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# BIPARTITE COALGEBRAS AND A REDUCTION FUNCTOR FOR CORADICAL SQUARE COMPLETE COALGEBRAS

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**Abstract.** Let $C$ be a coalgebra over an arbitrary field $K$. We show that the study of the category $C$-Comod of left $C$-comodules reduces to the study of the category of (co)representations of a certain bicomodule, in case $C$ is a bipartite coalgebra or a coradical square complete coalgebra, that is, $C = C_1$, the second term of the coradical filtration of $C$. If $C = C_1$, we associate with $C$ a $K$-linear functor $\mathbb{H}_C : C\text{-Comod} \to H_C\text{-Comod}$ that restricts to a representation equivalence $\mathbb{H}_C : C\text{-comod} \to H_C\text{-comod}^{\bullet}_{\mathrm{sp}}$, where $H_C$ is a coradical square complete hereditary bipartite $K$-coalgebra such that every simple $H_C$-comodule is injective or projective. Here $H_C\text{-comod}^{\bullet}_{\mathrm{sp}}$ is the full subcategory of $H_C$-comod whose objects are finite-dimensional $H_C$-comodules with projective socle having no injective summands of the form $\left[\begin{smallmatrix} S(i') \\ 0 \end{smallmatrix}\right]$ (see Theorem 5.11). Hence, we conclude that a coalgebra $C$ with $C = C_1$ is left pure semisimple if and only if $H_C$ is left pure semisimple. In Section 6 we get a diagrammatic characterisation of coradical square complete coalgebras $C$ that are left pure semisimple. Tameness and wildness of such coalgebras $C$ is also discussed.

**1. Introduction.** Throughout this paper we fix an arbitrary field $K$ and we use the coalgebra representation theory notation and terminology introduced in [14], [29]–[35]. The reader is referred to [1], [2], [12], [27], [37], and [38] for the representation theory terminology and notation, and to [16], [39] for the coalgebra and comodule terminology. In particular, given a finite-dimensional $K$-algebra $R$, we denote by $\mathrm{mod}(R)$ the category of all finite-dimensional $R$-modules.

Let $C$ be a $K$-coalgebra with comultiplication $\Delta$ and counit $\varepsilon$. We recall that a *left $C$-comodule* is a $K$-vector space $X$ together with a $K$-linear map $\delta_X : X \to C \otimes X$ such that $(\Delta \otimes \mathrm{id}_X)\delta_X = (\mathrm{id}_C \otimes \delta_X)\delta_X$ and $(\varepsilon \otimes \mathrm{id}_X)\delta_X$ is the canonical isomorphism $X \cong K \otimes X$, where $\otimes = \otimes_K$. Given a left $C$-comodule $X$, we denote by $X_0 = \mathrm{soc}\, X$ the *socle* of $X$, that is, the sum of all simple $C$-subcomodules of $X$.

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A $K$-linear map $f : X \to Y$ between two left $C$-comodules $X$ and $Y$ is a $C$-*comodule homomorphism* if $\delta_Y f = (\mathrm{id}_C \otimes f)\delta_X$. The $K$-vector space of all $C$-comodule homomorphisms from $X$ to $Y$ is denoted by $\mathrm{Hom}_C(X, Y)$. The $K$-algebra of all $C$-comodule endomorphisms of $X$ is denoted by $\mathrm{End}_C X$.

We denote by $C$-Comod the category of all left $C$-comodules, and by $C$-comod the full subcategory of $C$-Comod formed by $C$-comodules of finite $K$-dimension.

We recall that a $K$-coalgebra $C$ is *semisimple* (resp. *hereditary*) if $\mathrm{Ext}^1_C(M, N) = 0$ (resp. $\mathrm{Ext}^2_C(M, N) = 0$) for all $M$ and $N$ in $C$-Comod, or equivalently, if $M = \mathrm{soc}\, M$ for all $M$ in $C$-Comod (resp. if epimorphic images of injective $C$-comodules are injective $C$-comodules). A $K$-coalgebra $C$ is said to be *indecomposable* (or connected) if $C$ is not a product of two subcoalgebras, or equivalently, if $C$-Comod is not a direct sum of two non-trivial subcategories.

