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# *ON DERIVED EQUIVALENCE CLASSIFICATION OF GENTLE TWO-CYCLE ALGEBRAS*

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**Abstract.** We classify, up to derived (equivalently, tilting-cotilting) equivalence, all nondegenerate gentle two-cycle algebras. We also give a partial classification and formulate a conjecture in the degenerate case.

**Introduction and the main result.** Throughout the paper, $k$ denotes a fixed algebraically closed field. By an *algebra* we mean a finite-dimensional basic connected $k$-algebra and by a *module* a finite-dimensional left module. By $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{N}_+$, we denote the sets of integers, nonnegative integers, and positive integers, respectively. Finally, if $i, j \in \mathbb{Z}$, then $[i, j] = \{l \in \mathbb{Z} \mid i \leq l \leq j\}$.

With an algebra $\Lambda$ we may associate its bounded derived category $\mathcal{D}^b(\Lambda)$ (in the sense of Verdier [29]) of bounded complexes of $\Lambda$-modules, which has the structure of a triangulated category (see [17]). The bounded derived category is an important homological invariant of the module category of an algebra and attracts a lot of interest (see for example [5, 8, 15, 16, 18, 22, 24, 25]). In particular, the derived equivalence classes of algebras have been investigated (see for example [1, 9, 11, 14, 20]), where two algebras are said to be derived equivalent if their bounded derived categories are equivalent as triangulated categories.

A handy way of proving derived equivalence between algebras $\Lambda$ and $\Lambda'$ is to construct a (co)tilting $\Lambda$-module $T$ such that $\Lambda'$ is (isomorphic to) the opposite of the endomorphism algebra of $T$. Here a $\Lambda$-module $T$ is called (co)tilting if $\operatorname{pd}_\Lambda T \leq 1$ ($\operatorname{id}_\Lambda T \leq 1$, respectively), $\operatorname{Ext}^1_\Lambda(T, T) = 0$, and $T$ is a direct sum of precisely $\operatorname{rk} K_0(\Lambda)$ pairwise nonisomorphic indecomposable $\Lambda$-modules, where $K_0(\Lambda)$ denotes the Grothendieck group of the category of $\Lambda$-modules. The transitive closure of the relation defined in this way is called tilting-cotilting equivalence. For many classes of algebras tilting-cotilting equivalence and derived equivalence coincide.

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Results of this type have been obtained for gentle algebras, introduced by Assem and Skowroński in [4] (see Section 1 for a precise definition), which form an important subclass of the class of special biserial algebras in the sense of [27]. We note that a representation-infinite algebra admits a simply connected Galois covering all of whose finite convex subcategories are representation-finite if and only if it is a special biserial algebra and its simply connected Galois covering is the repetitive category of the union of a countable chain of gentle tree algebras (see [23]).

The class of algebras derived equivalent to a hereditary algebra of Dynkin type $\mathbb{A}_n$ for some $n \in \mathbb{N}_+$ coincides with the class of algebras tilting-cotilting equivalent to a hereditary algebra of type $\mathbb{A}_n$ and consists of the gentle algebras whose Gabriel quivers have $n$ vertices and $n-1$ arrows (see [2]). Moreover, for a given $n$ all such algebras form one derived equivalence class.

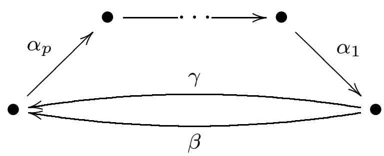
Similarly, the class of algebras derived equivalent to a hereditary algebra of Euclidean type $\widetilde{\mathbb{A}}_n$ for some $n \in \mathbb{N}_+$ coincides with the class of algebras tilting-cotilting equivalent to a hereditary algebra of type $\widetilde{\mathbb{A}}_n$ and consists of the gentle algebras whose Gabriel quivers have $n$ vertices and $n$ arrows and which satisfy the so-called clock condition on the unique cycle. In this case, there are exactly $\lfloor n/2 \rfloor$ derived (equivalently, tilting-cotilting) equivalence classes for a given $n$.

The algebras with the same numbers of vertices and arrows in the Gabriel quiver are called one-cycle algebras. The remaining gentle one-cycle algebras form the class of derived discrete algebras which are not derived (equivalently, tilting-cotilting) equivalent to a hereditary algebra of Dynkin type (see [30]). The derived equivalence classes of these algebras were described in [10].

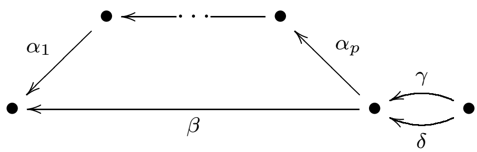
The aim of this paper it to extend the above classification to the class of gentle two-cycle algebras, where we call an algebra a two-cycle algebra if the number of arrows in the Gabriel quiver exceeds the number of vertices by one. An additional motivation for this research is the proof by Schröer and Zimmermann in [26] that the gentle algebras are closed under derived equivalences. Moreover, for gentle algebras the numbers of vertices and arrows in the Gabriel quiver are derived invariants (see [7, Corollary 15]). However, we obtain a full classification only for nondegenerate gentle two-cycle algebras, where we call a gentle two-cycle algebra $\Lambda$ *nondegenerate* if $\sum_{(n,m)\in\mathbb{N}\times\mathbb{N}} \phi_\Lambda(n,m) = 3$. Here $\phi_\Lambda : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ is the derived invariant introduced by Avella-Alaminos and Geiss in [7] (see Section 3). For the remaining gentle two-cycle algebras $\Lambda$, which we call *degenerate*, we have $\sum_{(n,m)\in\mathbb{N}\times\mathbb{N}} \phi_\Lambda(n,m) = 1$. Obviously, both these classes of gentle two-cycle algebras are closed under derived (hence also tilting-cotilting) equivalences.

Before formulating the main results of the paper we define the following families of algebras.

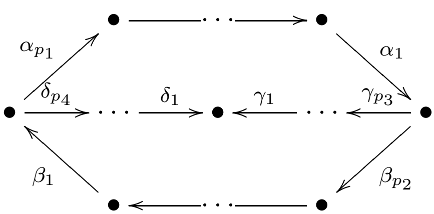
- $\Lambda_0(p,r)$ for $p \in \mathbb{N}_+$ and $r \in [0,p-1]$ is the algebra of the quiver



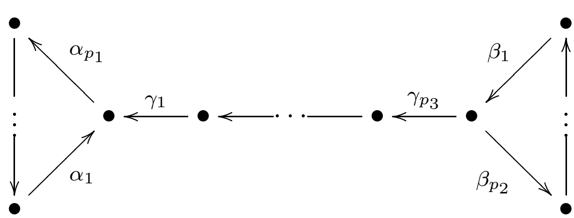

  bound by $\alpha_p\beta$, $\alpha_i\alpha_{i+1}$ for $i \in [1,r]$ and $\gamma\alpha_1$.
- $\Lambda_0'(p,r)$ for $p \in \mathbb{N}_+$ and $r \in [0,p-1]$ is the algebra of the quiver




  bound by $\alpha_i\alpha_{i+1}$ for $i \in [1,r]$, $\alpha_p\gamma$, and $\beta\delta$.
- $\Lambda_1(p_1,p_2,p_3,p_4,r_1)$ for $p_1,p_2 \in \mathbb{N}_+$, $p_3,p_4 \in \mathbb{N}$, and $r_1 \in [0,p_1-1]$ such that $p_2+p_3 \geq 2$ and $p_4+r_1 \geq 1$ is the algebra of the quiver




  bound by $\alpha_i\alpha_{i+1}$ for $i \in [p_1-r_1,p_1-1]$, $\alpha_{p_1}\beta_1$, $\beta_i\beta_{i+1}$ for $i \in [1,p_2-1]$, and $\beta_{p_2}\alpha_1$.
- $\Lambda_2(p_1,p_2,p_3,r_1,r_2)$ for $p_1,p_2 \in \mathbb{N}_+$, $p_3 \in \mathbb{N}$, $r_1 \in [0,p_1-1]$, and $r_2 \in [0,p_2-1]$ such that $p_3+r_1+r_2 \geq 1$ is the algebra of the quiver




  bound by $\alpha_i\alpha_{i+1}$ for $i \in [p_1-r_1,p_1-1]$, $\alpha_{p_1}\alpha_1$, $\beta_i\beta_{i+1}$ for $i \in [p_2-r_2, p_2-1]$, and $\beta_{p_2}\beta_1$.

The main results of the paper are the following.

THEOREM 1. *If $\Lambda$ is a nondegenerate gentle two-cycle algebra, then $\Lambda$ is derived* (*equivalently, tilting-cotilting*) *equivalent to one of the following algebras*:

- $\Lambda_1(p_1, p_2, p_3, p_4, r_1)$ *for some* $p_1, p_2 \in \mathbb{N}_+$, $p_3, p_4 \in \mathbb{N}$, *and* $r_1 \in [0, p_1-1]$ *such that* $p_2 + p_3 \geq 2$, $p_4 + r_1 \geq 1$, *and either* $p_3 > p_4$, *or* $p_3 = p_4$ *and* $p_2 > r_1$,
- $\Lambda_2(p_1, p_2, p_3, r_1, r_2)$ *for some* $p_1, p_2 \in \mathbb{N}_+$, $p_3 \in \mathbb{N}$, $r_1 \in [0, p_1 - 1]$, *and* $r_2 \in [0, p_2 - 1]$ *such that* $p_3 + r_1 + r_2 \geq 1$ *and either* $p_1 > p_2$, *or* $p_1 = p_2$ *and* $r_1 \geq r_2$.

*Moreover, different algebras from the above list are not derived (equivalently, tilting-cotilting) equivalent.*

THEOREM 2. *If* $\Lambda$ *is a degenerate gentle two-cycle algebra, then* $\Lambda$ *is derived (equivalently, tilting-cotilting) equivalent to one of the following algebras:*

- $\Lambda_0(p, r)$ *for some* $p \in \mathbb{N}_+$ *and* $r \in [0, p - 1]$,
- $\Lambda_0'(p, 0)$ *for some* $p \in \mathbb{N}_+$.

Moreover, we have the following conjecture concerning the minimality of the list in the above theorem.

CONJECTURE. *Different algebras from the list in Theorem 2 are not derived (equivalently, tilting-cotilting) equivalent.*

Obviously, if $p_1, p_2 \in \mathbb{N}_+$, $r_1 \in [0, p_1 - 1]$, $r_2 \in [0, p_2 - 1]$, and $p_1 \neq p_2$, then $\Lambda_0(p_1, r_1)$ and $\Lambda_0(p_2, r_2)$ ($\Lambda_0'(p_1, 0)$ and $\Lambda_0'(p_2, 0)$, respectively) are not derived equivalent. Similarly, if $p_1, p_2 \in \mathbb{N}_+$, $r_1 \in [0, p_1 - 1]$, and $p_1 \neq p_2 + 1$, then $\Lambda_0(p_1, r_1)$ and $\Lambda_0'(p_2, 0)$ are not derived equivalent. Thus it is enough to prove that $\Lambda_0(p+1, 0), \ldots, \Lambda_0(p+1, p)$ and $\Lambda_0'(p, 0)$ are pairwise not derived equivalent for a fixed $p \in \mathbb{N}_+$. It follows easily by investigating the Euler quadratic forms that $\Lambda_0(p+1, r_1)$ and $\Lambda(p+1, r_2)$ ($\Lambda_0'(p, 0)$ and $\Lambda_0(p+1, r_2)$) are not derived equivalent if $r_1 \not\equiv r_2 \pmod 2$ ($r_2 \equiv 0 \pmod 2$, respectively).

The paper is organized as follows. In Section 1 we first present basic definitions, then describe main tools used in order to reduce an arbitrary gentle two-cycle algebra to one of the algebras listed in Theorems 1 and 2: passing to the opposite algebra, (generalized) APR-(co)reflections, and HW-(co)reflections. Finally, we describe an operation of shifting relations, which is a basic application of the above operations, and investigate two particular families of gentle two-cycle algebras. In Section 2, the technical heart of the paper, we prove, in a sequence of steps, that the lists of representatives of the tilting-cotilting equivalence classes of gentle two-cycle algebras given in Theorems 1 and 2 are complete, while in Section 3 we show that different algebras from the list given in Theorem 1 are not derived equivalent. The last property follows by calculating the derived invariant introduced by Avella-Alaminos and Geiss in [7].

For basic background on representation theory of finite-dimensional algebras we refer to [3].

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**1. Basic tools and auxiliary results.** By a (finite) *quiver* $\Delta$ we mean a finite set $\Delta_0$ of vertices together with a finite set $\Delta_1$ of arrows and two maps $s = s_\Delta, t = t_\Delta : \Delta_1 \to \Delta_0$ which assign to an arrow $\alpha$ its starting and terminating vertex, respectively. We say that an arrow $\alpha$ is *adjacent* to a vertex $x$ if either $s\alpha = x$ or $t\alpha = x$. By a *path* of length $n \in \mathbb{N}_+$ we mean a sequence $\sigma = \alpha_1 \cdots \alpha_n$ of arrows such that $s\alpha_i = t\alpha_{i+1}$ for all $i \in [1, n-1]$. In the above situation we denote $s\alpha_n$ and $t\alpha_1$ by $s\sigma$ and $t\sigma$, respectively. We also call $\alpha_1$ and $\alpha_n$ the *terminating* and *starting arrow* of $\sigma$, respectively. Additionally, for each $x \in \Delta_0$ we consider the *trivial* path of length 0, also denoted by $x$, such that $sx = x = tx$. The length of a path $\sigma$ will be denoted by $\ell(\sigma)$. A path $\sigma$ is called *maximal* if there exists no arrow $\alpha$ such that either $s\alpha = t\sigma$ or $t\alpha = s\sigma$. Similarly, we define maximal paths starting (or terminating) at a given vertex. A connected quiver is said to be a *c-cycle* if $|\Delta_1| = |\Delta_0| + c - 1$.

With a quiver $\Delta$ we associate its *path algebra* $k\Delta$, which as a $k$-vector space has a basis formed by all paths in $\Delta$ and whose multiplication is induced by composition of paths. By a *relation* $\varrho$ in $\Delta$ we mean a linear combination of paths of length at least 2 with common starting and terminating vertices. The common starting vertex is denoted by $s\varrho$ and the common terminating vertex by $t\varrho$. A set $R$ of relations is called *minimal* if $\varrho$ does not belong to the ideal $\langle R \setminus \{\varrho\} \rangle$ of $k\Delta$ generated by $R \setminus \{\varrho\}$ for every $\varrho \in R$. A pair $(\Delta, R)$ consisting of a quiver $\Delta$ and a minimal set of relations $R$ such that there exists $n \in \mathbb{N}$ with $\sigma \in \langle R \rangle$ for each path $\sigma$ in $\Delta$ of length at least $n$ is called a *bound quiver*. If $(\Delta, R)$ is a bound quiver, then the algebra $k\Delta/\langle R \rangle$ is called the *bound quiver algebra* of $(\Delta, R)$.

Let $(\Delta, R)$ be a bound quiver and assume that $R$ consists of paths. A path $\sigma$ in $\Delta$ is said to be a *path* in $(\Delta, R)$ if $\sigma \notin \langle R \rangle$ (in other words, none of the paths from $R$ is a subpath of $\sigma$). A path $\sigma$ in $(\Delta, R)$ is said to be *maximal* if there is no $\alpha \in \Delta_1$ such that either $s\alpha = t\sigma$ and $\alpha\sigma \notin \langle R \rangle$ or $t\alpha = s\sigma$ and $\sigma\alpha \notin \langle R \rangle$. Again we define maximal paths starting and terminating at a given vertex. If additionally $R$ consists of paths of length two, then we say that $\alpha \in \Delta_1$ is a *free arrow* provided there exists no $\beta \in \Delta_1$ such that either $s\beta = t\alpha$ and $\beta\alpha \in R$, or $t\beta = s\alpha$ and $\alpha\beta \in R$.

Following [4] we say that a connected bound quiver $(\Delta, R)$ is *gentle* if the following conditions are satisfied:

(1) for each $x \in \Delta_0$ there are at most two arrows $\alpha$ such that $s\alpha = x$ ($t\alpha = x$),
(2) $R$ consists of paths of length two,

(3) for each $\alpha \in \Delta_1$ there is at most one arrow $\beta$ such that $t\beta = s\alpha$ and $\alpha\beta \notin R$ (of $s\beta = t\alpha$ and $\beta\alpha \notin R$),
(4) for each $\alpha \in \Delta_1$ there is at most one arrow $\beta$ such that $t\beta = s\alpha$ and $\alpha\beta \in R$ ($s\beta = t\alpha$ and $\beta\alpha \in R$).

An algebra which is isomorphic to the bound quiver algebra of a gentle bound quiver is called gentle.

With an abelian category $\mathcal{A}$ we may associate its *bounded derived category* $\mathcal{D}^b(\mathcal{A})$ in the following way (see for example [29] for details). The objects of $\mathcal{D}^b(\mathcal{A})$ are the bounded complexes of objects of $\mathcal{A}$ and the morphisms are obtained from the morphisms in the homotopy category by formally inversing the quasi-isomorphisms (more precisely, by localizing with respect to the quasi-isomorphisms), where by a *quasi-isomorphism* we mean a morphism of complexes which induces an isomorphism of homology groups. The derived category together with the shift functor sending $X$ to the shifted complex $X[1]$, where $X[1]_n = X_{n+1}$ and $d^n_{X[1]} = -d^{n+1}_{X[1]}$ for $n \in \mathbb{Z}$, is a triangulated category (see for example [17]). We say that two abelian categories $\mathcal{A}$ and $\mathcal{B}$ are *derived equivalent* if there exists a triangle equivalence

$$\mathcal{D}^b(\mathcal{A}) \to \mathcal{D}^b(\mathcal{B}).$$

We say that two algebras $\Lambda$ and $\Lambda'$ (bound quivers $(\Delta, R)$ and $(\Delta', R')$) are derived equivalent if their categories of modules (representations, respectively) are derived equivalent. It follows from [26, Corollary 1.2] and [7, Corollary 15] that for $c \in \mathbb{Z}$ the gentle $c$-cycle algebras (bound quivers) are closed under derived equivalences.

Recall from [12, 19] that if $\Lambda$ is an algebra, then a $\Lambda$-module $T$ is called *tilting* if $\operatorname{pd}_\Lambda T \leq 1$, $\operatorname{Ext}^1_\Lambda(T, T) = 0$, and $T$ is a direct sum of $n$ pairwise nonisomorphic indecomposable modules, where $n$ is the rank of the Grothendieck group of $\Lambda$. Dually, we define the notion of a cotilting module. Algebras $\Lambda$ and $\Lambda'$ are said to be *tilting-cotilting equivalent* if there exists a sequence $\Lambda = \Lambda_0$, $\Lambda_1, \ldots, \Lambda_n = \Lambda'$ of algebras such that for each $i \in [0, n-1]$ there exists a (co)tilting $\Lambda_{i-1}$-module $T_{i-1}$ such that $\Lambda_i \simeq \operatorname{End}_{\Lambda_{i-1}}(T_{i-1})^{\mathrm{op}}$. It was proved by Happel [16, Corollary 1.7] that if $\Lambda$ and $\Lambda'$ are tilting-cotilting equivalent, then they are derived equivalent.

