle* $Q(n)$ by trees $T_1, \ldots, T_m$ with roots $\omega_1, \ldots, \omega_m$ at the vertices $z_1, \ldots, z_m$. Throughout this section we keep the notation introduced above and we denote by Q the quiver $Q(n, \underline{T}, \underline{\omega}, \underline{z})$.

Given an algebra R and a quiver $\Delta = (\Delta_0, \Delta_1)$, the concept of R-representation $M = (M_x, M_\beta)_{x \in \Delta_0, \beta \in \Delta_1}$ of Δ is defined as usual: we associate right R-modules M_x to the vertices x of Δ and R-homomorphisms M_β to the arrows β. The category of R-representations (with morphisms defined in the usual way) is denoted by $\mathrm{Rep}_R(\Delta)$. The full subcategory of $\mathrm{Rep}_R(\Delta)$ consisting of objects $(M_x, M_\beta)_{x\in\Delta_0,\,\beta\in\Delta_1}$ such that M_x is finitely generated, for every vertex x of Δ, is denoted by $\mathrm{rep}_R(\Delta)$.

If $f : R \to R'$ is an algebra homomorphism then the functor Φ_f induces an equivalence of the categories $\mathrm{Rep}_R(\Delta)$ and $\mathrm{Rep}_{R'}(\Delta)$ (and of $\mathrm{rep}_R(\Delta)$ and $\mathrm{rep}_{R'}(\Delta)$), which we also denote by Φ_f.

Given a directed quiver Δ and its vertex x, we denote by x^+ the set of all vertices which are successors of x with respect to the path order in Δ.

LEMMA 4.3. *Let $\Lambda = KQ(n)/J$ for some nonzero ideal J. Assume that Δ is a tree and x_0 is a source in a tree Δ. Assume also that $M = (M_x, M_\beta)$ and $M' = (M'_x, M'_\beta)$ are objects of $\mathrm{rep}_\Lambda(\Delta)$ satisfying the following conditions*:

(i) $M_x = 0 = M'_x$ *for* $x \notin x_0^+$,
(ii) M_α *and* M'_α *are epimorphisms for any arrow* α *such that* $s(\alpha) \in x_0^+$,
(iii) M_{x_0} (*resp.* M'_{x_0}) *has at most one indecomposable direct summand in each connected component of* Γ_Λ.

If $M_x \cong M'_x$ as Λ-modules for each $x \in \Delta_0$, then $M \cong M'$ as Λ-representations.

Proof. Thanks to the orthogonality of the tubes in Γ_Λ it is enough to prove the statement for M, M' such that all the indecomposable direct summands of M_x and M'_x, $x \in \Delta_0$, lie in one connected component of Γ_Λ. Then M_{x_0} and M'_{x_0} are indecomposable by (iii). Condition (ii), together with Corollary 2.3, implies that M_x (resp. M'_x) is indecomposable or 0, for all $x \in \Delta_0$.

By induction with respect to the path order on the vertices of Δ we shall define a system of Λ-isomorphisms $\Theta_x : M_x \to M'_x$, $x \in \Delta_0$.

Fix an arbitrary isomorphism $\Theta_{x_0} : M_{x_0} \to M'_{x_0}$. Assume that $x \in x_0^+$ and $\Theta_x : M_x \to M'_x$ is defined and there exists an arrow $x \xrightarrow{\alpha} y$ in Δ. Therefore we have a Λ-module epimorphism $M_\alpha : M_x \to M_y$ and a Λ-module epimorphism $M'_\alpha : M'_x \to M'_y$. The maps $M'_\alpha \circ \Theta_x$ and M_α are epimorphisms, therefore, by Corollary 2.5, there exists a Λ-isomorphism $\Theta_y : M_y \to M'_y$ such that the diagram

$$\begin{array}{ccc} M_x & \xrightarrow{M_\alpha} & M_y \\ {\scriptstyle\Theta_x}\downarrow & & \downarrow{\scriptstyle\Theta_y} \\ M'_x & \xrightarrow{M'_\alpha} & M'_y \end{array}$$

is commutative. We set $\Theta_x = 0$, for $x \notin x_0^+$. Then $\Theta = (\Theta_x)_{x\in\Delta_0} : M \to M'$ is an isomorphism of Λ-representations. ∎