A vertex $x$ in a quiver $\Delta$ is called a *sink* (*source*) if there is no $\alpha \in \Delta_1$ with $s\alpha = x$ ($t\alpha = x$, respectively). If $x$ is a sink in a gentle bound quiver $(\Delta, R)$, then we define a new gentle bound quiver $(\Delta', R')$, called the bound quiver obtained from $(\Delta, R)$ by applying the *APR-reflection* at $x$, in the following way: $\Delta'_0 = \Delta_0$, $\Delta'_1 = \Delta_1$,

$$s_{\Delta'}\alpha = \begin{cases} x & \text{if } t_\Delta\alpha = x, \\ s_\Delta\alpha & \text{otherwise,} \end{cases}$$

$$t_{\Delta'}\alpha = \begin{cases} s_\Delta\alpha & \text{if } t_\Delta\alpha = x, \\ x & \text{if } \exists\beta \in \Delta_1 : t_\Delta\beta = x \wedge s_\Delta\beta = t_\Delta\alpha \wedge \beta\alpha \in R, \\ t_\Delta\alpha & \text{otherwise,} \end{cases}$$

and

$$R' = \{\varrho \in R \mid t_\Delta\varrho \neq x\} \cup \{\alpha\beta \mid t_\Delta\alpha = x \\ \wedge \exists\gamma \in \Delta_1 : \gamma \neq \alpha \wedge t_\Delta\gamma = x \wedge s_\Delta\gamma = t_\Delta\beta \wedge \gamma\beta \in R\}.$$

It follows that the bound quiver algebra of $(\Delta', R')$ is isomorphic to the opposite algebra of the endomorphism algebra of the APR-tilting module (see [6]) at $x$ defined as

$$\bigoplus_{\substack{a\in\Delta_0 \\ a\neq x}} P(a) \oplus \Big(\bigoplus_{\substack{\alpha\in\Delta_1 \\ t\alpha=x}} P(s\alpha)\Big)/P(x)$$

(see [4, 2.1]).

We now present a generalization of the above construction due to Brenner and Butler (see [13, Chapter 2]). Let $x$ be a vertex in a gentle bound quiver $(\Delta, R)$ such that there is no $\alpha \in \Delta_1$ with $s\alpha = x = t\alpha$ and for each $\alpha \in \Delta_1$ with $s\alpha = x$ there exists $\beta_\alpha \in \Delta_1$ with $t\beta_\alpha = x$ and $\alpha\beta_\alpha \notin R$. We define a bound quiver $(\Delta', R')$ in the following way: $\Delta'_0 = \Delta_0$, $\Delta'_1 = \Delta_1$,

$$s_{\Delta'}\alpha = \begin{cases} x & \text{if } t_\Delta\alpha = x, \\ s\beta_\alpha & \text{if } s_\Delta\alpha = x, \\ s_\Delta\alpha & \text{otherwise,} \end{cases}$$

$$t_{\Delta'}\alpha = \begin{cases} s_\Delta\alpha & \text{if } t_\Delta\alpha = x, \\ x & \text{if } \exists\beta \in \Delta_1 : t_\Delta\beta = x \wedge s_\Delta\beta = t_\Delta\alpha \wedge \beta\alpha \in R, \\ t_\Delta\alpha & \text{otherwise,} \end{cases}$$

and set

$$R' = \{\alpha\beta \in R \mid t_\Delta\alpha \neq x \wedge s_\Delta\alpha \neq x\} \cup \{\alpha\beta_\alpha \mid s_\Delta\alpha = x\} \\ \cup \{\alpha\beta \mid t_\Delta\alpha = x \wedge \exists\gamma \in \Delta_1 : \gamma \neq \alpha \wedge t_\Delta\gamma = x \wedge s_\Delta\gamma = t_\Delta\beta \wedge \gamma\beta \in R\}.$$

We will say that $(\Delta', R')$ is obtained from $(\Delta, R)$ by applying the *generalized APR-reflection* at $x$. As in the previous situation, it follows easily that the bound quiver algebra of $(\Delta', R')$ is the opposite algebra of the endomorphism algebra of the tilting module defined in the same way as before. Obviously all APR-reflections are generalized APR-reflections.

We also have a version of the above construction for a vertex $x$ of a gentle bound quiver $(\Delta, R)$ such that there exists $\alpha \in \Delta_1$ with $s\alpha = x = t\alpha$. Observe that then $\alpha^2 \in R$. We additionally assume that there exists $\beta_0 \in \Delta_1$ with $s\beta_0 \neq x$ and $t\beta_0 = x$. We define a bound quiver $(\Delta', R')$ in the following way: $\Delta'_0 = \Delta_0$, $\Delta'_1 = \Delta_1$,

$$s_{\Delta'}\alpha = \begin{cases} x & \text{if } t_\Delta\alpha = x, \\ s_\Delta\beta_0 & \text{if } s_\Delta\alpha = x \wedge t_\Delta\alpha \neq x, \\ s_\Delta\alpha & \text{otherwise,} \end{cases}$$

$$t_{\Delta'}\alpha = \begin{cases} s_\Delta\alpha & \text{if } t_\Delta\alpha = x, \\ x & \text{if } \exists\beta \in \Delta_1 : t_\Delta\beta = x \wedge s_\Delta\beta = t_\Delta\alpha \wedge \beta\alpha \in R, \\ t_\Delta\alpha & \text{otherwise,} \end{cases}$$

and $R' = R$. We will say again that $(\Delta', R')$ is obtained from $(\Delta, R)$ by applying the generalized APR-reflection at $x$. It follows that the bound quiver algebra of $(\Delta', R')$ is the opposite algebra of the endomorphism algebra of the tilting module

$$\bigoplus_{\substack{a\in\Delta_0 \\ a\neq x}} P(a) \oplus (P(y) \oplus P(y))/P(x),$$

where $y = s\beta_0$ and $P(x)$ is embedded in $P(y) \oplus P(y)$ in such a way that the quotient module is indecomposable.

Let again $x$ be a sink in a gentle bound quiver $(\Delta, R)$. We define the *HW-reflection* of $(\Delta, R)$ at $x$ as the bound quiver $(\Delta', R')$ constructed in the following way. If $\Delta_0 = \{x\}$ (equivalently, $\Delta_1 = \emptyset$), then $(\Delta', R') = (\Delta, R)$, hence assume this is not the case. Then we put $\Delta'_0 = \Delta_0$ and $\Delta'_1 = \Delta_1$. For each arrow $\alpha$ terminating at $x$ let $\beta_\alpha$ be the starting arrow of the maximal path in $(\Delta, R)$ terminating at $x$ whose terminating arrow is $\alpha$. We put

$$s_{\Delta'}\alpha = \begin{cases} x & \text{if } t_\Delta\alpha = x, \\ s_\Delta\alpha & \text{otherwise,} \end{cases} \qquad t_{\Delta'}\alpha = \begin{cases} s_\Delta\beta_\alpha & \text{if } t_\Delta\alpha = x, \\ t_\Delta\alpha & \text{otherwise.} \end{cases}$$

Finally, let

$$\begin{aligned} R' = \{\varrho \in R \mid t_\Delta\varrho \neq x\} \\ \cup \{\beta\alpha \mid t_\Delta\alpha = x \wedge s_\Delta\beta = s_\Delta\beta_\alpha \wedge \beta \neq \beta_\alpha \wedge t_\Delta\beta \neq x\}. \end{aligned}$$

It is known that the bound quiver algebra of $(\Delta', R')$ is (isomorphic to) the algebra obtained from the bound quiver algebra of $(\Delta, R)$ by the HW-reflection at $x$ (defined in [21]), hence in particular it is tilting-cotilting equivalent to $(\Delta, R)$ (see [28]). Dually, one defines the quiver obtained from $(\Delta, R)$ by applying the HW-coreflection at a source.

Before we present basic applications of the above transformations, we describe one more construction. Let $\Sigma$ be a subquiver of a quiver $\Delta$. Assume that $\Sigma'$ is a quiver such that $\Sigma'_0 = \Sigma_0$ and $\Sigma'_1 = \Sigma_1$ (but, usually, $s_{\Sigma'} \neq s_\Sigma$ and $t_{\Sigma'} \neq t_\Sigma$). We say that a quiver $\Delta'$ is obtained from $\Delta$ by *replacing* $\Sigma$

*by* $\Sigma'$ if $\Delta'_0 = \Delta_0$, $\Delta'_1 = \Delta_1$, and

$$s_{\Delta'}\alpha = \begin{cases} s_\Delta\alpha & \text{if } \alpha \in \Delta_1 \setminus \Sigma_1, \\ s_{\Sigma'}\alpha & \text{if } \alpha \in \Sigma_1, \end{cases} \qquad t_{\Delta'}\alpha = \begin{cases} t_\Delta\alpha & \text{if } \alpha \in \Delta_1 \setminus \Sigma_1, \\ t_{\Sigma'}\alpha & \text{if } \alpha \in \Sigma_1, \end{cases}$$

for $\alpha \in \Delta_1$.

We now describe operations of shifting relations.

LEMMA 1.1. *If*

$$\Sigma = \underset{u}{\bullet} \xleftarrow{\alpha_1} \underset{x}{\bullet} \xleftarrow{\alpha_2} \underset{y}{\bullet} \xleftarrow{\alpha_3} \underset{v}{\bullet}$$

*is a subquiver of a gentle bound quiver* $(\Delta, R)$ *such that* $\alpha_1\alpha_2 \in R$, $\alpha_2\alpha_3 \notin R$, *and there are no other arrows adjacent to* $y$, *then* $(\Delta, R)$ *is tilting-cotilting equivalent to the bound quiver* $(\Delta', R')$, *where* $R' = (R \setminus \{\alpha_1\alpha_2\}) \cup \{\alpha_2\alpha_3\}$ *and* $\Delta'$ *is obtained from* $\Delta$ *by replacing* $\Sigma$ *by the quiver*

$$\underset{u}{\bullet} \xleftarrow{\alpha_1} \underset{y}{\bullet} \xleftarrow{\alpha_2} \underset{x}{\bullet} \xleftarrow{\alpha_3} \underset{z}{\bullet}\,.$$

*Proof.* Apply the generalized APR-coreflection at $y$. ∎

We remark that it may happen that one of the following equalities holds: $u = y$, $x = v$ or $u = v$. Moreover, $u = y$ if and only if $x = v$, and in this case $\alpha_1 = \alpha_3$. We call the above operation *shifting the relation* $\alpha_1\alpha_2$ *to the right.* Dually, one defines the operation of shifting relations to the left.

We will need the following generalization of the above lemma.

LEMMA 1.2. *If*

$$\Sigma = \underset{u}{\bullet} \xleftarrow{\alpha_1} \underset{x}{\bullet} \xleftarrow{\alpha_2} \underset{y_n}{\bullet} \xrightarrow{\beta_n} \underset{y_{n-1}}{\bullet} \longrightarrow \cdots \longrightarrow \underset{y_1}{\bullet} \xrightarrow{\beta_1} \underset{y_0}{\bullet} \xleftarrow{\alpha_3} \underset{v}{\bullet}\,, \qquad n \in \mathbb{N}_+,$$

*is a subquiver of a gentle bound quiver* $(\Delta, R)$ *such that* $\alpha_1\alpha_2 \in R$, $\beta_1, \ldots, \beta_n$ *are free arrows, and there are no other arrows adjacent to* $x, y_0, \ldots, y_n$, *then* $(\Delta, R)$ *is tilting-cotilting equivalent to the bound quiver* $(\Delta', R')$, *where* $R' = (R \setminus \{\alpha_1\alpha_2\}) \cup \{\alpha_2\alpha_3\}$ *and* $\Delta'$ *is obtained from* $\Delta$ *by replacing* $\Sigma$ *by the quiver*

$$\underset{u}{\bullet} \xleftarrow{\alpha_1} \underset{y_0}{\bullet} \xrightarrow{\beta_1} \underset{y_1}{\bullet} \longrightarrow \cdots \longrightarrow \underset{y_{n-1}}{\bullet} \xrightarrow{\beta_n} \underset{y_n}{\bullet} \xleftarrow{\alpha_2} \underset{x}{\bullet} \xleftarrow{\alpha_3} \underset{z}{\bullet}\,.$$

*Proof.* We leave it to the reader to verify that the following sequence of operations leads from $(\Delta, R)$ to $(\Delta', R')$: first for each $i = n, \ldots, 1$ we apply the APR-coreflections at $y_i, \ldots, y_n, x$, and next we apply the generalized APR-coreflections at $y_0, \ldots, y_n$. ∎

We will also shift a group of relations in the following sense.

LEMMA 1.3. *Let*

$$\Sigma = \underset{y}{\bullet} \xrightarrow{\beta} \underset{x_0}{\bullet} \xleftarrow{\alpha_1} \underset{x_1}{\bullet} \longleftarrow \cdots \longleftarrow \underset{x_{n-1}}{\bullet} \xleftarrow{\alpha_n} \underset{x_n}{\bullet}\,, \qquad n \geq 2,$$

*be a subquiver of a gentle bound quiver* $(\Delta, R)$ *such that* $\beta$ *is a free arrow,* $\alpha_i\alpha_{i+1} \in R$ *for all* $i \in [1, n-1]$, *and there are no other arrows adjacent to* $x_0, \ldots, x_{n-1}$. *If there is no* $\alpha \in \Delta_1$ *with* $t\alpha = x_n$ *and* $\alpha_n\alpha \in R$, *then* $(\Delta, R)$ *is tilting-cotilting equivalent to the bound quiver* $(\Delta', R')$, *where* $R' = (R \setminus \{\alpha_{n-1}\alpha_n\}) \cup \{\beta\alpha_1\}$ *and* $\Delta'$ *is obtained from* $\Delta$ *by replacing* $\Sigma$ *by the quiver*

$$\underset{y}{\bullet} \xleftarrow{\beta} \underset{x_1}{\bullet} \xleftarrow{\alpha_1} \underset{x_2}{\bullet} \longleftarrow \cdots \longrightarrow \underset{x_{n-1}}{\bullet} \xleftarrow{\alpha_{n-1}} \underset{x_0}{\bullet} \xrightarrow{\alpha_n} \underset{x_n}{\bullet} \,.$$

*Proof.* We apply the APR-reflection at $x_0$, followed by the composition of the APR-reflection at $x_i$ and the generalized APR-reflection at $x_0$ applied for $i = 1, \ldots, n-1$. ∎

Observe that in the above lemma we shift relations to the left. Dually we define the operation of shifting a group of relations to the right.

We now present a reduction, resulting from the above lemmas, which will appear a few times in our proofs. Let

$$\Sigma = \underset{x_0}{\bullet} \overset{\alpha_1}{\text{———}} \underset{x_1}{\bullet} \text{———} \cdots \text{———} \underset{x_{n-1}}{\bullet} \overset{\alpha_n}{\text{———}} \underset{x_n}{\bullet}\,, \qquad n \in \mathbb{N}_+,$$

be a subquiver of a gentle bound quiver $(\Delta, R)$ such that there are no other arrows adjacent to $x_1, \ldots, x_{n-1}$ (it may happen that $x_0 = x_n$). We divide $\Sigma_1$ into two disjoint subsets $\Sigma_{1,+}$ and $\Sigma_{1,-}$ in such a way that, for each $i \in [1, n-1]$, $\alpha_i$ and $\alpha_{i+1}$ belong to the same subset if and only if either $s\alpha_i = t\alpha_{i+1}$ or $t\alpha_i = s\alpha_{i+1}$. We additionally assume that there exists $\varepsilon \in \{-,+\}$ such that $\alpha\beta \notin R$ for all $\alpha, \beta \in \Sigma_{1,\varepsilon}$ with $s\alpha = t\beta$. If $x_0 = t\alpha_1$, then by applying APR-reflections and shifts of relations (we leave the details to the reader), hence by passing to a tilting-cotilting equivalent bound quiver, we may replace $\Sigma$ by the quiver

$$\underset{x_0}{\bullet} \xleftarrow{\alpha'_1} \cdots \xleftarrow{\alpha'_{l_1}} \bullet \xrightarrow{\alpha''_{l_2}} \cdots \xrightarrow{\alpha''_1} \bullet \xleftarrow{\alpha'''_1} \cdots \xleftarrow{\alpha'''_{l_3}} \underset{x_n}{\bullet}$$

for some $l_1, l_2, l_3 \in \mathbb{N}$ with $l_1 + l_2 + l_3 = n$. Moreover, we may additionally assume that $l_3 = 0$ if either $x_n = t\alpha_n$ or $x_n = s\alpha_n$ and there is no $\alpha \in \Delta_1$ with $t\alpha = x_n$ and $\alpha_n\alpha \in R$. Obviously, we have the dual statement if $x_0 = s\alpha_1$.

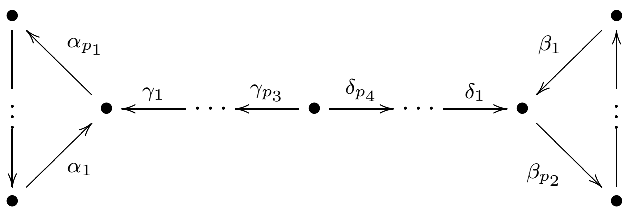
The next observation is the following.

Lemma 1.4. *If* $p_1, p_2 \in \mathbb{N}_+$, $p_3, p_4 \in \mathbb{N}$, *and* $r_1 \in [0, p_1 - 1]$ *are such that* $p_2 + p_3 \geq 2$ *and* $p_4 + r_1 \geq 1$, *then* $\Lambda_1(p_1, p_2, p_3, p_4, r_1)$ *and* $\Lambda_1(p_1 + p_2 - r_1 - 1, r_1 + 1, p_4, p_3, p_2 - 1)$ *are tilting-cotilting equivalent.*

*Proof.* This follows immediately by shifting relations. ∎

In order to formulate the next lemma we introduce a new family of algebras. Namely, for $p_1, p_2 \in \mathbb{N}_+$, $p_3, p_4 \in \mathbb{N}$, $r_1 \in [0, p_1 - 1]$, and $r_2 \in [0, p_2 - 1]$

such that $p_3 + p_4 + r_1 + r_2 \geq 1$, let $\Lambda_2'(p_1, p_2, p_3, p_4, r_1, r_2)$ be the algebra of the quiver




bound by $\alpha_i\alpha_{i+1}$ for $i \in [p_1 - r_1, p_1 - 1]$, $\alpha_{p_1}\alpha_1$, $\beta_i\beta_{i+1}$ for $i \in [p_2 - r_2, p_2 - 1]$, and $\beta_{p_2}\beta_1$.

LEMMA 1.5. *If $p_1, p_2, p_3 \in \mathbb{N}_+$, $p_4 \in \mathbb{N}$, $r_1 \in [0, p_1 - 1]$, and $r_2 \in [0, p_2 - 1]$, then $\Lambda_2'(p_1, p_2, p_3, p_4, r_1, r_2)$ and $\Lambda_2'(p_1, p_2, p_3 - 1, p_4 + 1, r_1, r_2)$ are tilting-cotilting equivalent.*

*Proof.* Put $a_i = s\delta_i$, $i \in [1, p_4]$, and $b_i = s\beta_i$, $i \in [1, p_2]$. We first apply the APR-coreflections at $a_{p_4}, \ldots, a_1$, followed by the generalized APR-coreflection at $b_{p_2}$ (we only apply the generalized APR-coreflection at $b_{p_2}$ if $p_4 = 0$). Next we apply the APR-coreflection at $b_{p_2 - i}$ followed by the generalized APR-coreflection at $b_{p_2}$ for $i = 1, \ldots, r_2$ (we do nothing in this step if $r_2 = 0$, hence in particular if $p_2 = 1$), and finally we apply the APR-coreflections at $b_{p_2 - r_2 - 1}, \ldots, b_1$ (there is nothing to do if $r_2 = p_2 - 1$, hence again if $p_2 = 1$). ∎

COROLLARY 1.6. *If $p_1, p_2 \in \mathbb{N}_+$, $p_3 \in \mathbb{N}$, $r_1 \in [0, p_1 - 1]$ and $r_2 \in [0, p_2 - 1]$, are such that $p_3 + r_1 + r_2 \geq 1$, then $\Lambda_2(p_1, p_2, p_3, r_1, r_2)$ and $\Lambda_2(p_2, p_1, p_3, r_2, r_1)$ are tilting-cotilting equivalent.*

*Proof.* This follows immediately from the above lemma, since it is easily seen that $\Lambda_2(p_1, p_2, p_3, r_1, r_2)$ and $\Lambda_2(p_2, p_1, p_3, r_2, r_1)$ are isomorphic to $\Lambda_2'(p_1, p_2, p_3, 0, r_1, r_2)$ and $\Lambda_2'(p_1, p_2, 0, p_3, r_1, r_2)$, respectively. ∎

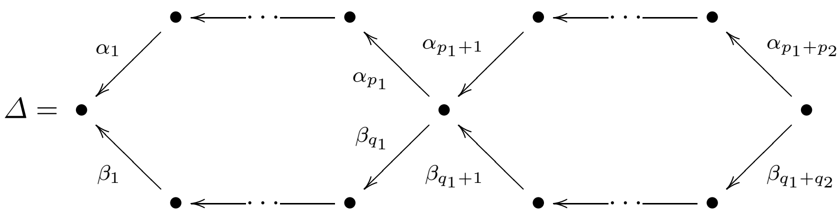
PROPOSITION 1.7. *If $\Lambda$ is one of the algebras listed in Theorems 1 and 2, then $\Lambda$ and $\Lambda^{\mathrm{op}}$ are tilting-cotilting equivalent.*

*Proof.* If either $\Lambda = \Lambda_0(p, r)$ for some $p \in \mathbb{N}_+$ and $r \in [0, p - 1]$, or $\Lambda = \Lambda_0'(p, 0)$ for some $p \in \mathbb{N}_+$, then the claim follows immediately by shifting relations. If $\Lambda = \Lambda_1(p_1, p_2, p_3, p_4, r_1)$ for some $p_1, p_2 \in \mathbb{N}_+$, $p_3, p_4 \in \mathbb{N}$, and $r_1 \in [0, p_1]$ such that $p_2 + p_3 \geq 2$ and $r_1 + p_4 \geq 1$, then we have to additionally apply APR-coreflections. Finally, if $\Lambda = \Lambda_2(p_1, p_2, p_3, r_1, r_2)$ for some $p_1, p_2 \in \mathbb{N}_+$, $p_3 \in \mathbb{N}$, $r_1 \in [0, p_1 - 1]$, and $r_2 \in [0, p_2 - 1]$ such that $p_3 + r_1 + r_2 \geq 1$, then the claim follows from Corollary 1.6. ∎

An important consequence of the above lemma is that in our considerations we may always replace an algebra by its opposite algebra. Indeed, if for an algebra $\Gamma$ we are able to prove that $\Gamma^{\mathrm{op}}$ is tilting-cotilting equivalent to an algebra $\Lambda$ listed in Theorems 1 and 2, then obviously $\Gamma$ is tilting-cotilting equivalent to $\Lambda^{\mathrm{op}}$, hence also to $\Lambda$. In particular, once Theorems 1 and 2 are proved, we know that if $\Gamma$ is a gentle two-cycle algebra, then $\Gamma$ and $\Gamma^{\mathrm{op}}$ are tilting-cotilting equivalent.

We finish this section by analyzing two particular families of gentle two-cycle bound quivers. First, we prove the following.

Proposition 1.8. *If $(\Delta, R)$ is a gentle bound quiver such that*




*for some $p_1, p_2, q_1, q_2 \in \mathbb{N}_+$, then the bound quiver algebra of $(\Delta, R)$ is tilting-cotilting equivalent to $\Lambda_0'(p, r)$ for some $p \in \mathbb{N}_+$ and $r \in [0, p-1]$.*

We first show that also in the proof of this theorem we may pass to opposite algebras.

Lemma 1.9. *If $p \in \mathbb{N}_+$ and $r \in [1, p-1]$, then $\Lambda_0'(p, r)$ and $\Lambda_0(p+1, r-1)$ are tilting-cotilting equivalent.*

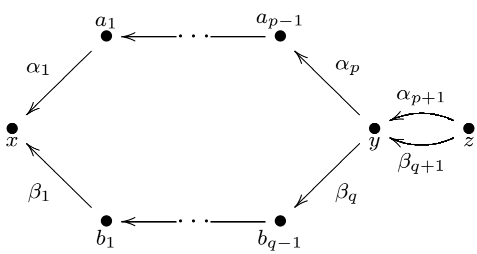
*Proof.* In order to prove this equivalence, we put $x = t\beta$, $z = s\delta$ and $x_1 = s\alpha_1$, and apply the APR-reflection at $x$ followed by the APR-coreflection at $z$ and the APR-reflection at $x_1$ to $\Lambda_0'(p, r)$. Then the claim follows by shifting relations. ∎

Corollary 1.10. *If $p \in \mathbb{N}_+$ and $r \in [0, p-1]$, then $\Lambda_0'(p, r)$ and $\Lambda_0'(p, r)^{\mathrm{op}}$ are tilting-cotilting equivalent.*

*Proof.* This follows either from Proposition 1.7 (if $r = 0$), or from the previous lemma and Proposition 1.7 (if $r > 0$). ∎

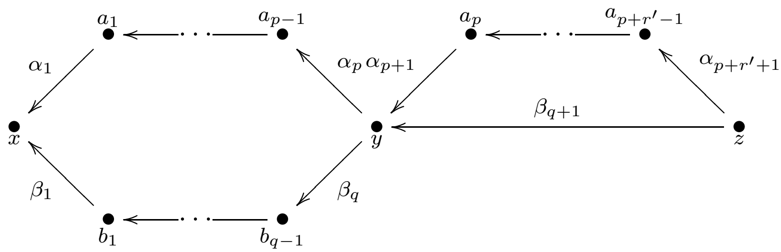
In the proof of Proposition 1.8 we will need the following families of algebras:

- $\Gamma_0(p, q, r)$ for $p, q \in \mathbb{N}_+$ and $r \in [0, p-1]$ is the algebra of the quiver

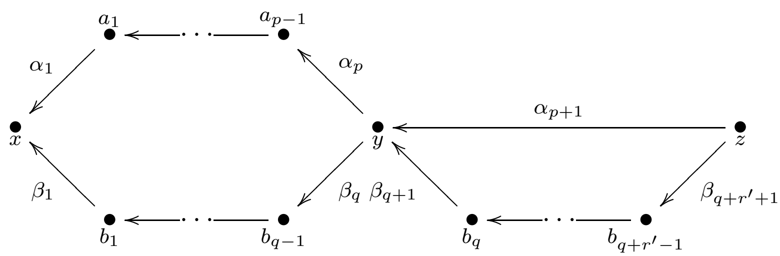



bound by $\alpha_i \alpha_{i+1}$ for $i \in [p-r, p]$ and $\beta_q \beta_{q+1}$,

- $\Gamma_1(p,q,r,r')$ for $p,q \in \mathbb{N}_+$, $r \in [0,p-1]$, and $r' \in \mathbb{N}$, is the algebra of the quiver




  bound by $\alpha_i\alpha_{i+1}$ for $i \in [p-r, p+r']$ and $\beta_q\beta_{q+1}$,
- $\Gamma_2(p,q,r,r')$ for $p,q \in \mathbb{N}_+$, $r \in [0,p-1]$, and $r' \in \mathbb{N}$ is the algebra of the quiver




  bound by $\alpha_i\alpha_{i+1}$ for $i \in [p-r, p]$ and $\beta_i\beta_{i+1}$ for $i \in [q, q+r']$,

and the following series of lemmas.

LEMMA 1.11. *If $p,q \in \mathbb{N}_+$, $r \in [0,p-1]$, and $q > 1$, then $\Gamma_0(p,q,r)$ is tilting-cotilting equivalent to $\Gamma_0(p+1,q-1,r)$.*

*Proof.* It is enough to apply the generalized APR-reflection at $b_{q-1}$, followed by the APR-coreflection at $z$, the generalized APR-coreflection at $y$, and the APR-coreflections at $b_{q-2},\dots,b_1$ (we omit the last step if $q=2$). ■

LEMMA 1.12. *If $p,q,\in \mathbb{N}_+$, $r \in [0,p-1]$, $r' \in \mathbb{N}$, and $r' \geq r$, then $\Gamma_1(p,q,r,r')$ is tilting-cotilting equivalent to $\Gamma_2(q+r'-r,p,r'-r,r)$.*

*Proof.* First for each $i \in [1,r]$ we apply the HW-coreflection at $z$ followed by the APR-reflection at $z$, and the generalized APR-coreflection at $a_{p+r'-i}$ applied $r+r'+1-i$ times. Next we apply the HW-coreflections at $z, a_{p+r'-r-1},\dots,a_p$ (only at $z$ if $r=r'$) and we obtain a bound quiver whose bound quiver algebra is easily seen to be tilting-cotilting equivalent to $\Gamma_2(q+r'-r,p,r'-r,r)$. ■

LEMMA 1.13. *If $p,q,\in \mathbb{N}_+$, $r \in [0,p-1]$, $r' \in \mathbb{N}$, and $r \geq r'$, then $\Gamma_1(p,q,r,r')$ is tilting-cotilting equivalent to $\Gamma_2(p+2r'-r,q,r',r-r')$.*

*Proof.* Since $\Gamma_1(p,q,r,r')$ is tilting-cotilting equivalent to $\Gamma_1(p+r'-r, q,r',r)^{\mathrm{op}}$ and $\Gamma_2(p+2r'-r,q,r',r-r')$ is tilting-cotilting equivalent to

$\Gamma_2(q+r-r', p+r'-r, r-r', r')^{\mathrm{op}}$, hence the claim follows from the previous lemma. ∎

LEMMA 1.14. *If $p, q \in \mathbb{N}_+$, $r \in [0, p-1]$, $r' \in \mathbb{N}$, and $r \geq r'$, then $\Gamma_2(p, q, r, r')$ is tilting-cotilting equivalent to $\Gamma_2(p, q, r-r', r')$.*

*Proof.* By applying the APR-coreflection at $z$ followed by the generalized APR-coreflection at $z$ applied $r'$ times, we replace $\Gamma_2(p, q, r, r')$ by (an algebra isomorphic to) the bound quiver algebra of the quiver

$$\begin{tikzcd}
 & a'_1 \arrow[dl, "\alpha'_1"'] & \cdots \arrow[l] & a_{p-r'-1'} \arrow[l] & & & & & & a'_{p-r'} \arrow[dl, "\alpha'_{p-r'+2}"'] & \cdots \arrow[l] & a'_{p-1} \arrow[l] \\
x' & & & & y' \arrow[ul, "\alpha'_{p-r'}"'] \arrow[dl, "\beta'_q"] & & & & z \arrow[llll, "\alpha'_{p-r'+1}"'] \arrow[dl, "\beta'_{q+r'+1}"] & & & \\
 & b_1 \arrow[ul, "\beta'_1"] & \cdots \arrow[l] & b'_{q-1} \arrow[l] & & b'_q \arrow[ul, "\beta'_{q+1}"] & \cdots \arrow[l] & b'_{q+r'-1} \arrow[l] & & & &
\end{tikzcd}$$

bound by $\alpha'_i\alpha'_{i+1}$ for $i \in [p-r, p]$ and $\beta'_i\beta'_{i+1}$ for $i \in [q, q+r']$. It is easily seen that this algebra is tilting-cotilting equivalent to $\Gamma_2(p, q, r-r', r')$ (we just shift relations sufficiently many times). ∎

LEMMA 1.15. *If $p, q \in \mathbb{N}$, $r \in [0, p-1]$, $r' \in \mathbb{N}$, and $r' \geq r$, then $\Gamma_2(p, q, r, r')$ is tilting-cotilting equivalent to $\Gamma_2(p, q+r, r, r'-r)$.*

*Proof.* Since $\Gamma_2(p, q, r, r')$ is tilting-cotilting equivalent to $\Gamma_2(q+r', p-r, r', r)^{\mathrm{op}}$ and $\Gamma_2(p, q+r, r, r'-r)$ is tilting-cotilting equivalent to $\Gamma_2(q+r', p-r, r'-r, r)^{\mathrm{op}}$, the claim follows from the previous lemma. ∎

*Proof of Proposition 1.8.* Without loss of generality we may assume that $\alpha_{p_1}\alpha_{p_1+1} \in R$ and $\beta_{q_1}\beta_{q_1+1} \in R$. We first show that either $\alpha_i\alpha_{i+1} \notin R$ for all $i \in [1, p_1-1]$, or $\beta_i\beta_{i+1} \notin R$ for all $i \in [1, q_1-1]$. Assume this is not the case. In particular, $p_1, q_1 \geq 2$. By shifting relations we may assume that $\alpha_1\alpha_2 \in R$ and $\beta_1\beta_2 \in R$. If $(\Delta', R')$ is the quiver obtained from $(\Delta, R)$ by applying the HW-reflection at $x$ followed by the APR-reflection at $x$, where $x = t\alpha_1$, then $\Delta' = \Delta$ and $R' = R \setminus \{\alpha_1\alpha_2, \beta_1\beta_2\}$, hence the claim follows by induction. Similarly, we prove that either $\alpha_i\alpha_{i+1} \notin R$ for all $i \in [p_1+1, p_1+p_2-1]$, or $\beta_i\beta_{i+1} \notin R$ for all $i \in [q_1+1, q_1+q_2-1]$. Consequently, by shifting relations one easily observes that the bound quiver algebra of $(\Delta, R)$ is tilting-cotilting equivalent either to $\Gamma_1(p, q, r, r')$ or to $\Gamma_2(p, q, r, r')$ for some $p, q \in \mathbb{N}_+$, $r \in [0, p-1]$, and $r' \in \mathbb{N}$. Since

$$\Gamma_1(p, q, r, 0) = \Gamma_0(p, q, r) = \Gamma_2(p, q, r, 0)$$

for all $p, q \in \mathbb{N}_+$ and $r \in \mathbb{N}$, $\Gamma_1(p, q, 0, r') \simeq \Gamma_0(p+r', q, r')^{\mathrm{op}}$ and

$$\Gamma_2(p, q, 0, r') \simeq \Gamma_0(q+r', p, r')^{\mathrm{op}}$$

for all $p, q \in \mathbb{N}_+$ and $r' \in \mathbb{N}$, and $\Gamma_0(p, 1, r)$ is tilting-cotilting equivalent to $\Lambda_0'(p, r)$ for all $p \in \mathbb{N}_+$ and $r \in [0, p-1]$, the claim follows from the above series of lemmas. ∎

We finish this section with the following.

PROPOSITION 1.16. *If $(\Delta, R)$ is a gentle bound quiver such that*

$$\Delta = \begin{tikzcd} & x_1 \arrow[dl, "\alpha_1"'] & \cdots \arrow[l] & x_{p_1-1} \arrow[l] & \\ u \arrow[r, "\beta_{p_2}"] & y_{p_2-1} \arrow[r] & \cdots \arrow[r] & y_1 \arrow[r, "\beta_1"] & v \arrow[ul, "\alpha_{p_1}"'] \arrow[dl, "\gamma_{p_3}"] \\ & z_1 \arrow[ul, "\gamma_1"] & \cdots \arrow[l] & z_{p_3-1} \arrow[l] & \end{tikzcd}$$

*for some $p_1, p_2, p_3 \in \mathbb{N}_+$, and $\beta_{p_2}\alpha_1, \gamma_{p_3}\beta_1 \in R$, then the bound quiver algebra of $(\Delta, R)$ is tilting-cotilting equivalent to $\Lambda_0(p, r)$ for some $p \in \mathbb{N}_+$ and $r \in [0, p-1]$.*

*Proof.* Let $r_1$ be the number of $i \in [1, p_1 - 1]$ such that $\alpha_i\alpha_{i+1} \in R$, let $r_2$ be the number of $i \in [1, p_2 - 1]$ such that $\beta_i\beta_{i+1} \in R$, and let $r_3$ be the number of $i \in [1, p_3 - 1]$ such that $\gamma_i\gamma_{i+1} \in R$. We prove the claim by induction on $r_1 + r_2 + r_3$.

If $r_1 = 0 = r_3$, then it follows by shifting relations that the bound quiver algebra of $(\Delta, R)$ is tilting-cotilting equivalent to $\Lambda_0(p_1 + p_2 + p_3 - 2, r_2)$.

If $r_1 > 0$ and $r_3 = 0$, then by shifting relations we may assume that $p_3 = 1$ and $\alpha_1\alpha_2 \in R$. If $(\Delta', R')$ is the bound quiver obtained from $(\Delta, R)$ by applying the generalized APR-reflection at $u$ followed by the APR-reflection at $x_1$, then $R' = (R \setminus \{\alpha_1\alpha_2, \beta_{p_2}\alpha_1, \gamma_1\beta_1\}) \cup \{\gamma_1\alpha_2, \beta_{p_2}\gamma_1\}$ and

$$\Delta' = \begin{tikzcd} & x_2 \arrow[dl, "\alpha_2"'] & & \cdots \arrow[ll] & x_{p_1-1} \arrow[l] & \\ u \arrow[rrrrr, "\gamma_1"] & & & & & v \arrow[ul, "\alpha_{p_1}"'] \arrow[dl, "\beta_{p_2}"] \\ & x_1 \arrow[ul, "\alpha_1"] & y_1 \arrow[l, "\beta_1"'] & \cdots \arrow[l] & y_{p_2-1} \arrow[l] & \end{tikzcd}$$

hence the claim follows by induction. Dually, the claim follows if $r_1 = 0$ and $r_3 > 0$.

Assume finally that $r_1 > 0$ and $r_3 > 0$. By shifting relations we may assume that $\alpha_1\alpha_2 \in R$ and $\gamma_1\gamma_2 \in R$. If $(\Delta', R')$ is obtained from $(\Delta, R)$ by applying the generalized APR-reflection at $u$ followed by the APR-reflection at $x_1$, then $R' = (R \setminus \{\alpha_1\alpha_2, \beta_{p_2}\alpha_1, \gamma_1\gamma_2\}) \cup \{\beta_{p_2}\gamma_1, \gamma_1\alpha_2\}$ and

$$\Delta' = \begin{tikzcd}
 & \underset{x_2}{\bullet} \arrow[dl, "\alpha_2"'] & & \cdots \arrow[ll] & \underset{x_{p_1-1}}{\bullet} \arrow[l] & \\
u\bullet \arrow[r, "\gamma_1"] & \underset{z_1}{\bullet} \arrow[r, "\beta_{p_2}"] & \underset{y_{p_2-1}}{\bullet} \arrow[r] & \cdots \arrow[r] & \underset{y_1}{\bullet} \arrow[r, "\beta_1"] & \bullet v \arrow[ul, "\alpha_{p_1}"'] \arrow[dl, "\gamma_{p_3}"] \\
 & \underset{x_1}{\bullet} \arrow[ul, "\alpha_1"] & \underset{z_2}{\bullet} \arrow[l, "\gamma_2"'] & \cdots \arrow[l] & \underset{z_{p_2-1}}{\bullet} \arrow[l] &
\end{tikzcd}$$

and the claim again follows by induction. ∎

**2. Completeness of the list.** We start our considerations in this section by extending the list of algebras in Theorems 1 and 2. Namely, as a consequence of Lemmas 1.4 and 1.9 and Corollary 1.6, to show the completeness of the lists in Theorems 1 and 2, it is enough to prove the following.