THEOREM 4.4. *Let* $Q = Q(n, \underline{T}, \underline{\omega}, \underline{z})$ *be the tree extension of the cycle* $Q(n)$ *by trees* $\underline{T}$ *with roots* $\underline{\omega}$ *at the vertices* $\underline{z}$, $B = KQ/I$, $B' = KQ/I'$, *where* I *and* I' *are two-sided ideals in* KQ *such that* $KT_j \cap I = KT_j \cap I'$ *for* $j = \{1, \dots, m\}$. *Let*

$$\Lambda = KQ(n)/(I \cap KQ(n)), \qquad \Lambda' = KQ(n)/(I' \cap KQ(n)).$$

If there exists an isomorphism of algebras $f : \Lambda \to \Lambda'$ *such that* $\Phi_f(e_i B|_\Lambda) \cong e_i B|_{\Lambda'}$ *as* Λ'*-modules for any* $i \in Q_0 \setminus Q(n)_0$, *then the algebras* B *and* B' *are isomorphic.*

In the formulation of Theorem 4.4, given a B-module, we denote by

$$X|_\Lambda = X \cdot \Big(\sum_{x \in Q(n)_0} e_x \Big)$$

the restriction of X to Λ. Analogously, we denote by $Y|_{\Lambda'}$ the restriction of a B'-module Y to Λ'.

Proof. We define $M^j \in \mathrm{rep}_\Lambda(T_j^{\mathrm{op}})$ as follows. To a vertex x of T_j, we associate the Λ-module $e_x B|_\Lambda$, and given an arrow $x \xrightarrow{\alpha} y$ of T_j we define the map $M^j_\alpha : M^j_y \to M^j_x$ by setting $M^j_\alpha(r) = \alpha r$ for $r \in M^j_y$. Similarly, we define $M'^j \in \mathrm{rep}_{\Lambda'}(T_j^{\mathrm{op}})$ by associating the Λ'-module $e_x B|_{\Lambda'}$ to a vertex x and we define the map $M'^j_\alpha : M'^j_y \to M'^j_x$ by setting $M'^j_\alpha(r) = \alpha r$ for $r \in M'^j_y$. We shall prove that the representations $\Phi_f(M^j)$ and M'^j satisfy conditions (i)–(iii) of Lemma 4.3 with $\Delta = T_j^{\mathrm{op}}$ and $x_0 = \omega_j$. Condition (i) is clear. Let $x \xrightarrow{\alpha} y$ be an arrow in T_j such that y is a predecessor of ω_j in the path order in T_j. Since each path in KQ starting at x and ending in $Q(n)$ passes through α it follows that M^j_α , M'^j_α are epimorphisms and (ii) follows. For any $y \in (T_j)_0$ the module $e_y B|_\Lambda$ is a factor of the projective Λ-module $e_{z_j}\Lambda$ (resp. $e_y B'|_{\Lambda'}$ is a factor of the projective Λ'-module $e_{z_j}\Lambda'$), which has at most one indecomposable direct summand in each connected component of Γ_Λ (resp. $\Gamma_{\Lambda'}$); see Proposition 3.7. Hence we get (iii) by Corollary 2.3. Then it follows from Lemma 4.3 that $M'^j \cong \Phi_f(M^j)$.

We treat any representation of T_j^{op} as a representation of T^{op} in the obvious way, where T is the disjoint union of $T_1, \dots, T_m$.

Then we set $M = \bigoplus_{j=1}^m M^j \in \mathrm{rep}_\Lambda(T^{\mathrm{op}})$, $M' = \bigoplus_{j=1}^m M'^j \in \mathrm{rep}_{\Lambda'}(T^{\mathrm{op}})$.