PROPOSITION 2.1. *If $(\Delta, R)$ is a gentle two-cycle bound quiver, then the bound quiver algebra of $(\Delta, R)$ is tilting-cotilting equivalent to one of the following algebras*:

- $\Lambda_0(p, r)$ *for some* $p \in \mathbb{N}_+$ *and* $r \in [0, p-1]$,
- $\Lambda'_0(p, r)$ *for some* $p \in \mathbb{N}_+$ *and* $r \in [0, p-1]$,
- $\Lambda_1(p_1, p_2, p_3, p_4, r_1)$ *for some* $p_1, p_2 \in \mathbb{N}_+$, $p_3, p_4 \in \mathbb{N}$, *and* $r_1 \in [0, p_1 - 1]$ *such that* $p_2 + p_3 \geq 2$ *and* $p_4 + r_1 \geq 1$,
- $\Lambda_2(p_1, p_2, p_3, r_1, r_2)$ *for some* $p_1, p_2 \in \mathbb{N}_+$, $p_3 \in \mathbb{N}$, $r_1 \in [0, p_1 - 1]$, $r_2 \in [0, p_2 - 1]$ *such that* $p_3 + r_1 + r_2 \geq 1$.

For the rest of the section we assume that $(\Delta, R)$ is a gentle two-cycle bound quiver. We show, in a sequence of steps, that the bound quiver algebra of $(\Delta, R)$ is tilting-cotilting equivalent to one of the algebras listed in the above proposition.

We divide the arrows in $\Delta$ into three disjoint groups:

- $\alpha \in \Delta_1$ is called a *cycle arrow* if the quiver $(\Delta_0, \Delta_1 \setminus \{\alpha\})$ is connected,
- $\alpha \in \Delta_1$ is called a *branch arrow* if the quiver $(\Delta_0, \Delta_1 \setminus \{\alpha\})$ has a connected component which is a two-cycle quiver,
- $\alpha \in \Delta_1$ is called a *connecting arrow* if the quiver $(\Delta_0, \Delta_1 \setminus \{\alpha\})$ has two connected components which are one-cycle quivers.

A vertex $x$ of $\Delta$ is called a *connecting vertex* if there exist at least three arrows adjacent to $x$ which are not branch arrows. We call $\alpha\beta \in R$ a *branch relation* if either $\alpha$ or $\beta$ is a branch arrow.

STEP 1. *We may assume that there are no branch relations in $R$.*

*Proof.* If there exists a branch relation in $(\Delta, R)$, then by passing to the opposite algebra if necessary, we may assume that there exists a subquiver

$$\Sigma = \underset{x_0}{\bullet} \xrightarrow{\ \alpha_1\ } \cdots \xrightarrow{\ \alpha_{n-2}\ } \underset{x_{n-2}}{\bullet} \xleftarrow{\ \alpha_{n-1}\ } \underset{x_{n-1}}{\bullet} \xleftarrow{\ \alpha_n\ } \underset{x_n}{\bullet}$$

of $\Delta$ for some $n \geq 2$, where $\alpha_1, \ldots, \alpha_{n-2}$ are free arrows, $\alpha_{n-1}\alpha_n \in R$, and there are no other arrows adjacent to $x_0, \ldots, x_{n-2}$ (in particular, $\alpha_{n-1}$ is a branch arrow, hence $\alpha_{n-1}\alpha_n$ is a branch relation). By applying APR-coreflections we may assume that $s\alpha_i = x_i$ for all $i \in [1, n-2]$. If $(\Delta', R')$ is the bound quiver obtained from $(\Delta, R)$ by applying the generalized APR-reflections at $x_{n-2}, \ldots, x_1$ followed by the APR-reflection at $x_0$, then $R' = R \setminus \{\alpha_{n-1}\alpha_n\}$ and $\Delta'$ is obtained from $\Delta$ by replacing $\Sigma$ by the quiver

$$\underset{x_{n-1}}{\bullet} \xleftarrow{\ \alpha_1\ } \underset{x_0}{\bullet} \xleftarrow{\ \alpha_2\ } \cdots \xleftarrow{\ \alpha_{n-1}\ } \underset{x_{n-2}}{\bullet} \xleftarrow{\ \alpha_n\ } \underset{x_n}{\bullet} .$$

In particular, the number of branch relations decreases, hence the claim follows by induction. ∎

By a *branch* in $\Delta$ we mean a maximal nontrivial (i.e. with nonempty set of arrows) connected subquiver of $\Delta$ all of whose arrows are branch arrows. We say that a branch $B$ in $\Delta$ is *rooted* at $x$ if $x \in B_0$ and there exists $\alpha \in \Delta_1$ adjacent to $x$ which is not a branch arrow. An immediate consequence of the assumption made in the above step is that each branch $B$ in $\Delta$ is a linear quiver rooted at one of its ends. Moreover, by applying APR-reflections we may assume that $B$ is equioriented and rooted at its unique sink.

STEP 2. *We may assume that there are no branch arrows in $\Delta$.*

*Proof.* We say that $x \in \Delta_0$ is an *insertion vertex* if either $x$ is a connecting vertex, or there exists $\alpha \in \Delta_1$ such that $s\alpha = x$, $\alpha$ is not a branch arrow, and there is no $\beta \in \Delta_1$ with $t\beta = x$ and $\alpha\beta \in R$. Observe that no branch is rooted at an insertion vertex. Moreover, for each $x \in \Delta_0$ there exists a path in $\Delta$ starting at an insertion vertex and terminating at $x$. In particular, if $B$ is a branch rooted at $x$, then we call the minimal length of such a path the distance between $B$ and an insertion vertex. We prove our claim by induction on the number of branches in $(\Delta, R)$ and, for a given branch $B$, by induction on the distance between $B$ and an insertion vertex.

Let

$$B = \underset{x_0}{\bullet} \xleftarrow{\ \alpha_1\ } \underset{x_1}{\bullet} \longleftarrow \cdots \longrightarrow \underset{x_{n-1}}{\bullet} \xleftarrow{\ \alpha_n\ } \underset{x_n}{\bullet} , \qquad n \in \mathbb{N}_+,$$

be a branch in $\Delta$. Let $\alpha$ and $\beta$ be the arrows in $\Delta$ with $s\alpha = x_0 = t\beta$ and $\beta \neq \alpha_1$. Observe that $\alpha\beta \in R$ and there are no other arrows adjacent to $x_0$. Put $y = t\alpha$ and $z = s\beta$.

Assume first that there is no $\gamma \in \Delta_1$ with $t\gamma = z$ and $\beta\gamma \in R$. If $(\Delta', R')$ is the bound quiver obtained from $(\Delta, R)$ by applying the generalized APR-reflections at $x_0, \ldots, x_{n-1}$, then $R' = (R \setminus \{\alpha\beta\}) \cup \{\alpha\alpha_n\}$ and $\Delta'$ is obtained from $\Delta$ by replacing the subquiver

$$\begin{tikzcd} & y & & & & \\ z \arrow[r, "\beta"] & x_0 \arrow[u, "\alpha"'] & x_1 \arrow[l, "\alpha_1"'] & \cdots \arrow[l] & x_{n-1} \arrow[l] & x_n \arrow[l, "\alpha_n"'] \end{tikzcd}$$

by the quiver

$$\begin{tikzcd} z & x_0 \arrow[l, "\beta"'] & x_1 \arrow[l, "\alpha_1"'] & \cdots \arrow[l] & x_{n-1} \arrow[l] \arrow[r, "\alpha_n"] & x_n \arrow[r, "\alpha"] & y \end{tikzcd},$$

hence the claim follows in this case.

Assume now that there exists $\gamma \in \Delta_1$ with $t\gamma = z$ and $\beta\gamma \in R$, and $z$ is a connecting vertex in $\Delta_1$. Put $v = s\gamma$. If $(\Delta', R')$ is the bound quiver obtained from $(\Delta, R)$ by applying the generalized APR-reflections at $x_0, \ldots, x_{n-1}$, then $R' = (R \setminus \{\alpha\beta, \beta\gamma\}) \cup \{\alpha\alpha_n, \alpha_n\gamma\}$ and $\Delta'$ is obtained from $\Delta$ by replacing the subquiver

$$\begin{tikzcd} & & y & & & & \\ v \arrow[r, "\gamma"] & z \arrow[r, "\beta"] & x_0 \arrow[u, "\alpha"'] & x_1 \arrow[l, "\alpha_1"'] & \cdots \arrow[l] & x_{n-1} \arrow[l] & x_n \arrow[l, "\alpha_n"'] \end{tikzcd}$$

by the quiver

$$\begin{tikzcd} & & & & v \arrow[d, "\gamma"] & & \\ z & x_0 \arrow[l, "\beta"'] & x_1 \arrow[l, "\alpha_1"'] & \cdots \arrow[l] & x_{n-1} \arrow[l] \arrow[r, "\alpha_n"] & x_n \arrow[r, "\alpha"] & y \end{tikzcd}$$

Observe that the assumption that $z$ is a connecting vertex in $\Delta$ implies that $\beta, \alpha_1, \ldots, \alpha_{n-1}$ are not branch arrows in $\Delta'$.

Finally, assume that there exists $\gamma \in \Delta_1$ with $t\gamma = z$ and $\beta\gamma \in R$, but $z$ is not a connecting vertex in $\Delta_1$. By induction we may assume that there is no branch rooted at $z$. If $(\Delta', R')$ is the bound quiver obtained from $(\Delta, R)$ by applying the HW-coreflection at $x_i$ followed by the APR-reflection at $x_i$ for $i = n, \ldots, 1$, then $R' = R$ and $\Delta'$ is obtained from $\Delta$ by replacing the subquiver

$$\begin{tikzcd} z \arrow[r, "\beta"] & x_0 & x_1 \arrow[l, "\alpha_1"'] & \cdots \arrow[l] & x_{n-1} \arrow[l] & x_n \arrow[l, "\alpha_n"'] \end{tikzcd}$$

by the quiver

$$\begin{tikzcd} x_0 & z \arrow[l, "\beta"'] & x_1 \arrow[l, "\alpha_1"'] & \cdots \arrow[l] & x_{n-1} \arrow[l] & x_n \arrow[l, "\alpha_n"'] \end{tikzcd},$$

and the claim follows by induction. ∎

We say that $\Delta$ is *special* if either there is a unique connecting vertex in $\Delta$, or there is a connecting arrow in $\Delta$. Otherwise, we call $\Delta$ *proper*. We now concentrate on the case when $\Delta$ is special. We first describe its structure more precisely. We may divide the cycle arrows of $\Delta$ into two disjoint subsets $\Delta_1^{(1)}$ and $\Delta_1^{(2)}$ in such a way that cycle arrows $\alpha$ and $\beta$ belong to the same subset if and only if the quiver $(\Delta_0, \Delta_1 \setminus \{\alpha, \beta\})$ has a connected component which is a one-cycle quiver. For $j \in [1,2]$ we denote by $\Delta^{(j)}$ the minimal subquiver of $\Delta$ with the set of arrows $\Delta_1^{(j)}$. Observe that $\Delta^{(j)}$ is a (not necessarily oriented) cycle. We divide the arrows in $\Delta^{(j)}$ into disjoint subsets $\Delta_{1,-}^{(j)}$ and $\Delta_{1,+}^{(j)}$ in such a way that if $\alpha, \beta \in \Delta_1^{(j)}$, $\alpha \neq \beta$, are adjacent to the same vertex, then they belong to the same subset if and only if either $s\alpha = t\beta$ or $t\alpha = s\beta$. For $\varepsilon \in \{-,+\}$ we put

$$R_\varepsilon^{(j)} = \{\alpha\beta \in R \mid \alpha, \beta \in \Delta_{1,\varepsilon}^{(j)}\}.$$

STEP 3. *If $\Delta$ is special, then we may assume that for each $j \in [1,2]$ there exists $\varepsilon \in \{-,+\}$ such that $R_\varepsilon^{(j)} = \emptyset$.*

*Proof.* If $\Delta^{(j)}$ is an oriented cycle, then there is nothing to prove, hence assume that $\Delta^{(j)}$ is not an oriented cycle and $R_-^{(j)} \neq \emptyset \neq R_+^{(j)}$. There exists a subquiver

$$\Sigma = \underset{y_1}{\bullet} \xleftarrow{\alpha_1} \underset{y_2}{\bullet} \xleftarrow{\alpha_2} \underset{x_0}{\bullet} \overset{\gamma_1}{\text{———}} \cdots \overset{\gamma_n}{\text{———}} \underset{x_n}{\bullet} \xrightarrow{\beta_2} \underset{z_2}{\bullet} \xrightarrow{\beta_1} \underset{z_1}{\bullet}$$

of $\Delta$ for some $n \in \mathbb{N}$ such that $\alpha_1\alpha_2 \in R_-^{(j)}$, $\beta_1\beta_2 \in R_+^{(j)}$, there are no other arrows adjacent to $x_0, \dots, x_n$, and $\gamma_1, \dots, \gamma_n$ are free arrows. By applying appropriate APR-reflections at $x_1, \dots, x_{n-1}$ (see the discussion after Lemma 1.3) we may assume that

$$\Sigma = \underset{y_1}{\bullet} \xleftarrow{\alpha_1} \underset{y_2}{\bullet} \xleftarrow{\alpha_2} \underset{x_0}{\bullet} \xleftarrow{\gamma_1} \cdots \xleftarrow{\gamma_k} \underset{x_k}{\bullet} \xrightarrow{\gamma_{k+1}} \cdots \xrightarrow{\gamma_n} \underset{x_n}{\bullet} \xrightarrow{\beta_2} \underset{z_2}{\bullet} \xrightarrow{\beta_1} \underset{z_1}{\bullet}$$

for some $k \in [0,n]$. By shifting the relations $\alpha_1\alpha_2$ and $\beta_1\beta_2$ to the right, we may assume that $n = 0$, i.e.

$$\Sigma = \underset{y_1}{\bullet} \xleftarrow{\alpha_1} \underset{y_2}{\bullet} \xleftarrow{\alpha_2} \underset{x}{\bullet} \xrightarrow{\beta_2} \underset{z_2}{\bullet} \xrightarrow{\beta_1} \underset{z_1}{\bullet} .$$

Assume first that neither $y_2$ nor $z_2$ is a connecting vertex. If $(\Delta', R')$ is the bound quiver obtained from $(\Delta, R)$ by applying the APR-coreflections at $x$, $y_2$, and $z_2$, then $R' = R \setminus \{\alpha_1\alpha_2, \beta_1\beta_2\}$ and $\Delta'$ is obtained from $\Delta$ by replacing $\Sigma$ by the quiver

$$\underset{y_1}{\bullet} \xleftarrow{\alpha_1} \underset{z_2}{\bullet} \xleftarrow{\beta_2} \underset{x}{\bullet} \xrightarrow{\alpha_2} \underset{y_2}{\bullet} \xrightarrow{\beta_1} \underset{z_1}{\bullet} ,$$

and the claim follows by induction. Otherwise, we may assume without loss of generality that $y_2$ is a connecting vertex and $z_2$ is not a connecting vertex. If $(\Delta', R')$ is the bound quiver obtained from $(\Delta, R)$ by applying the APR-coreflections at $x$ and $z_2$, then $R' = (R \setminus \{\alpha_1\alpha_2, \beta_1\beta_2\}) \cup \{\beta_1\alpha_2\}$ and $\Delta'$ is obtained from $\Delta$ by replacing $\Sigma$ by the quiver

$$\begin{tikzcd} & & y_2 \arrow[d, "\alpha_2"] & \\ y_1 & z_2 \arrow[l, "\alpha_1"'] & x \arrow[l, "\beta_2"'] \arrow[r, "\beta_1"] & z_1 \end{tikzcd}$$

Observe that $\alpha_2$ is a connecting arrow in $\Delta'$, hence the claim again follows by induction. ■

STEP 4. *If $\Delta$ is special, then for each $j \in [1, 2]$ we may assume that either $\Delta^{(j)}$ is an oriented cycle, or there is a unique source* (*equivalently, unique sink*) *in $\Delta^{(j)}$.*

*Proof.* This follows easily by applying APR-reflections and shifts of relations (see the discussion after Lemma 1.3). ■

STEP 5. *If $\Delta$ is special, then we may assume that either there is no connecting arrow in $\Delta$, or, for each $j \in [1, 2]$, $\Delta^{(j)}$ is an oriented cycle and $\alpha\beta \in R$ for all $\alpha, \beta \in \Delta_1^{(j)}$ with $s\alpha = t\beta$.*

*Proof.* We prove the claim by induction on the sum of the number of connecting arrows and the number of connecting relations, where we say that $\alpha\beta \in R$ is a *connecting relation* if both $\alpha$ and $\beta$ are connecting arrows. We may assume without loss of generality that either $\Delta^{(1)}$ is not an oriented cycle, or there exist $\alpha, \beta \in \Delta_1^{(1)}$ with $s\alpha = t\beta$ and $\alpha\beta \notin R$. Let $x \in \Delta_0^{(1)}$ be a connecting vertex. Let $\alpha$ be the connecting arrow adjacent to $x$. Without loss of generality we may assume that $x = s\alpha$. Let $\beta$ and $\gamma$ be the arrows adjacent to $x$ different from $\alpha$. Again we may assume without loss of generality that $x = t\beta$. By symmetry we may also assume that $\alpha\beta \in R$ if $x = t\gamma$. Put $y = t\alpha$ and $z = s\beta$. In order to make it easier to follow the proof we will number the cases.

(1) Assume that $\alpha\beta \notin R$. According to our assumptions this implies that $x = s\gamma$ and $\gamma\beta \in R$. Put $v = t\gamma$. If $\Delta^{(1)}$ is not an oriented cycle, then by applying APR-reflections and the dual of Lemma 1.2 we may assume that $v$ is a sink. In particular, there is no $\gamma' \in \Delta_1$ with $s\gamma' = v$ and $\gamma'\gamma \in R$. By shifting relations we may also assume that this condition is satisfied if $\Delta^{(1)}$ is an oriented cycle. Let $(\Delta', R')$ be the bound quiver obtained from $(\Delta, R)$ by applying the generalized APR-coreflection at $x$. If there is no $\alpha' \in \Delta_1$ with $s\alpha' = y$ and $\alpha'\alpha \in R$, then $R' = (R \setminus \{\gamma\beta\}) \cup \{\alpha\beta\}$ and $\Delta'$ is obtained

from $\Delta$ by replacing the subquiver

$$\begin{tikzcd} & \underset{z}{\bullet} \arrow[d, "\beta"] & \\ \underset{v}{\bullet} & \underset{x}{\bullet} \arrow[l, "\gamma"'] \arrow[r, "\alpha"] & \underset{y}{\bullet} \end{tikzcd}$$

by the quiver

$$\begin{tikzcd} \underset{v}{\bullet} \arrow[r, "\gamma"] & \underset{x}{\bullet} & \underset{y}{\bullet} \arrow[l, "\alpha"'] & \underset{z}{\bullet} \arrow[l, "\beta"'] \end{tikzcd}.$$

On the other hand, if there exists $\alpha' \in \Delta_1$ with $s\alpha' = y$ and $\alpha'\alpha \in R$, then $R = (R \setminus \{\gamma\beta, \alpha'\alpha\}) \cup \{\alpha\beta, \alpha'\gamma\}$ and $\Delta'$ is obtained from $\Delta$ by replacing the subquiver

$$\begin{tikzcd} & \underset{z}{\bullet} \arrow[d, "\beta"] & & \\ \underset{v}{\bullet} & \underset{x}{\bullet} \arrow[l, "\gamma"'] \arrow[r, "\alpha"] & \underset{y}{\bullet} \arrow[r, "\alpha'"] & \underset{y'}{\bullet} \end{tikzcd}$$

by the quiver

$$\begin{tikzcd} & \overset{y'}{\bullet} & & \\ \underset{v}{\bullet} \arrow[r, "\gamma"] & \underset{x}{\bullet} \arrow[u, "\alpha'"'] & \underset{y}{\bullet} \arrow[l, "\alpha"'] & \underset{z}{\bullet} \arrow[l, "\beta"'] \end{tikzcd}$$

where $y' = t\alpha'$. Observe that either $\Delta'$ is proper (if $y$ is a connecting vertex in the second case), or we decrease the number of connecting arrows (otherwise), hence the claim follows by induction.

(2) Assume that $\alpha\beta \in R$.

(2.1) Assume that there is no $\alpha' \in \Delta'_1$ with $s\alpha' = y$ and $\alpha'\alpha \in R$.

(2.1.1) Assume that $y$ is a connecting vertex. If either $\Delta^{(2)}$ is not an oriented cycle, or there exist $\delta', \delta'' \in \Delta_1^{(2)}$ with $s\delta' = t\delta''$ and $\delta'\delta'' \notin R$, then the claim follows by symmetry from (1), thus we may assume that $\Delta^{(2)}$ is an oriented cycle such that $\delta'\delta'' \in R$ for all $\delta', \delta'' \in \Delta_1^{(2)}$ with $s\delta' = t\delta''$.

(2.1.1.1) Assume that $|\Delta_1^{(2)}| = 1$. If $(\Delta', R')$ is the bound quiver obtained from $(\Delta, R)$ by applying the generalized APR-reflection at $y$, then $R' = R$ and $\Delta'$ is obtained from $\Delta$ by replacing the subquiver

$$\begin{tikzcd} \underset{z}{\bullet} \arrow[r, "\beta"] & \underset{x}{\bullet} \arrow[r, "\alpha"] & \underset{y}{\bullet} \end{tikzcd}$$

by the quiver

$$\begin{tikzcd} \underset{z}{\bullet} \arrow[r, "\beta"] & \underset{y}{\bullet} \arrow[r, "\alpha"] & \underset{x}{\bullet} \end{tikzcd},$$

hence the claim follows.

(2.1.1.2) Assume that $|\Delta_1^{(2)}| > 1$. Let $\alpha'$ and $\beta'$ be the arrows in $\Delta^{(2)}$ with $s\alpha' = y = t\beta'$. Put $v' = t\alpha'$ and $x' = s\beta'$. Let $\gamma'$ be the arrow in $\Delta^{(2)}$ with $t\gamma' = x'$. Put $z' = s\gamma'$. Recall that $\alpha'\beta', \beta'\gamma' \in R$. If $(\Delta', R')$ is the bound quiver obtained from $(\Delta, R)$ by applying the generalized APR-reflection at $y$ followed by the APR-reflection at $x'$, then $R' = (R \setminus \{\alpha\beta, \alpha'\beta', \beta'\gamma'\}) \cup \{\alpha\gamma', \alpha'\alpha\}$ and $\Delta'$ is obtained from $\Delta$ by replacing the subquiver

$$\begin{tikzcd} & & \underset{v'}{\bullet} & & \\ \underset{z}{\bullet} \arrow[r, "\beta"] & \underset{x}{\bullet} \arrow[r, "\alpha"] & \underset{y}{\bullet} \arrow[u, "\alpha'"'] & \underset{x'}{\bullet} \arrow[l, "\beta'"'] & \underset{z'}{\bullet} \arrow[l, "\gamma'"'] \end{tikzcd}$$

by the quiver

$$\begin{tikzcd} & & \underset{z'}{\bullet} \arrow[d, "\gamma'"] & & \\ \underset{z}{\bullet} \arrow[r, "\beta"] & \underset{x'}{\bullet} \arrow[r, "\beta'"] & \underset{y}{\bullet} \arrow[r, "\alpha"] & \underset{x}{\bullet} \arrow[r, "\alpha'"] & \underset{v'}{\bullet} \end{tikzcd}$$

hence the claim follows in this case.

(2.1.2) Assume that $y$ is not a connecting vertex.

(2.1.2.1) Assume that there exists $\alpha' \in \Delta_1$ with $s\alpha' = y$. Our assumptions imply that $\alpha'\alpha \notin R$. Put $y' = t\alpha'$. If $(\Delta', R')$ is the bound quiver obtained from $(\Delta, R)$ by applying the generalized APR-reflection at $y$, then $R' = (R \setminus \{\alpha\beta\}) \cup \{\alpha'\alpha\}$ and $\Delta'$ is obtained from $\Delta$ by replacing the subquiver

$$\begin{tikzcd} \underset{z}{\bullet} \arrow[r, "\beta"] & \underset{x}{\bullet} \arrow[r, "\alpha"] & \underset{y}{\bullet} \arrow[r, "\alpha'"] & \underset{y'}{\bullet} \end{tikzcd}$$

by the quiver

$$\begin{tikzcd} \underset{z}{\bullet} \arrow[r, "\beta"] & \underset{y}{\bullet} \arrow[r, "\alpha"] & \underset{x}{\bullet} \arrow[r, "\alpha'"] & \underset{y'}{\bullet} \end{tikzcd},$$

hence the claim follows by induction.

(2.1.2.2) Assume there exists $\alpha' \in \Delta'$ with $t\alpha' = y$. Put $x' = s\alpha'$.

(2.1.2.2.1) Assume that either $x'$ is a connecting vertex or $\alpha'$ is a free arrow. Moreover, if $x'$ is a connecting arrow and $\alpha'$ is not a free arrow, then let $\beta'$ be the arrow in $\Delta$ with $t\beta' = x'$ and $\alpha'\beta' \in R$, and put $z' = s\beta'$. Let $(\Delta', R')$ be the bound quiver obtained from $(\Delta, R)$ by applying the APR-reflection at $y$. If $\alpha'$ is a free arrow, then $R' = (R \setminus \{\alpha\beta\}) \cup \{\alpha'\beta\}$ and $\Delta'$ is obtained from $\Delta$ by replacing the subquiver

$$\begin{tikzcd} \underset{z}{\bullet} \arrow[r, "\beta"] & \underset{x}{\bullet} \arrow[r, "\alpha"] & \underset{y}{\bullet} & \underset{x'}{\bullet} \arrow[l, "\alpha'"'] \end{tikzcd}$$

by the quiver

$$\begin{tikzcd} & z \arrow[d, "\beta"] & \\ x & y \arrow[l, "\alpha"'] \arrow[r, "\alpha'"] & x' \end{tikzcd}$$

hence the claim follows by induction. On the other hand, if $x'$ is a connecting arrow and $\alpha'$ is not a free arrow, then $R' = (R \setminus \{\alpha\beta, \alpha'\beta'\}) \cup \{\alpha\beta', \alpha'\beta\}$ and $\Delta'$ is obtained from $\Delta$ by replacing the subquiver

$$\begin{tikzcd} z \arrow[r, "\beta"] & x \arrow[r, "\alpha"] & y & x' \arrow[l, "\alpha'"'] & z' \arrow[l, "\beta'"'] \end{tikzcd}$$

by the quiver

$$\begin{tikzcd} z \arrow[dr, "\beta"] & & z' \arrow[dl, "\beta'"'] \\ x & y \arrow[l, "\alpha"'] \arrow[r, "\alpha'"] & x' \end{tikzcd}$$

hence the claim follows.

(2.1.2.2.2) Assume that $x'$ is not a connecting vertex and there exists $\beta' \in \Delta_1$ with $t\beta' = x'$ and $\alpha'\beta' \in R$. Put $z' = s\beta'$. If $(\Delta', R')$ is the bound quiver obtained from $(\Delta, R)$ by applying the APR-reflections at $y$ and $x'$, then $R' = (R \setminus \{\alpha\beta, \alpha'\beta'\}) \cup \{\alpha\beta'\}$ and $\Delta'$ is obtained from $\Delta$ by replacing the subquiver

$$\begin{tikzcd} z \arrow[r, "\beta"] & x \arrow[r, "\alpha"] & y & x' \arrow[l, "\alpha'"'] & z' \arrow[l, "\beta'"'] \end{tikzcd}$$

by the quiver

$$\begin{tikzcd} & & x & \\ z \arrow[r, "\beta"] & x' \arrow[r, "\alpha'"] & y \arrow[u, "\alpha"'] & z' \arrow[l, "\beta'"'] \end{tikzcd}$$

hence the claim follows by induction.

(2.2) Assume that there exists $\alpha' \in \Delta_1$ with $s\alpha' = y$ and $\alpha'\alpha \in R$. Put $y' = t\alpha'$.

(2.2.1) Assume that $x = t\gamma$. Let $\beta_1 \cdots \beta_n$ and $\gamma_1 \cdots \gamma_m$ be the maximal paths in $\Delta$ terminating at $x$ with $\beta_1 = \beta$ and $\gamma_1 = \gamma$. Put $u = s\beta_n$, $u'_i = s\beta_i$ for $i \in [1, n-1]$ and $u''_i = s\gamma_i$ for $i \in [1, m-1]$.

(2.2.1.1) Assume that there exists $i \in [1, m-1]$ such that $\gamma_i\gamma_{i+1} \in R$. By shifting relations we may assume that $\gamma_{m-1}\gamma_m \in R$. Observe that $\beta_i\beta_{i+1} \notin R$ for all $i \in [1, n-1]$. If $(\Delta', R')$ is the bound quiver obtained from $(\Delta, R)$ by applying the HW-coreflection at $u$ followed by the composition of the HW-coreflection at $u'_i$ and the APR-reflection at $u'_i$ for $i = n-1, \dots, 1$, then

$R' = (R \setminus \{\gamma_{m-1}\gamma_m, \alpha\beta\}) \cup \{\beta_n\gamma\}$ and $\Delta'$ is obtained from $\Delta$ by replacing the subquiver

$$u''_{m-1} \xleftarrow{\gamma_m} u \xrightarrow{\beta_n} u'_{n-1} \longrightarrow \cdots \longrightarrow u'_1 \xrightarrow{\beta_1} x$$

by the quiver

$$u''_{m-1} \xrightarrow{\gamma_m} u'_{n-1} \longrightarrow \cdots \longrightarrow u'_1 \xrightarrow{\beta_1} u \xleftarrow{\beta_n} x\ ,$$

hence we reduce the proof to (1).

(2.2.1.2) Assume that $\gamma_i\gamma_{i+1} \notin R$ for all $i \in [1, m-1]$. Let $r$ be the number of $i \in [1, n-1]$ such that $\beta_i\beta_{i+1} \in R$. By shifting relations we may assume $\beta_i\beta_{i+1} \in R$ for all $i \in [n-r, n-1]$. Put $\beta_0 = \alpha$. If $(\Delta', R')$ is the bound quiver obtained from $(\Delta, R)$ by applying the generalized APR-coreflections at $u'_1, \dots, u'_{n-r-1}$, then $R' = (R \setminus \{\alpha\beta\}) \cup \{\beta_{n-r-1}\beta_{n-r}\}$ and $\Delta'$ is obtained from $\Delta$ by replacing the subquiver

$$u \xrightarrow{\beta_n} \cdots \xrightarrow{\beta_1} x \xrightarrow{\beta_0} y$$

by the quiver

$$u \xrightarrow{\beta_n} \cdots \xrightarrow{\beta_{n-r}} x \xrightarrow{\beta_{n-r-1}} \cdots \xrightarrow{\beta_0} y\ .$$

Let $\gamma'_1 \cdots \gamma'_l$ be the maximal path in $(\Delta', R')$ with $\gamma'_l = \alpha$. Observe that $l > 1$ implies that $y$ is a connecting vertex. Put

$$u' = \begin{cases} u'_{n-1} & \text{if } r \geq 1, \\ x & \text{if } r = 0, \end{cases} \qquad v' = t\gamma'_1.$$

Let $(\Delta'', R'')$ be the bound quiver obtained from $(\Delta', R')$ by applying the HW-coreflection at $u$ followed by the composition of the HW-coreflection at $u''_i$ and the APR-reflection at $u''_i$ for $i = m-1, \dots, 1$. If there exists $\delta$ in $\Delta$ with $t\delta = v'$ and $\delta \neq \gamma'_1$, then $R'' = (R' \setminus \{\beta_{n-1}\beta_n\}) \cup \{\gamma_m\delta\}$, while $R'' = R' \setminus \{\beta_{n-1}\beta_n\}$, otherwise. Moreover, $\Delta''$ is obtained from $\Delta'$ by replacing the subquiver

$$u' \xleftarrow{\beta_n} u \xrightarrow{\gamma_m} u''_{m-1} \xrightarrow{\gamma_{m-1}} \cdots \xrightarrow{\gamma_1} x \xrightarrow{\beta_{n-r-1}} \cdots \xrightarrow{\beta_0} y \xrightarrow{\gamma'_{l-1}} \cdots \xrightarrow{\gamma'_1} v'$$

by the quiver

$$u' \xrightarrow{\beta_n} u''_{m-1} \xrightarrow{\gamma_{m-1}} \cdots \xrightarrow{\gamma_1} u \xleftarrow{\gamma_m} v' \xleftarrow{\gamma'_1} \cdots \xleftarrow{\gamma'_{l-1}} y \xleftarrow{\beta_0} \cdots \xleftarrow{\beta_{n-r-1}} x\ ,$$

and the claim follows (by induction if $y$ is not a connecting vertex).

(2.2.2) Assume that $x = s\gamma$. Put $v = t\gamma$.

(2.2.2.1) Assume that there exists $\gamma' \in \Delta_1$ with $s\gamma' = v$ and $\gamma'\gamma \in R$ (by shifting relations we may assume that this condition is satisfied if $\Delta^{(1)}$ is an oriented cycle). Put $v' = t\gamma'$. If $(\Delta', R')$ is the bound quiver obtained from $(\Delta, R)$ by applying the generalized coreflection at $x$ followed, if $y$ is not a connecting vertex, by the APR-coreflection at $y$, then

$$R' = \begin{cases} (R \setminus \{\alpha'\alpha, \alpha\beta, \gamma'\gamma\}) \cup \{\gamma'\alpha, \gamma\beta, \alpha'\gamma\} & \text{if } y \text{ is a connecting vertex,} \\ (R \setminus \{\alpha'\alpha, \alpha\beta, \gamma'\gamma\}) \cup \{\gamma\beta, \alpha'\gamma\} & \text{otherwise} \end{cases}$$

and $\Delta'$ is obtained from $\Delta$ by replacing the subquiver

$$\begin{tikzcd}
& & z \arrow[d, "\beta"] & & \\
v' & v \arrow[l, "\gamma'"'] & x \arrow[l, "\gamma"'] \arrow[r, "\alpha"] & y \arrow[r, "\alpha'"] & y'
\end{tikzcd}$$

by the quiver

$$\begin{tikzcd}
& v' & & y' \\
z \arrow[r, "\beta"] & v \arrow[r, "\gamma"] & x \arrow[ul, "\gamma'"] \arrow[ur, "\alpha'"'] & y \arrow[l, "\alpha"']
\end{tikzcd}$$

if $y$ is a connecting vertex, and by

$$\begin{tikzcd}
& & y' & & \\
v' & y \arrow[l, "\gamma'"'] & x \arrow[l, "\alpha"'] \arrow[u, "\alpha'"'] & v \arrow[l, "\gamma"'] & z \arrow[l, "\beta"']
\end{tikzcd}$$

otherwise, hence the claim again follows.

(2.2.2.2) Assume that $\Delta^{(1)}$ is not an oriented cycle. Let $\gamma_1 \cdots \gamma_n$ be the maximal path in $\Delta$ with $\gamma_n = \gamma$. We may additionally assume that $\gamma_i\gamma_{i+1} \notin R$ for all $i \in [1, n-1]$. Consequently, we may reduce the proof in this case to (2.2.1) by applying APR-reflections and shifts of relations. ∎

STEP 6. *If $\Delta$ is special, then we may assume that for each $j \in [1, 2]$, $\Delta^{(j)}$ is an oriented cycle or either the source or the sink in $\Delta^{(j)}$ is a connecting vertex.*

*Proof.* If both $\Delta^{(1)}$ and $\Delta^{(2)}$ are oriented cycles, then there is nothing to prove, so without loss of generality we may assume that $\Delta^{(1)}$ is not an oriented cycle. Observe that our assumptions imply that there are no connecting arrows in $\Delta$. Let $x$ be the connecting vertex in $\Delta$ and assume that $x$ is neither a source nor a sink in $\Delta^{(1)}$. Observe that $x \in \Delta_0^{(1)} \cap \Delta_0^{(2)}$. Let $\alpha$, $\beta$, $\alpha'$ and $\beta'$ be the arrows in $\Delta$ with $s\alpha = t\beta = x = s\alpha' = t\beta'$, $\alpha, \beta \in \Delta_1^{(1)}$, and $\alpha', \beta' \in \Delta_1^{(2)}$. Put $y = t\alpha$, $y' = t\alpha'$, $z = s\beta$, and $z' = s\beta'$. By applying

APR-coreflections, shifts of relations and Lemma 1.2 we may assume that $z$ is a source in $\Delta^{(1)}$.

Assume first that $\alpha' = \beta'$. Then $\alpha\beta \in R$ and $\alpha'\beta' \in R$. Let $\gamma_1 \cdots \gamma_m$ be the maximal path in $\Delta$ starting at $z$ with $\gamma_m \neq \beta$. Observe that $\gamma_i\gamma_{i+1} \notin R$ for all $i \in [1, m-1]$. Put $v_i = s\gamma_i$ for $i \in [1, m-1]$. The bound quiver algebra of the bound quiver obtained from $(\Delta, R)$ by applying the APR-coreflections at $z, v_{m-1}, \dots, v_1$ is easily seen to be tilting-cotilting equivalent to $\Lambda_2(p, 1, m, r, 0)$ for some $p \in \mathbb{N}_+$ and $r \in [0, p-1]$, hence the claim follows in this case.

Assume now that $\alpha\beta \in R$ and $\alpha'\beta' \in R$, but $\alpha' \neq \beta'$. Let $(\Delta', R')$ be the bound quiver obtained from $(\Delta, R)$ by applying the generalized APR-reflection at $x$. If there exists $\beta'' \in \Delta_1$ with $t\beta'' = z'$ and $\beta'\beta'' \in R$, then $R' = (R \setminus \{\alpha\beta, \alpha'\beta', \beta'\beta''\}) \cup \{\alpha\beta', \alpha'\beta, \beta\beta''\}$ and $\Delta'$ is obtained from $\Delta$ by replacing the subquiver

$$\begin{tikzcd}
y & & y' & \\
z \arrow[r, "\beta"] & x \arrow[ul, "\alpha"'] \arrow[ur, "\alpha'"] & z' \arrow[l, "\beta'"'] & z'' \arrow[l, "\beta''"']
\end{tikzcd}$$

by the quiver

$$\begin{tikzcd}
 & & z'' \arrow[d, "\beta''"] & & \\
y & z' \arrow[l, "\alpha"'] & x \arrow[l, "\beta'"'] \arrow[r, "\beta"] & z \arrow[r, "\alpha'"] & y'
\end{tikzcd}$$

where $z'' = s\beta''$. Otherwise, $R' = (R \setminus \{\alpha\beta, \alpha'\beta'\}) \cup \{\alpha\beta', \alpha'\beta\}$ and $\Delta'$ is obtained from $\Delta$ by replacing the subquiver

$$\begin{tikzcd}
y & & y' \\
z \arrow[r, "\beta"] & x \arrow[ul, "\alpha"'] \arrow[ur, "\alpha'"] & z' \arrow[l, "\beta'"']
\end{tikzcd}$$

by the quiver

$$\begin{tikzcd}
y & z' \arrow[l, "\alpha"'] & x \arrow[l, "\beta'"'] \arrow[r, "\beta"] & z \arrow[r, "\alpha'"] & y'
\end{tikzcd} .$$

In particular, in both cases $\Delta'$ proper.