Let $A = KT$. If we identify the Λ-representation M of T^{op} with an A-Λ-bimodule (resp. Λ'-representation M' of T^{op} with an A-Λ'-bimodule), then

$$B \cong \begin{bmatrix} A & M \\ 0 & \Lambda \end{bmatrix}, \qquad B' \cong \begin{bmatrix} A & M' \\ 0 & \Lambda' \end{bmatrix}.$$

Hence, by Proposition 4.2, the algebras B and B' are isomorphic. ∎

Dually, we obtain an analogous theorem for tree coextensions of the cycle $Q(n)$ defined in the obvious way.

REMARK. The whole category of finitely generated Λ-representations of a tree may be arbitrarily complicated even if T is just one arrow and $\Lambda \cong K[t]/(t^m)$ for some $m \geq 1$ (actually, that category is wild when $m \geq 7$, see [11]).

The last result of this article has a technical character, at least as regards its formulation. Let us explain its idea: if Q is a tree extension of the cycle $Q(n)$, and ideals I and I' of KQ have generating sets "of the same shape" (cf. Corollary 3.8), then the algebras KQ/I and KQ/I' are isomorphic.

Let us say that two polynomials in one indeterminate (over an algebraically closed field) *have the same shape* if there is a multiplicity preserving bijection between the sets of their roots. One can naturally extend this concept to sequences of polynomials.

Given an ideal in KQ, we can choose a generating set consisting of zero-relations (that is, paths) and relations of the form $v_a F_a(u_{i_a}) w_a$, where v_a, w_a are paths, u_{i_a} is a cyclic path in $Q(n)$ and F_a is a polynomial. Now one can say that two sets (of generators of two ideals) have the same shape if they contain the same zero-relations and the remaining relations can be ordered in such a way that the paths v_a, u_{i_a}, w_a are the same in both sets and the resulting sequences of F_a's have the same shape.

Precisely, let $Q = Q(n, \underline{T}, \underline{\omega}, \underline{z})$ be a tree extension of $Q(n)$ (we keep the notation introduced above). Assume that there are fixed integers $M > 0$, $-1 \leq N' < N$, $L \geq 0$ and

(i) pairwise relatively prime polynomials $F_1, \ldots, F_M$ of degree one such that $F_a(0) \neq 0$, $a = 1, \ldots, M$,

(i′) pairwise relatively prime polynomials $F'_1, \ldots, F'_M$ of degree one such that $F'_a(0) \neq 0$, $a = 1, \ldots, M$,

(ii) paths v_a, $a = -L, \ldots, -1, 0, 1, \ldots, N'$, starting in $T = T_1 \cup \cdots \cup T_m$. If the path v_a starts in T_i then v_a terminates in T_i for $a < 0$ and terminates at the vertex z_i for $a \geq 0$,

(iii) vertices i_a of $Q(n)$, $a = 0, \ldots, N$, such that $i_a = t(v_a)$ for $a = 0, \ldots, N'$,

(iv) numbers $r_a, n_{a,s} \in \mathbb{N} \cup \{0\}$, $a = 0, \ldots, N$, $s = 1, \ldots, M$,

(v) numbers j_a, $0 \leq j_a < n$, $0 = 1, \ldots, N$.

These data determine two triples of sets of elements of KQ:

$$\begin{aligned}
\mathcal{G}_0 &= \{v_a : a = -L, \ldots, -1\}, \\
\mathcal{G}_1 &= \{v_a u_{i_a}^{r_a} G_a(u_{i_a}) u_{i_a j_a} : a = 0, \ldots, N'\}, \\
\mathcal{G}_2 &= \{u_{i_a}^{r_a} G_a(u_{i_a}) u_{i_a j_a} : a = N' + 1, \ldots, N\}
\end{aligned}$$

and

$$\begin{aligned}\mathcal{G}'_0 &= \mathcal{G}_0,\\ \mathcal{G}'_1 &= \{v_a u_{i_a}^{r_a} G'_a(u_{i_a}) u_{i_a j_a} : a = 0, \dots, N'\},\\ \mathcal{G}'_2 &= \{u_{i_a}^{r_a} G'_a(u_{i_a}) u_{i_a j_a} : a = N'+1, \dots, N\},\end{aligned}$$

where $G_a = F_1^{n_{a,1}} \dots F_M^{n_{a,M}}$ and $G'_a = (F'_1)^{n_{a,1}} \dots (F'_M)^{n_{a,M}}$ for $a = 0, \dots, N$.