Assume finally that $\alpha\beta' \in R$ and $\alpha'\beta \in R$. Let $\gamma$ be the arrow in $\Delta$ with $s\gamma = z$ and $\gamma \neq \beta$. Put $v = t\gamma$. If there exists $\gamma' \in \Delta_1$ with $s\gamma' = v$ and $\gamma'\gamma \in R$, then let $(\Delta', R')$ be the bound quiver obtained from $(\Delta, R)$ by applying the APR-coreflections at $z$ and $v$, and let $v' = t\gamma'$. Observe that $R' = (R \setminus \{\alpha'\beta, \gamma'\gamma\}) \cup \{\gamma'\beta\}$ and $\Delta'$ is obtained from $\Delta$ by replacing the subquiver

$$\begin{tikzcd} v' & v \arrow[l, "\gamma'"'] & z \arrow[l, "\gamma"'] \arrow[r, "\beta"] & x \arrow[r, "\alpha'"] & y' \end{tikzcd}$$

by the quiver

$$\begin{tikzcd} & x \arrow[d, "\beta"] & & \\ v' & z \arrow[l, "\gamma'"'] \arrow[r, "\gamma"] & v \arrow[r, "\alpha'"] & y' \end{tikzcd}$$

Otherwise, if $(\Delta', R')$ is the bound quiver obtained from $(\Delta, R)$ by applying the APR-coreflection at $z$, then $R' = (R \setminus \{\alpha'\beta\}) \cup \{\alpha'\gamma\}$ and $\Delta'$ is obtained from $\Delta$ by replacing the subquiver

$$\begin{tikzcd} v & z \arrow[l, "\gamma"'] \arrow[r, "\beta"] & x \arrow[r, "\alpha'"] & y' \end{tikzcd}$$

by the quiver

$$\begin{tikzcd} & y' & \\ v \arrow[r, "\gamma"] & z \arrow[u, "\alpha'"'] & x \arrow[l, "\beta"'] \end{tikzcd}$$

Again in both cases $\Delta'$ is proper and this finishes the proof. ∎

STEP 7. *We may assume that* $(\Delta, R)$ *is proper.*

*Proof.* If $\Delta^{(1)}$ is not an oriented cycle, then neither is $\Delta^{(2)}$ and the claim follows from Proposition 1.8; thus assume that $\Delta^{(1)}$ (and consequently also $\Delta^{(2)}$) is an oriented cycle.

Assume first that there are no connecting arrows in $\Delta$ and let $x$ be the connecting vertex in $\Delta$. Let $\alpha$, $\beta$, $\alpha'$ and $\beta'$ be the arrows in $\Delta$ with $s\alpha = t\beta = x = s\alpha' = t\beta'$, $\alpha, \beta \in \Delta_1^{(1)}$, and $\alpha', \beta' \in \Delta_1^{(2)}$. If $\alpha\beta \in R$ and $\alpha'\beta' \in R$, then it follows by shifting relations that the bound quiver algebra of $(\Delta, R)$ is tilting-cotilting equivalent to $\Lambda_2(p_1, p_2, 0, r_1, r_2)$ for some $p_1, p_2 \in \mathbb{N}_+$, $r_1 \in [0, p_1 - 1]$, and $r_2 \in [0, p_2 - 1]$ such that $r_1 + r_2 \geq 1$. On the other hand, if $\alpha\beta' \in R$ and $\alpha'\beta \in R$, then it follows by shifting relations that the bound quiver algebra of $(\Delta, R)$ is tilting-cotilting equivalent to $\Lambda_1(p_1, p_2, 0, 0, r_1)$ for some $p_1, p_2 \in \mathbb{N}_+$, $p_1, p_2 \geq 2$, and $r_1 \in [1, p_1 - 1]$.

Now assume that there are connecting arrows in $\Delta$. Recall that in this case $\alpha\beta \in R$ for all cycle arrows $\alpha$ and $\beta$ with $s\alpha = t\beta$. Let $\Delta^{(0)}$ be the minimal subquiver of $\Delta$ with the set of arrows consisting of the connecting arrows. Let $x \in \Delta_0^{(1)}$ and $y \in \Delta_0^{(2)}$ be the connecting vertices. Observe that $\Delta^{(0)}$ is a linear quiver. We show that we may assume that $x$ is a unique sink in $\Delta^{(0)}$, $y$ is a unique source in $\Delta^{(0)}$, and there are no $\alpha, \beta \in \Delta_1^{(0)}$ with

$s\alpha = t\beta$ and $\alpha\beta \in R$. This will immediately imply that the bound quiver algebra of $(\Delta, R)$ is tilting-cotilting equivalent to $\Lambda_2(p_1, p_2, p_3, p_1 - 1, p_2 - 1)$ for some $p_1, p_2, p_3 \in \mathbb{N}_+$.

By repeating arguments from the proofs of Steps 3 and 4 and passing to the opposite algebra if necessary, we may assume that

$$\Delta = \begin{tikzcd} \underset{x}{\bullet} & \underset{x_1}{\bullet} \arrow[l, "\alpha_1"'] & \cdots \arrow[l] & \underset{x_{n-1}}{\bullet} \arrow[l] & \underset{z}{\bullet} \arrow[l, "\alpha_n"'] \arrow[r, "\beta_m"] & \underset{y_{m-1}}{\bullet} \arrow[r] & \cdots \arrow[r] & \underset{y_1}{\bullet} \arrow[r, "\beta_1"] & \underset{y}{\bullet} \end{tikzcd}$$

for some $n \in \mathbb{N}_+$ and $m \in \mathbb{N}$, and $\beta_i\beta_{i+1} \notin R$ for all $i \in [1, m-1]$. It is enough to show that we may additionally assume that $\alpha_i\alpha_{i+1} \notin R$ for all $i \in [1, n-1]$, since then the claim follows from Lemma 1.5. Assume this is not the case. By shifting relations we may assume that $\alpha_1\alpha_2 \in R$.

If $|\Delta_1^{(1)}| = 1$ and $(\Delta', R')$ is the bound quiver obtained from $(\Delta, R)$ by applying the generalized APR-reflection at $x$ followed by the APR-reflection at $x_1$, then $R' = R \setminus \{\alpha_1\alpha_2\}$ and $\Delta' = \Delta$. Otherwise, let $\gamma$, $\delta$ and $\delta'$ be the arrows in $\Delta^{(1)}$ with $s\gamma = x = t\delta$ and $t\delta' = s\delta$. Observe that our assumptions imply that $\gamma\delta, \delta\delta' \in R$. Put $u = t\gamma$, $v = s\delta$ and $v' = s\delta'$. If $(\Delta', R')$ is the bound quiver obtained from $(\Delta, R)$ by applying the generalized APR-reflection at $x$ followed by the APR-reflection at $v$, then $R' = (R \setminus \{\gamma\delta, \delta\delta', \alpha_1\alpha_2\}) \cup \{\gamma\alpha_1, \alpha_1\delta'\}$ and $\Delta'$ is obtained from $\Delta$ by replacing the subquiver

$$\begin{tikzcd} & & \overset{u}{\bullet} & & \\ \underset{v'}{\bullet} \arrow[r, "\delta'"] & \underset{v}{\bullet} \arrow[r, "\delta"] & \underset{x}{\bullet} \arrow[u, "\gamma"'] & \underset{x_1}{\bullet} \arrow[l, "\alpha_1"'] & \underset{x_2}{\bullet} \arrow[l, "\alpha_2"'] \end{tikzcd}$$

by the quiver

$$\begin{tikzcd} & & \overset{v'}{\bullet} \arrow[d, "\delta'"] & & \\ \underset{u}{\bullet} & \underset{x_1}{\bullet} \arrow[l, "\gamma"'] & \underset{x}{\bullet} \arrow[l, "\alpha_1"'] & \underset{v}{\bullet} \arrow[l, "\delta"'] & \underset{x_2}{\bullet} \arrow[l, "\alpha_2"'] \end{tikzcd}$$

where $x_2 = z$ if $n = 2$. Consequently, in both cases the claim follows by induction. ■

We now investigate the case when $\Delta$ is proper. In this case we may divide the arrows in $\Delta$ into three disjoint subsets $\Delta_1^{(1)}$, $\Delta_1^{(2)}$, $\Delta_1^{(3)}$ in such a way that $\alpha, \beta \in \Delta_1$ belong to the same subset if and only if the quiver $(\Delta_0, \Delta_1 \setminus \{\alpha, \beta\})$ has a connected component which is a one-cycle quiver. For $j \in [1, 3]$ we denote by $\Delta^{(j)}$ the minimal subquiver of $\Delta$ with the set of arrows $\Delta_1^{(j)}$. Observe that $\Delta^{(j)}$ is a linear quiver. We divide the arrows in $\Delta^{(j)}$ into disjoint subsets $\Delta_{1,-}^{(j)}$ and $\Delta_{1,+}^{(j)}$ in such a way that if $\alpha, \beta \in \Delta_1^{(j)}$, $\alpha \neq \beta$, are adjacent to the same vertex then they belong to the same subset if and only if either $s\alpha = t\beta$ or $t\alpha = s\beta$. For $j \in [1, 3]$ and $\varepsilon \in \{-, +\}$ we put

$$R_{\varepsilon}^{(j)} = \{\alpha\beta \in R \mid \alpha, \beta \in \Delta_{1,\varepsilon}^{(j)}\}.$$

STEP 8. *We may assume that either $R_{+}^{(j)} = \emptyset$ or $R_{-}^{(j)} = \emptyset$ for each $j \in [1, 3]$.*

*Proof.* Analogous to the proof of Step 3. ■

STEP 9. *We may assume that either there is at most one sink in $\Delta^{(j)}$ or there is at most one source in $\Delta^{(j)}$, for each $j \in [1, 3]$.*

*Proof.* We prove the claim by induction on $|R|$ and, for a fixed $j$, on $|\Delta_1^{(j)}|$. Fix $j \in [1, 3]$ and assume that there is either a unique source or a unique sink in $\Delta^{(l)}$ for each $l \in [1, j-1]$. Let $u$ and $v$ be the connecting vertices in $\Delta$, and let $\alpha$ and $\beta$ be the arrows in $\Delta^{(j)}$ adjacent to $u$ and $v$, respectively. The claim follows by the arguments presented after Lemma 1.3, unless the following condition (or its dual) is satisfied: $s\alpha = u$, $t\beta = v$, there exists $\alpha' \in \Delta_1$ with $t\alpha' = u$ and $\alpha\alpha' \in R$, and there exists $\beta' \in \Delta_1$ with $s\beta' = v$ and $\beta'\beta \in R$. Assume the above condition is satisfied. Put $x = s\beta$ and $v' = t\beta'$. If $\Delta^{(j)}$ is not an equioriented linear quiver, then by applying APR-coreflections, shifts of relations, and Lemma 1.2, we may assume that there exists $\gamma \in \Delta_1$ with $\gamma \neq \beta$ and $s\gamma = x$. Put $y = t\gamma$.

Assume there exists $\gamma' \in \Delta_1$ with $s\gamma' = y$ and $\gamma'\gamma \in R$. Put $y' = t\gamma'$. If $(\Delta', R')$ is the bound quiver obtained from $(\Delta, R)$ by applying the APR-coreflections at $x$ and $y$, then $R' = (R \setminus \{\beta'\beta, \gamma'\gamma\}) \cup \{\gamma'\beta\}$ and $\Delta'$ is obtained from $\Delta$ by replacing the subquiver

$$\begin{tikzcd} \underset{y'}{\bullet} & \underset{y}{\bullet} \arrow[l, "\gamma'"'] & \underset{x}{\bullet} \arrow[l, "\gamma"'] \arrow[r, "\beta"] & \underset{v}{\bullet} \arrow[r, "\beta'"] & \underset{v'}{\bullet} \end{tikzcd}$$

by the quiver

$$\begin{tikzcd} & \overset{v}{\bullet} \arrow[d, "\beta"] & & \\ \underset{y'}{\bullet} & \underset{x}{\bullet} \arrow[l, "\gamma'"'] \arrow[r, "\gamma"] & \underset{y}{\bullet} \arrow[r, "\beta'"] & \underset{v'}{\bullet} \end{tikzcd}$$

In particular, $|R'| < |R|$, hence the claim follows by induction in this case.

Otherwise, if $(\Delta', R')$ is the bound quiver obtained from $(\Delta, R)$ by applying the APR-coreflection at $x$, then $R' = (R \setminus \{\beta'\beta\}) \cup \{\beta'\gamma\}$ and $\Delta'$ is obtained from $\Delta$ by replacing the subquiver

$$\begin{tikzcd} \underset{y}{\bullet} & \underset{x}{\bullet} \arrow[l, "\gamma"'] \arrow[r, "\beta"] & \underset{v}{\bullet} \arrow[r, "\beta'"] & \underset{v'}{\bullet} \end{tikzcd}$$

by the quiver

$$\begin{tikzcd}
 & v \arrow[d, "\beta"] & \\
y \arrow[r, "\gamma"] & x \arrow[r, "\beta'"] & v'
\end{tikzcd}$$

Observe that if $l \in [1, j-1]$ and there is no $\delta \in \Delta_1^{(l)}$ with $s\delta = v$ and $\delta \neq \beta'$, then there is either a unique source or a unique sink in $\Delta'^{(l)}$. On the other hand, if there exists such an arrow, then we may assume that there is either a unique source or a unique sink in $\Delta'^{(l)}$, since $\beta$ is a free arrow in $(\Delta', R')$. In particular, in both cases the claim follows again by induction, since $|\Delta_1'^{(j)}| < |\Delta_1^{(j)}|$. ■

STEP 10. *We may assume that if either $s\alpha = x = s\beta$ or $t\alpha = x = t\beta$ for a connecting vertex $x$, $\alpha \in \Delta_{1,\varepsilon_1}^{(j_1)}$, and $\beta \in \Delta_{1,\varepsilon_2}^{(j_2)}$, with $j_1 \neq j_2$ and $\varepsilon_1, \varepsilon_2 \in \{-,+\}$, then either $R_{\varepsilon_1}^{(j_1)} = \emptyset$ or $R_{\varepsilon_2}^{(j_2)} = \emptyset$.*

*Proof.* Without loss of generality we may assume that $s\alpha = x = s\beta$. If $R_{\varepsilon_1}^{(j_1)} \neq \emptyset$ or $R_{\varepsilon_2}^{(j_2)} \neq \emptyset$, then by shifting relations we may assume that there exist arrows $\alpha'$ and $\beta'$ in $\Delta$ with $s\alpha' = t\alpha$, $s\beta' = t\beta$, and $\alpha'\alpha, \beta'\beta \in R$. Let $\gamma$ be the arrow in $\Delta$ with $t\gamma = x$. Without loss of generality we may assume that $\alpha\gamma \in R$ and $\beta\gamma \notin R$. Put $y' = t\alpha'$, $z' = t\beta'$, and $u = s\gamma$. If $(\Delta', R')$ is the bound quiver obtained from $(\Delta, R)$ by applying the generalized APR-coreflections at $x$ and $y$, then

$$R' = R(\setminus\{\alpha'\alpha, \beta'\beta, \alpha\gamma\}) \cup \{\alpha'\beta, \beta\gamma\}$$

and $\Delta'$ is obtained from $\Delta$ by replacing the subquiver

$$\begin{tikzcd}
 & & u \arrow[d, "\gamma"'] & & \\
y' & y \arrow[l, "\alpha'"'] & x \arrow[l, "\alpha"'] \arrow[r, "\beta"] & z \arrow[r, "\beta'"] & z'
\end{tikzcd}$$

by the quiver

$$\begin{tikzcd}
 & & y' & & \\
u \arrow[r, "\gamma"] & z \arrow[r, "\beta"] & x \arrow[u, "\alpha'"'] \arrow[r, "\alpha"] & y \arrow[r, "\beta'"] & z'
\end{tikzcd}$$

In particular, $|R_{\varepsilon_1}'^{(j_1)}| < |R_{\varepsilon_1}^{(j_1)}|$ and $|R_{\varepsilon_2}'^{(j_2)}| < |R_{\varepsilon_2}^{(j_2)}|$, hence the claim follows by induction. ■

STEP 11. *We may assume that there exists $j \in [1,3]$ such that $\Delta^{(j)}$ is equioriented.*

*Proof.* If the above condition is not satisfied, then without loss of generality we may assume that

$$\Delta^{(1)} = \underset{u}{\bullet} \xleftarrow{\alpha_1} \underset{x_1}{\bullet} \longleftarrow \cdots \longleftarrow \underset{x_{p_1-1}}{\bullet} \xleftarrow{\alpha_{p_1}} \underset{x}{\bullet} \xrightarrow{\alpha'_{q_1}} \underset{x'_{q_1-1}}{\bullet} \longrightarrow \cdots \longrightarrow \underset{x'_1}{\bullet} \xrightarrow{\alpha'_1} \underset{v}{\bullet},$$

$$\Delta^{(2)} = \underset{u}{\bullet} \xrightarrow{\beta_{p_2}} \underset{y_{p_2-1}}{\bullet} \longrightarrow \cdots \longrightarrow \underset{y_1}{\bullet} \xrightarrow{\beta_1} \underset{y}{\bullet} \xleftarrow{\beta'_1} \underset{y'_1}{\bullet} \longleftarrow \cdots \longleftarrow \underset{y'_{q_2-1}}{\bullet} \xleftarrow{\beta'_{q_2}} \underset{v}{\bullet},$$

$$\Delta^{(3)} = \underset{u}{\bullet} \xrightarrow{\gamma_{p_3}} \underset{z_{p_3-1}}{\bullet} \longrightarrow \cdots \longrightarrow \underset{z_1}{\bullet} \xrightarrow{\gamma_1} \underset{z}{\bullet} \xleftarrow{\gamma'_1} \underset{z'_1}{\bullet} \longleftarrow \cdots \longleftarrow \underset{z'_{q_3-1}}{\bullet} \xleftarrow{\gamma'_{q_3}} \underset{v}{\bullet},$$

for some $p_1, p_2, p_3, q_1, q_2, q_3 \in \mathbb{N}_+$. Moreover, we may assume that $\beta_{p_2}\alpha_1 \in R$. Consequently, by shifting relations we may assume that $\beta_i\beta_{i+1} \in R$ for all $i \in [1, p_2 - 1]$. There are two cases to consider.

Assume first $\gamma_i\gamma_{i+1} \notin R$ for all $i \in [1, p_3 - 1]$. If $(\Delta', R')$ is the bound quiver obtained from $(\Delta, R)$ by applying the generalized APR-coreflection at $u$ followed by the composition of the APR-coreflection at $y_i$ and the generalized APR-coreflection at $u$ for $i = p_2 - 1, \ldots, 1$, then

$$R' = \begin{cases} (R \setminus \{\beta_{p_2}\alpha_1, \beta_1\beta_2\}) \cup \{\gamma_{p_3}\alpha_1, \beta_{p_2}\gamma_{p_3}\} & \text{if } p_2 > 1, \\ (R \setminus \{\beta_{p_2}\alpha_1\}) \cup \{\gamma_{p_3}\alpha_1\} & \text{if } p_2 = 1, \end{cases}$$

$$\Delta'^{(1)} = \underset{z_{p_3-1}}{\bullet} \xleftarrow{\alpha_1} \underset{x_1}{\bullet} \longleftarrow \cdots \longleftarrow \underset{x_{p_1-1}}{\bullet} \xleftarrow{\alpha_{p_1}} \underset{x}{\bullet} \xrightarrow{\alpha'_{q_1}} \underset{x'_{q_1-1}}{\bullet} \longrightarrow \cdots \longrightarrow \underset{x'_1}{\bullet} \xrightarrow{\alpha'_1} \underset{v}{\bullet},$$

$$\Delta'^{(2)} = \underset{z_{p_3-1}}{\bullet} \xrightarrow{\gamma_{p_3}} \underset{y_{p_2-1}}{\bullet} \xrightarrow{\beta_{p_2}} \underset{y_{p_2-2}}{\bullet} \longrightarrow \cdots \longrightarrow \underset{u}{\bullet} \xleftarrow{\beta_1} \underset{y}{\bullet} \xleftarrow{\beta'_1} \underset{y'_1}{\bullet} \longleftarrow \cdots \longleftarrow \underset{y'_{q_2-1}}{\bullet} \xleftarrow{\beta'_{q_2}} \underset{v}{\bullet},$$

$$\Delta'^{(3)} = \underset{z_{p_3-1}}{\bullet} \longrightarrow \cdots \longrightarrow \underset{z_1}{\bullet} \xrightarrow{\gamma_1} \underset{z}{\bullet} \xleftarrow{\gamma'_1} \underset{z'_1}{\bullet} \longleftarrow \cdots \longleftarrow \underset{z'_{q_3-1}}{\bullet} \xleftarrow{\gamma'_{q_3}} \underset{v}{\bullet},$$

where $z_{p_3-1} = z$ if $p_3 = 1$. Consequently, the claim follows by an easy induction.

Assume now that there exists $i \in [1, p_3 - 1]$ such that $\gamma_i\gamma_{i+1} \in R$. Consequently, $p_2 = 1$. Moreover, by shifting relations we may assume that $\gamma_{p_3-1}\gamma_{p_3} \in R$. If $(\Delta', R')$ is the bound quiver obtained from $(\Delta, R)$ by applying the generalized APR-coreflection at $u$, then

$$R' = (R \setminus \{\beta_1\alpha_1, \gamma_{p_3-1}\gamma_{p_3}\}) \cup \{\gamma_{p_3}\alpha_1, \gamma_{p_3-1}\beta_1\},$$

$$\Delta'^{(1)} = \underset{u}{\bullet} \xleftarrow{\gamma_{p_3}} \underset{z_{p_3-1}}{\bullet} \xleftarrow{\alpha_1} \underset{x_1}{\bullet} \longleftarrow \cdots \longleftarrow \underset{x_{p_1-1}}{\bullet} \xleftarrow{\alpha_{p_1}} \underset{x}{\bullet} \xrightarrow{\alpha'_{q_1}} \underset{x'_{q_1-1}}{\bullet} \longrightarrow \cdots \longrightarrow \underset{x'_1}{\bullet} \xrightarrow{\alpha'_1} \underset{v}{\bullet},$$

$$\Delta'^{(2)} = \underset{u}{\bullet} \xleftarrow{\beta_1} \underset{y}{\bullet} \xleftarrow{\beta'_1} \underset{y'_1}{\bullet} \longleftarrow \cdots \longleftarrow \underset{y'_{q_2-1}}{\bullet} \xleftarrow{\beta'_{q_2}} \underset{u}{\bullet},$$

$$\Delta'^{(3)} = \underset{u}{\bullet} \xrightarrow{\gamma_{p_3-1}} \underset{z_{p_3-2}}{\bullet} \longrightarrow \cdots \longrightarrow \underset{z}{\bullet} \xleftarrow{\gamma_1'} \underset{z_1'}{\bullet} \longleftarrow \cdots \longleftarrow \underset{z_{q_3-1}'}{\bullet} \xleftarrow{\gamma_{q_3}'} \underset{u}{\bullet}\,,$$

thus the claim follows. ∎

STEP 12. *We may assume that there is at most one $j \in [1,3]$ such that $\Delta^{(j)}$ is not equioriented.*

*Proof.* If the above condition is not satisfied, then without loss of generality we may assume that

$$\Delta^{(1)} = \underset{u}{\bullet} \xleftarrow{\alpha_1} \underset{x_1}{\bullet} \longleftarrow \cdots \longleftarrow \underset{x_{p_1-1}}{\bullet} \xleftarrow{\alpha_{p_1}} \underset{x}{\bullet} \xrightarrow{\alpha_{q_1}'} \underset{x_{q_1-1}'}{\bullet} \longrightarrow \cdots \longrightarrow \underset{x_1'}{\bullet} \xrightarrow{\alpha_1'} \underset{v}{\bullet}\,,$$

$$\Delta^{(2)} = \underset{u}{\bullet} \xrightarrow{\beta_{p_2}} \underset{y_{p_2-1}}{\bullet} \longrightarrow \cdots \longrightarrow \underset{y_1}{\bullet} \xrightarrow{\beta_1} \underset{y}{\bullet} \xleftarrow{\beta_1'} \underset{y_1'}{\bullet} \longleftarrow \cdots \longleftarrow \underset{y_{q_2-1}'}{\bullet} \xleftarrow{\beta_{q_2}'} \underset{v}{\bullet}\,,$$

$$\Delta^{(3)} = \underset{u}{\bullet} \xrightarrow{\gamma_{p_3}} \underset{z_{p_3-1}}{\bullet} \longrightarrow \cdots \longrightarrow \underset{z_1}{\bullet} \xrightarrow{\gamma_1} \underset{v}{\bullet}\,,$$

for some $p_1, p_2, p_3, q_1, q_2 \in \mathbb{N}_+$. In this proof we will again number the cases. Up to symmetry, there are three main cases to consider: either $\beta_{p_2}\alpha_1 \in R$ and $\beta_{q_2}'\alpha_1' \in R$, or $\beta_{p_2}\alpha_1 \in R$ and $\beta_{q_2}'\gamma_1 \in R$, or $\gamma_{p_3}\alpha_1 \in R$ and $\beta_{q_2}'\gamma_1 \in R$.

(1) Assume $\beta_{p_2}\alpha_1 \in R$ and $\beta_{q_2}'\alpha_1' \in R$. In this case we may apply the same arguments as in the proof of the previous step. Note, however, that if $\gamma_i\gamma_{i+1} \notin R$ for all $i \in [1, p_3-1]$, then we obtain a gentle bound quiver whose bound quiver algebra is tilting-cotilting equivalent to $\Lambda_0'(p,r)$ for some $p \in \mathbb{N}_+$ and $r \in [0, p-1]$ according to Proposition 1.8.

(2) Assume that $\beta_{p_2}\alpha_1 \in R$ and $\beta_{q_2}'\gamma_1 \in R$.

(2.1) Assume that $\alpha_i'\alpha_{i+1}' \notin R$ for all $i \in [1, q_1-1]$ and $\beta_i'\beta_{i+1}' \notin R$ for all $i \in [1, q_2-1]$. By shifting the relation $\beta_{q_2}'\gamma_1$ to the left we may assume that $q_2 = 1$.

(2.1.1) Assume that $\beta_i\beta_{i+1} \notin R$ for all $i \in [1, p_2-1]$. By shifting the relation $\beta_{p_2}\alpha_1$ to the left we may assume that $p_2 = 1$. Consequently, the path algebra of the bound quiver obtained from $(\Delta, R)$ by application of the APR-reflections at $y, v, x_1', \ldots, x_{q_1-1}'$ is easily seen to be tilting-cotilting equivalent to $\Lambda_2(p_3+1, p_1+1, q_1, r_3, r_1)$, where $r_1$ is the number of $i \in [1, p_1-1]$ such that $\alpha_i\alpha_{i+1} \in R$ and $r_3$ is the number of $i \in [1, p_3-1]$ such that $\gamma_i\gamma_{i+1} \in R$.

(2.1.2) Assume that there exists $i \in [1, p_2-1]$ such that $\beta_i\beta_{i+1} \in R$. By shifting relations we may assume that $\beta_1\beta_2 \in R$. If $(\Delta', R')$ is the bound quiver obtained from $(\Delta, R)$ by applying the APR-reflections at $y, y_1, v, x_1', \ldots, x_{q_1-1}'$, then

$$R' = (R \setminus \{\beta_1\beta_2, \beta_1'\gamma_1\}) \cup \{\alpha_{q_1}'\beta_2\},$$

$$\Delta'^{(1)} = \underset{u}{\bullet} \xleftarrow{\alpha_1} \underset{x_1}{\bullet} \longleftarrow \cdots \longleftarrow \underset{x_{p_1-1}}{\bullet} \xleftarrow{\alpha_{p_1}} \underset{x}{\bullet} \xleftarrow{\alpha_{q_1}'} \underset{x_{q_1-1}'}{\bullet},$$

$$\Delta'^{(2)} = \underset{u}{\bullet} \longrightarrow \cdots \longrightarrow \underset{y_2}{\bullet} \xrightarrow{\beta_2} \underset{x_{q_1-1}'}{\bullet},$$

$$\Delta'^{(3)} = \underset{u}{\bullet} \xrightarrow{\gamma_{p_3}} \underset{z_{p_3-1}}{\bullet} \longrightarrow \cdots \longrightarrow \underset{z_1}{\bullet} \xrightarrow{\gamma_1} \underset{y_1}{\bullet} \xrightarrow{\beta_1} \underset{y}{\bullet} \xleftarrow{\beta_1'} \underset{v}{\bullet} \xleftarrow{\alpha_1'} \underset{x_1'}{\bullet} \longleftarrow \cdots \longleftarrow \underset{x_{q_1-1}'}{\bullet},$$

where $x_{q_1-1}' = v$ if $q_1 = 1$, hence the claim follows.

(2.2) Assume that there exists $i \in [1, q_2 - 1]$ such that $\beta_i'\beta_{i+1}' \in R$. By shifting relations we may assume that $\beta_1'\beta_2' \in R$. Moreover, this condition implies that $\beta_i\beta_{i+1} \notin R$ for all $i \in [1, p_2 - 1]$. By shifting the relation $\beta_{p_2}\alpha_1$ to the left we may assume $p_2 = 1$. If $(\Delta', R')$ is the bound quiver obtained from $(\Delta, R)$ by applying the APR-reflections at $y$ and $y_1'$, then

$$R' = (R \setminus \{\beta_1\alpha_1, \beta_1'\beta_2'\}) \cup \{\beta_1\beta_2'\},$$

$$\Delta'^{(1)} = \underset{y}{\bullet} \xleftarrow{\beta_1'} \underset{y_1'}{\bullet} \xleftarrow{\alpha_1} \underset{x_1}{\bullet} \longleftarrow \cdots \longleftarrow \underset{x_{p_1-1}}{\bullet} \xleftarrow{\alpha_{p_1}} \underset{x}{\bullet} \xrightarrow{\alpha_{q_1}'} \underset{x_{q_1-1}'}{\bullet} \longrightarrow \cdots \longrightarrow \underset{x_1'}{\bullet} \xrightarrow{\alpha_1'} \underset{v}{\bullet},$$

$$\Delta'^{(2)} = \underset{y}{\bullet} \xleftarrow{\beta_2'} \underset{y_2'}{\bullet} \longleftarrow \cdots \longleftarrow \underset{v}{\bullet},$$

$$\Delta'^{(3)} = \underset{y}{\bullet} \xrightarrow{\beta_1} \underset{u}{\bullet} \xrightarrow{\gamma_{p_3}} \underset{z_{p_3-1}}{\bullet} \longrightarrow \cdots \longrightarrow \underset{z_1}{\bullet} \xrightarrow{\gamma_1} \underset{v}{\bullet},$$

hence the claim follows again.

(2.3) Assume that $\beta_i'\beta_{i+1}' \notin R$ for all $i \in [1, q_2 - 1]$ and there exists $i \in [1, q_1 - 1]$ such that $\alpha_i'\alpha_{i+1}' \in R$. Observe that in this case $\alpha_i\alpha_{i+1} \notin R$ for all $i \in [1, p_1 - 1]$, hence by shifting the relation $\beta_{p_2}\alpha_1$ to the right we may assume that $p_1 = 1$. Similarly, $\gamma_i\gamma_{i+1} \notin R$ for all $i \in [1, p_3 - 1]$.

(2.3.1) Assume that there exists $i \in [1, p_2 - 1]$ such that $\beta_i\beta_{i+1} \in R$, then by shifting relations we may assume that $\beta_1\beta_2 \in R$. Moreover, by shifting the relation $\beta_{q_2}'\gamma_1$ to the left we may assume that $q_2 = 1$. Additionally, by shifting relations we may assume that $\alpha_1'\alpha_2' \in R$. If $(\Delta', R')$ is the bound quiver obtained from $(\Delta, R)$ by applying the HW-reflection at $y$ followed by the APR-coreflection at $y$ and the APR-reflections at $v, z_1, \ldots, z_{p_3-1}$, then

$$R' = (R \setminus \{\alpha_1'\alpha_2', \beta_1\beta_2, \beta_1'\gamma_1\}) \cup \{\gamma_{p_3}\alpha_2'\},$$

$$\Delta'^{(1)} = \underset{u}{\bullet} \xleftarrow{\alpha_1} \underset{x}{\bullet} \longrightarrow \cdots \longrightarrow \underset{x_2'}{\bullet} \xrightarrow{\alpha_2'} \underset{z_{p_3-1}}{\bullet},$$

$$\Delta'^{(2)} = u \xrightarrow{\beta_{p_2}} y_{p_2-1} \longrightarrow \cdots \longrightarrow y_1 \xrightarrow{\beta_1} y \xleftarrow{\beta'_1} x'_1 \xleftarrow{\alpha'_1} v \xleftarrow{\gamma_1} z_1 \longleftarrow \cdots \longleftarrow z_{p_3-1},$$

$$\Delta'^{(3)} = u \xleftarrow{\gamma_{p_3}} z_{p_3-1},$$

where $z_{p_3-1} = v$ if $p_3 = 1$. In particular, we reduce the proof to the situation dual either to (2.1) or to (2.2).

(2.3.2) Assume that $\beta_i\beta_{i+1} \notin R$ for all $i \in [1, p_2 - 1]$. By shifting the relation $\beta'_{q_2}\gamma_1$ to the right we may assume that $p_3 = 1$. Additionally, by shifting relations we may assume that $\alpha'_{q_1-1}\alpha'_{q_1} \in R$. The bound quiver algebra of the bound quiver obtained from $(\Delta, R)$ by applying the APR-coreflections at $x, x'_{q_1-1}, u, x, x'_{q_1-1}, y_{p_2-1}, \ldots, y_1$ is easily seen to be tilting-cotilting equivalent to $\Lambda_2(q_2+1, q_1, p_2+1, 0, r'_1 - 1)$, where $r'_1$ is the number of $i \in [1, q_1 - 1]$ such that $\alpha'_i\alpha'_{i+1} \in R$.

(3) Assume that $\gamma_{p_3}\alpha_1 \in R$ and $\beta'_{q_2}\gamma_1 \in R$.

(3.1) Assume that there exists $i \in [1, p_2 - 1]$ such that $\beta_i\beta_{i+1} \in R$. By shifting relations we may assume that $\beta_{p_2-1}\beta_{p_2} \in R$. Since in this case $\gamma_i\gamma_{i+1} \notin R$ for all $i \in [1, p_3-1]$, we may assume, by shifting the relation $\gamma_{p_3}\alpha_1$ to the left, that $p_3 = 1$. Consequently, the bound quiver algebra of the bound quiver obtained from $(\Delta, R)$ by applying the generalized APR-coreflection at $u$ is tilting-cotilting equivalent to $\Lambda'_0(p, r)$ for some $p \in \mathbb{N}_+$ and $r \in [0, p-1]$ according to Proposition 1.8.

(3.2) Assume that $\beta_i\beta_{i+1} \notin R$ for all $i \in [1, p_2 - 1]$. By shifting relations we may also assume that $\gamma_i\gamma_{i+1} \in R$ for all $i \in [1, p_3 - 1]$. If $(\Delta', R')$ is the bound quiver obtained from $(\Delta, R)$ by applying the APR-coreflection at $u$ followed by the composition of the APR-coreflection at $z_i$ and the generalized APR-coreflection at $u$ for $i = p_3 - 1, \ldots, 1$, then

$$R' = \begin{cases} (R \setminus \{\gamma_{p_3}\alpha_1, \gamma_1\gamma_2, \beta'_{q_2}\gamma_1\}) \cup \{\beta_{p_2}\alpha_1, \gamma_{p_3}\beta_{p_2}, \beta'_{q_2}\gamma_2\} & \text{if } p_3 > 1, \\ (R \setminus \{\gamma_1\alpha_1, \beta'_{q_2}\gamma_1\}) \cup \{\beta_{p_2}\alpha_1, \beta'_{q_2}\beta_{p_2}\} & \text{if } p_3 = 1, \end{cases}$$

$$\Delta'^{(1)} = y_{p_2-1} \xleftarrow{\alpha_1} x_1 \longleftarrow \cdots \longleftarrow x_{p_1-1} \xleftarrow{\alpha_{p_1}} x \xrightarrow{\alpha'_{q_1}} x'_{q_1-1} \longrightarrow \cdots \longrightarrow x'_1 \xrightarrow{\alpha'_1} v \xrightarrow{\gamma_{p_3}} u,$$

$$\Delta'^{(2)} = y_{p_2-1} \longrightarrow \cdots \longrightarrow y_1 \xrightarrow{\beta_1} y \xleftarrow{\beta'_1} y'_1 \longleftarrow \cdots \longleftarrow y'_{q_2-1} \xleftarrow{\beta'_{q_2}} u,$$

$$\Delta'^{(3)} = y_{p_2-1} \xrightarrow{\beta_{p_2}} z_{p_3-1} \longrightarrow \cdots \longrightarrow z_1 \xrightarrow{\gamma_2} u,$$

where $y_{p_2-1} = y$ if $p_2 = 1$. Consequently, the claim follows by induction. ∎

For $p_1, p_2, p_3 \in \mathbb{N}_+$, $p_2 \geq 2$, $r_1 \in [0, p_1 - 1]$, and $r_2 \in [1, p_2 - 1]$, let $\Lambda_2'(p_1, p_2, p_3, r_1, r_2)$ be the algebra of the quiver

$$\begin{tikzcd}
& \bullet \arrow[rr, dotted] & & \bullet \arrow[dr, "\alpha_1"] & \\
\bullet \arrow[ur, "\alpha_{p_1}"] \arrow[d, "\delta_{p_3}"'] & & & & \bullet \arrow[llll, "\beta"'] \arrow[d, "\gamma_{p_2}"] \\
\bullet \arrow[r, dotted] & \bullet \arrow[r, "\delta_1"] & \bullet & \bullet \arrow[l, "\gamma_1"'] & \bullet \arrow[l, dotted]
\end{tikzcd}$$

bound by $\alpha_i \alpha_{i+1}$ for $i \in [p_1 - r_1, p_1 - 1]$, $\alpha_{p_1}\beta$, $\beta\alpha_1$, $\gamma_i\gamma_{i+1}$ for $i \in [1, r_2]$. Observe that $\Lambda_2'(p_1, p_2, p_3, r_1, r_2)$ is tilting-cotilting equivalent to $\Lambda_2(p_2, p_1+1, p_3, r_2 - 1, r_1 + 1)$. Indeed, it is enough to apply the HW-reflection at $x_i$ followed by the APR-coreflection at $x_i$ for $i = 1, \ldots, p_3$, where $x_i = t\delta_i$ for $i \in [1, p_3]$.