Corollary 4.5. *Keep the above notation and let I (resp. I') be the ideal in KQ generated by $\mathcal{G}_0 \cup \mathcal{G}_1 \cup \mathcal{G}_2$ (resp. $\mathcal{G}'_0 \cup \mathcal{G}'_1 \cup \mathcal{G}'_2$). Then*

$$KQ/I \cong KQ/I'.$$

Proof. We set

$$B = KQ/I, \quad \Lambda = KQ(n)/J, \quad B' = KQ/I', \quad \Lambda' = KQ(n)/J',$$

where J (resp. J') is the ideal in $KQ(n)$ generated by $\mathcal{G}_2$ (resp. $\mathcal{G}'_2$). Observe that the sets $\mathcal{G}_2$ and $\mathcal{G}'_2$ are of the same shape in the sense of Corollary 3.8. Hence there is an isomorphism of algebras $f : \Lambda \to \Lambda'$. It is enough to prove that one can choose f so that $\Phi_f(e_y B|_\Lambda) \cong e_y B'|_{\Lambda'}$ for every vertex y of T. Then the statement follows from Theorem 4.4.

To do this let us decompose the modules $e_y B|_\Lambda$ and $e_y B'|_{\Lambda'}$ into indecomposables. Fix $k \in \{1, \dots, m\}$ and let y be a vertex of T_k. Recall that z_k is the vertex of $Q(n)_0$ the tree T_k is attached at. First observe that $e_y B|_\Lambda = 0$ and $e_y B'|_{\Lambda'} = 0$ provided that either there is no oriented path from y to z_k, or there is such a path (then it is unique) but it contains v_a as a subpath for some $a = -L, \dots, -1$.

Now suppose that w_* is the unique oriented path from y to z_j, and w_* does not contain any v_a, $a = -L, \dots, -1$, as a subpath. Then there is an epimorphism of Λ-modules $h : e_{z_k}\Lambda \to e_y B|_\Lambda$ such that $\operatorname{Ker} h$ is a submodule of $e_{z_k}\Lambda$ generated by

$$\{u_{z_k}^{r_a} G_a(u_{z_k}) u_{z_k, j_a} : a \in \mathcal{J}_y\},$$

where $\mathcal{J}_y = \{a = 0, \dots, N' : v_a \preceq w_*\}$. We have an analogous description of $e_y B'|_{\Lambda'}$.

Let $G = \gcd\{G_a : a = N'+1, \dots, N\}$, $G' = \gcd\{G'_a : a = N'+1, \dots, N\}$. Without loss of generality, we can assume that $G = F_1^{c_1} \dots F_{M'}^{c_{M'}}$ and $G' = (F'_1)^{c_1} \dots (F'_{M'})^{c_{M'}}$ for some $M' \le M$ and $c_1, \dots, c_{M'} > 0$. Moreover, put $H_y = \gcd\{G, G_a : a \in \mathcal{J}_y\}$ and $H'_y = \gcd\{G', G'_a : a \in \mathcal{J}_y\}$.

For $a = 1, \dots, M'$, let q_a be the multiplicity of F_a in H_y (equal to the multiplicity of F'_a in H'_y). Let also $b = \min(\{nr_a + j_a : a \in \mathcal{J}_y\} \cup \{\dim_K \Pi_{z_k}\})$, where Π_{z_k} is the projective Λ-module (as well as Λ'-module) described in Theorem 3.6(b). Then

$$\begin{aligned} e_y B|_\Lambda &\cong V(F_1^{q_1}) \oplus \dots \oplus V(F_{M'}^{q_{M'}}) \oplus X_{z_k, b},\\ e_y B'|_{\Lambda'} &\cong V((F'_1)^{q_1}) \oplus \dots \oplus V((F'_{M'})^{q_{M'}}) \oplus X_{z_k, b}.\end{aligned}$$

Hence, by Corollary 3.8, the isomorphism f can be chosen in such a way that $\Phi_f(e_yB|_\Lambda) \cong e_yB|_{\Lambda'}$ for any vertex y of T. ∎

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