STEP 13. *We may assume that $\Delta^{(j)}$ is equioriented for each $j \in [1, 3]$.*

*Proof.* Suppose that there exists $j \in [1, 3]$ such that $\Delta^{(j)}$ is not equioriented. Without loss of generality we may assume that

$$\Delta^{(1)} = \begin{tikzcd}[column sep=small]
\underset{u}{\bullet} & \underset{x_1}{\bullet} \arrow[l, "\alpha_1"'] & \cdots \arrow[l] & \underset{x_{p_1-1}}{\bullet} \arrow[l] & \underset{x}{\bullet} \arrow[l, "\alpha_{p_1}"'] \arrow[r, "\alpha'_{q_1}"] & \underset{x'_{q_1-1}}{\bullet} \arrow[r] & \cdots \arrow[r] & \underset{x'_1}{\bullet} \arrow[r, "\alpha'_1"] & \underset{v}{\bullet}
\end{tikzcd},$$

$$\Delta^{(2)} = \begin{tikzcd}[column sep=small]
\underset{u}{\bullet} \arrow[r, "\beta_{p_2}"] & \underset{y_{p_2-1}}{\bullet} \arrow[r] & \cdots \arrow[r] & \underset{y_1}{\bullet} \arrow[r, "\beta_1"] & \underset{v}{\bullet}
\end{tikzcd},$$

$$\Delta^{(3)} = \begin{tikzcd}[column sep=small]
\underset{u}{\bullet} & \underset{z_1}{\bullet} \arrow[l, "\gamma_1"'] & \cdots \arrow[l] & \underset{z_{p_3-1}}{\bullet} \arrow[l] & \underset{v}{\bullet} \arrow[l, "\gamma_{p_3}"']
\end{tikzcd},$$

for some $p_1, p_2, p_3, q_1 \in \mathbb{N}_+$. We may additionally assume that $\alpha_{i+1}\alpha_i \notin R$ for all $i \in [1, p_1 - 1]$. Let $r_1'$ be the number of $i \in [1, q_1 - 1]$ such that $\alpha_i'\alpha_{i+1}' \in R$, let $r_2$ be the number of $i \in [1, p_2 - 1]$ such that $\beta_i\beta_{i+1} \in R$, and let $r_3$ be the number of $i \in [1, p_3 - 1]$ such that $\gamma_i\gamma_{i+1} \in R$. Observe that by symmetry we may assume that $r_1' > 0$ if $\gamma_{p_3}\alpha_1' \in R$ and $\beta_{p_2}\alpha_1 \notin R$.

Assume first that $\beta_{p_2}\alpha_1 \in R$. In this case by shifting the relation $\beta_{p_2}\alpha_1$ to the right we may assume that $p_1 = 1$. Observe that either $r_1' = 0$ or $r_2 = 0$.

If $r_3 \geq 1$, then by shifting relations we may assume that $\gamma_1\gamma_2 \in R$. If $(\Delta', R')$ is the bound quiver obtained from $(\Delta, R)$ by applying the generalized APR-reflection at $u$, then

$$R' = (R \setminus \{\beta_{p_2}\alpha_1, \gamma_1\gamma_2\}) \cup \{\alpha_1\gamma_2, \beta_{p_2}\gamma_1\},$$

$$\Delta'^{(1)} = \begin{tikzcd}[column sep=small]
\underset{u}{\bullet} \arrow[r, "\alpha_1"] & \underset{x}{\bullet} \arrow[r, "\alpha'_{q_1}"] & \underset{x'_{q_1-1}}{\bullet} \arrow[r] & \cdots \arrow[r] & \underset{x'_1}{\bullet} \arrow[r, "\alpha'_1"] & \underset{v}{\bullet}
\end{tikzcd},$$

$$\Delta'^{(2)} = \begin{tikzcd}[column sep=small]
\underset{u}{\bullet} \arrow[r, "\gamma_1"] & \underset{z_1}{\bullet} \arrow[r, "\beta_{p_2}"] & \underset{y_{p_2-1}}{\bullet} \arrow[r] & \cdots \arrow[r] & \underset{y_1}{\bullet} \arrow[r, "\beta_1"] & \underset{v}{\bullet}
\end{tikzcd},$$

$$\Delta'^{(3)} = \underset{u}{\bullet} \xleftarrow{\gamma_2} \underset{z_2}{\bullet} \longleftarrow \cdots \longleftarrow \underset{v}{\bullet}\,,$$

hence the claim follows in this case.

Assume now that $r_3 = 0$. There are two additional possibilities in this case. If $\gamma_{p_3}\alpha'_1 \in R$, then $r_2 \geq 1$ (since $(\Delta, R)$ is a bound quiver). Consequently, $r'_1 = 0$ and we have the situation symmetric to the previous one. On the other hand, if $\gamma_{p_3}\beta_1 \in R$, then by shifting the relation $\gamma_{p_3}\beta_1$ to the left we may assume that $p_3 = 1$. Consequently, if $(\Delta', R')$ is the bound quiver obtained from $(\Delta, R)$ by applying the generalized APR-reflection at $u$, then

$$R' = (R \setminus \{\beta_{p_2}\alpha_1, \gamma_1\beta_1\}) \cup \{\alpha_1\beta_1, \beta_{p_2}\gamma_1\},$$

$$\Delta'^{(1)} = \underset{u}{\bullet} \xrightarrow{\alpha_1} \underset{x}{\bullet} \xrightarrow{\alpha'_{q_1}} \underset{x'_{q_1-1}}{\bullet} \longrightarrow \cdots \longrightarrow \underset{x'_1}{\bullet} \xrightarrow{\alpha'_1} \underset{v}{\bullet}\,,$$

$$\Delta'^{(2)} = \underset{u}{\bullet} \xleftarrow{\beta_1} \underset{y_1}{\bullet} \longleftarrow \cdots \longleftarrow \underset{y_{p_2-1}}{\bullet} \xleftarrow{\beta_{p_2}} \underset{v}{\bullet}\,,$$

$$\Delta'^{(3)} = \underset{u}{\bullet} \xrightarrow{\gamma_1} \underset{v}{\bullet}\,,$$

hence the claim follows.

Assume now that $\beta_{p_2}\gamma_1 \in R$. If $\gamma_{p_3}\beta_1 \in R$, then it follows easily that the bound quiver algebra of $(\Delta, R)$ is tilting-cotilting equivalent either to $\Lambda_2(p_2+p_3-r_2-1, r_2+1, q_1, p_1, r_3)^{\mathrm{op}}$ if $r'_1 = 0$, or to $\Lambda'_2(p_2+p_3-1, q_1, p_1, r_3, r'_1)^{\mathrm{op}}$ if $r'_1 \geq 1$. Since $\Lambda'_2(p_2+p_3-1, q_1, p_1, r_3, r'_1)$ is tilting-cotilting equivalent to $\Lambda_2(q_1, p_2+p_3, p_1, r'_1-1, r_3+1)$, we may assume that $\gamma_{p_3}\alpha'_1 \in R$. Consequently, by shifting relations we may assume that $\alpha'_i\alpha'_{i+1} \in R$ for all $i \in [1, q_1-1]$. Recall that $q_1 > 1$ in this case. If $(\Delta', R')$ is the bound quiver obtained from $(\Delta, R)$ by applying the APR-coreflection at $x$ followed by the composition of the HW-coreflection at $x'_{q_1-1}$ and the APR-reflection at $x'_{q_1-1}$ applied $q_1 - 1$ times, then

$$R' = (R \setminus \{\alpha'_{q_1-1}\alpha'_{q_1}\}) \cup \{\alpha'_{q_1-1}\alpha_{p_1}\},$$

$$\Delta'^{(1)} = \begin{cases} \underset{u}{\bullet} \xleftarrow{\alpha_1} \underset{x_1}{\bullet} \longleftarrow \cdots \longleftarrow \underset{x_{p_1-1}}{\bullet} \xrightarrow{\alpha_{p_1}} \underset{x}{\bullet} \xrightarrow{\alpha'_{q_1-1}} \underset{x'_{q_1-2}}{\bullet} \longrightarrow \cdots \longrightarrow \underset{v}{\bullet} & \text{if } p_1 > 1, \\ \underset{u}{\bullet} \xrightarrow{\alpha_1} \underset{x}{\bullet} \xrightarrow{\alpha'_{q_1-1}} \underset{x'_{q_1-2}}{\bullet} \longrightarrow \cdots \longrightarrow \underset{v}{\bullet} & \text{if } p_1 = 1, \end{cases}$$

$$\Delta'^{(2)} = \underset{u}{\bullet} \xrightarrow{\beta_1} \underset{y_1}{\bullet} \longrightarrow \cdots \longrightarrow \underset{y_{p_2-1}}{\bullet} \xrightarrow{\beta_{p_2}} \underset{x'_{q_1-1}}{\bullet} \xrightarrow{\alpha'_{q_1}} \underset{v}{\bullet}\,,$$

$$\Delta'^{(3)} = \underset{u}{\bullet} \xleftarrow{\gamma_1} \underset{z_1}{\bullet} \longleftarrow \cdots \longleftarrow \underset{z_{p_3-1}}{\bullet} \xleftarrow{\gamma_{p_3}} \underset{v}{\bullet}\,,$$

thus the claim follows by induction. ∎

We may now prove Proposition 2.1. According to our considerations we may assume that $(\Delta, R)$ is proper, and

$$\Delta^{(1)} = \underset{u}{\bullet} \xleftarrow{\alpha_1} \underset{x_1}{\bullet} \longleftarrow \cdots \longleftarrow \underset{x_{p_1-1}}{\bullet} \xleftarrow{\alpha_{p_1}} \underset{v}{\bullet}\,,$$

$$\Delta^{(2)} = \underset{u}{\bullet} \xrightarrow{\beta_{p_2}} \underset{y_{p_2-1}}{\bullet} \longrightarrow \cdots \longrightarrow \underset{y_1}{\bullet} \xrightarrow{\beta_1} \underset{v}{\bullet}\,,$$

$$\Delta^{(3)} = \underset{u}{\bullet} \xleftarrow{\gamma_1} \underset{z_1}{\bullet} \longleftarrow \cdots \longleftarrow \underset{z_{p_3-1}}{\bullet} \xleftarrow{\gamma_{p_3}} \underset{v}{\bullet}\,,$$

for some $p_1, p_2, p_3 \in \mathbb{N}_+$. Moreover, we may additionally assume that $\alpha_i\alpha_{i+1} \notin R$ for all $i \in [1, p_1 - 1]$. Let $r_2$ be the number of $i \in [1, p_2 - 1]$ such that $\beta_i\beta_{i+1} \in R$ and let $r_3$ be the number of $i \in [1, p_3 - 1]$ such that $\gamma_i\gamma_{i+1} \in R$.

Observe that if either $\beta_{p_2}\alpha_1 \in R$ and $\gamma_{p_3}\beta_1 \in R$, or $\beta_{p_2}\gamma_1 \in R$ and $\alpha_{p_1}\beta_1 \in R$, then the claim follows from Proposition 1.16, thus we have to consider two remaining cases.

Assume first that $\beta_{p_2}\alpha_1 \in R$ and $\alpha_{p_1}\beta_1 \in R$. In this case by shifting the relation $\beta_{p_2}\alpha_1$ to the right we may assume that $p_1 = 1$. If $r_3 = 0$, then the bound quiver algebra of $(\Delta, R)$ is tilting-cotilting equivalent to $\Lambda_1(p_2, 1, p_3, 0, r_2)$ (observe that $r_2 \geq 1$ since $(\Delta, R)$ is a bound quiver). On the other hand, if there exists $i \in [1, p_3 - 1]$ such that $\gamma_i\gamma_{i+1} \in R$, then by shifting relations we may assume that $\gamma_1\gamma_2 \in R$. Consequently, the bound quiver algebra of the bound quiver obtained from $(\Delta, R)$ by applying the generalized APR-reflection at $u$ is tilting-cotilting equivalent to $\Lambda_2(p_2 + 1, p_3, 0, r_2 + 1, r_3 - 1)$, and this finishes the proof in this case.

Assume now that $\beta_{p_2}\gamma_1 \in R$ and $\gamma_{p_3}\beta_1 \in R$. In this case it follows by shifting relations that the bound quiver algebra of $(\Delta, R)$ is tilting-cotilting equivalent to $\Lambda_1(p_2 + p_3 - r_3 - 1, r_3 + 1, p_1, 0, r_2)$ (again $r_2 \geq 1$ since $(\Delta, R)$ is a bound quiver), and this finishes the proof.

**3. Minimality of the list.** In this section we prove that different algebras from the list in Theorem 1 are not derived equivalent. We also check that the algebras listed in Theorem 1 are nondegenerate, while the algebras listed in Theorem 2 are degenerate. A tool used in order to distinguish between derived equivalence classes of these algebras will be the derived invariant introduced by Avella-Alaminos and Geiss in [7].

Let $(\Delta, R)$ be a gentle quiver. By a *permitted thread* in $(\Delta, R)$ we mean either a maximal path in $(\Delta, R)$, or $x \in \Delta_0$ such that there is at most one arrow $\alpha$ with $s\alpha = x$, there is at most one arrow $\beta$ with $t\beta = x$, and $\alpha\beta \notin R$ for all $\alpha, \beta \in \Delta_1$ with $s\alpha = x = t\beta$. Similarly, we define the notion of a forbidden thread in $(\Delta, R)$. Namely, first we say that by an *anti-path* in $(\Delta, R)$ we mean a path $\alpha_1 \cdots \alpha_n$ in $\Delta$ such that $\alpha_i\alpha_{i+1} \in R$ for all $i \in [1, n-1]$. In particular, every trivial path is an anti-path. By a *forbidden* *thread* we mean either a maximal anti-path in $(\Delta, R)$, or $x \in \Delta_0$ such that there is at most one arrow $\alpha$ with $s\alpha = x$, there is at most one arrow $\beta$ with $t\beta = x$, and $\alpha\beta \in R$ for all $\alpha, \beta \in \Delta_1$ with $s\alpha = x = t\beta$.

By a *characteristic sequence* in a gentle bound quiver $(\Delta, R)$ we mean a sequence $(\sigma_i, \tau_i)_{i\in\mathbb{Z}}$ of permitted threads $\sigma_i$, $i \in \mathbb{Z}$, and forbidden threads $\tau_i$, $i \in \mathbb{Z}$, such that for each $i \in \mathbb{Z}$ the following conditions are satisfied:

(1) $t\tau_i = t\sigma_i$ and $s\sigma_{i+1} = s\tau_i$,
(2) if $\sigma_i = x = \tau_i$ for $x \in \Delta_0$, then $\sigma_{i+1} \neq x$, unless $\Delta_1 = \emptyset$,
(3) if $\tau_i = x = \sigma_{i+1}$ for $x \in \Delta_0$, then $\tau_{i+1} \neq x$, unless $\Delta_1 = \emptyset$,
(4) if neither $\sigma_i$ nor $\tau_i$ is a trivial path, then the terminating arrow of $\tau_i$ differs from the terminating arrow of $\sigma_i$,
(5) if neither $\tau_i$ nor $\sigma_{i+1}$ is a trivial path, then the starting arrow of $\sigma_{i+1}$ differs from the starting arrow of $\tau_i$.

We identify characteristic sequences $(\sigma_i, \tau_i)_{i\in\mathbb{Z}}$ and $(\sigma'_i, \tau'_i)_{i\in\mathbb{Z}}$ if there exists $l \in \mathbb{Z}$ such that $\sigma_i = \sigma'_{i+l}$ and $\tau_i = \tau'_{i+l}$ for all $i \in \mathbb{Z}$. By the *type* of the characteristic sequence $(\sigma_i, \tau_i)_{i\in\mathbb{Z}}$ we mean a pair $(n, m) \in \mathbb{N} \times \mathbb{N}$ defined by $n = \min\{l \in \mathbb{N}_+ \mid \sigma_l = \sigma_0\}$ and $m = \sum_{i\in[1,n]} \ell(\tau_i)$. In the above situation we also write $(\sigma_1, \tau_1, \ldots, \sigma_n, \tau_n)$ instead of $(\sigma_i, \tau_i)_{i\in\mathbb{Z}}$. Additionally, we also call a sequence $(\alpha_i)_{i\in\mathbb{Z}}$ of arrows in $\Delta$ a characteristic sequence if $s\alpha_i = t\alpha_{i+1}$ and $\alpha_i\alpha_{i+1} \in R$ for all $i \in \mathbb{Z}$. Again we identify sequences $(\alpha_i)_{i\in\mathbb{Z}}$ and $(\alpha'_i)_{i\in\mathbb{Z}}$ if there exists $l \in \mathbb{Z}$ such that $\alpha_i = \alpha'_{i+l}$ for all $i \in \mathbb{Z}$. The type of a characteristic sequence $(\alpha_i)_{i\in\mathbb{Z}}$ of the above type is by definition $(0, m)$, where $m = \min\{l \in \mathbb{N}_+ \mid \alpha_l = \alpha_0\}$. In the above situation we also write $\alpha_1 \cdots \alpha_m$ instead of $(\alpha_i)_{i\in\mathbb{Z}}$.

If $(\Delta, R)$ is a gentle bound quiver, then the function $\phi_{\Delta,R} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, where $\phi_{\Delta,R}(n, m)$ is the number of the characteristic sequences of type $(n, m)$ for $(n, m) \in \mathbb{N} \times \mathbb{N}$, is a derived invariant, i.e. if $(\Delta, R)$ and $(\Delta', R')$ are derived equivalent gentle bound quivers, then $\phi_{\Delta,R} = \phi_{\Delta',R'}$. If $\Lambda$ is the bound quiver algebra of a gentle bound quiver $(\Delta, R)$, then we also write $\phi_\Lambda$ instead of $\phi_{\Delta,R}$. We will write $\phi_{\Delta,R}$ as a "multi-set" $[(n_1, m_1), \ldots, (n_l, m_l)]$, where $(n, m)$ appears $\phi_{\Delta,R}(n, m)$ times.

We calculate the values of the above invariant for algebras appearing in Theorems 1 and 2, and this will finish the proofs of these theorems. The proof of the following lemma is left to the reader as an easy exercise.

LEMMA 3.1. *We have the following.*

(1) *If* $p \in \mathbb{N}_+$ *and* $r \in [0, p-1]$, *then*
$$\phi_{\Lambda_0(p,r)} = [(p, p+2)].$$
(2) *If* $p \in \mathbb{N}_+$, *then*
$$\phi_{\Lambda'_0(p,0)} = [(p+1, p+3)].$$

(3) *If* $p_1, p_2 \in \mathbb{N}_+$, $p_3, p_4 \in \mathbb{N}$, *and* $r_1 \in [0, p_1 - 1]$ *are such that* $p_2 + p_3 \geq 2$ *and* $r_1 + p_4 \geq 1$, *then*

$$\phi_{\Lambda_1(p_1,p_2,p_3,p_4,r_1)} = [(p_1 - r_1 - 1, p_1 + p_2), (p_2 + p_3 - 1, p_3), (r_1 + p_4, p_4)].$$

(4) *If* $p_1, p_2 \in \mathbb{N}_+$, $p_3 \in \mathbb{N}$, $r_1 \in [0, p_1 - 1]$, $r_2 \in [0, p_2 - 1]$, *are such that* $p_3 + r_1 + r_2 \geq 1$, *then*

$$\phi_{\Lambda_2(p_1,p_2,p_3,r_1,r_2)} = [(p_1 - r_1 - 1, p_1), (p_2 - r_2 - 1, p_2), (r_1 + r_2 + p_3, p_3)].$$

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