Given a coalgebra $C$, we denote by $C_0 \subseteq C_1 \subseteq \cdots \subseteq C$ the *coradical filtration* of $C$, where $C_0 = \mathrm{soc}\,_C C$ (or equivalently, the sum of all simple subcoalgebras of $C$), $C_1 = C_0 \wedge C_0$ is the wedge of two copies of $C_0$, and $C_{m+1} = C_0 \wedge C_m$ for $m \geq 1$.

We call $C$ *basic* if there is a decomposition $\mathrm{soc}\,_C C = \bigoplus_{j \in I_C} S(j)$ such that $\{S(j);\, j \in I_C\}$ is a complete set of pairwise non-isomorphic simple left $C$-comodules (see [4], [6], [26] and [29]).

One of the aims of this paper is to study the comodule categories and the valued Gabriel quiver of the following class of coalgebras that are topologically dual (see [29]) to the class of (Jacobson) radical square zero algebras.

Definition 1.1. A $K$-coalgebra $C$ is defined to be *coradical square complete* if $C = C_1 = C_0 \wedge C_0$.

Following an idea of Gabriel [10] (see also [2, Section X.2]), we reduce the study of $C$-comodules over any coradical square complete coalgebra $C$ to the study of comodules over a coradical square complete hereditary coalgebra $H_C$ which is a bipartite coalgebra in the sense of Definition 2.0 below. Moreover, every simple subcomodule of $H_C$ is projective or injective. This is one of the motivations for our investigations in this paper, because the representation theory of hereditary coalgebras is well understood by a reduction to the study of nilpotent representations of quivers or $K$-species (see [14], [20], [29]–[35]), and therefore we get an efficient tool for the study of $C$-comod.

We recall from [1], [2], [10], [12], [15], [27], [37], and [38] that triangular matrix algebras play an important role in the representation theory of finite-dimensional algebras. In particular, we know from [10] and [2, Section X.2] that the representation theory of radical square zero algebras of finite $K$-dimension reduces to the representation theory of hereditary triangular matrix algebras. In Section 2 we follow this idea and, in analogy to triangular matrix algebras and bipartite rings [27, Section 17.4], we introduce a concept of a bipartite $K$-coalgebra

$$H = \begin{bmatrix} H' & {}_{H'}U_{H''} \\ 0 & H'' \end{bmatrix},$$

where $(H', \Delta', \varepsilon')$ and $(H'', \Delta'', \varepsilon'')$ are $K$-coalgebras and ${}_{H'}U_{H''}$ is a $H'$-$H''$-*bicomodule*, that is, ${}_{H'}U_{H''}$ is a left $H'$-comodule $(U, \delta'_U : U \to H' \otimes U)$ equipped with a right $H''$-comodule structure given by a right $H''$-comodule homomorphism $\delta''_U : U \to U \otimes H''$, which is a homomorphism of left $H'$-comodules. Moreover, given $H$ as above, we define an equivalence of categories between $H$-Comod and the category $\mathrm{Rep}_{\square}({}_{H'}U_{H''})$ of (co)representations of ${}_{H'}U_{H''}$.

In Section 4, following Gabriel [10], with each coradical square complete coalgebra $C$ we associate a coradical square complete hereditary bipartite $K$-coalgebra $H_C$ and a $K$-linear functor

$$\mathbb{H}_C : C\text{-Comod} \to H_C\text{-Comod}. \tag{1.2}$$

We prove in Theorem 5.11 that $\mathbb{H}_C$ is full, carries injectives to injectives, does not vanish on non-zero comodules, but vanishes on the $C$-comodule homomorphisms $f : X \to Y$ such that $f(\mathrm{soc}\ X) = 0$. Moreover, $\mathbb{H}_C$ restricts to a representation equivalence of categories (i.e. it is full, dense, and reflects isomorphisms, see [27], [28], and [38])

$$\mathbb{H}_C : C\text{-comod} \to H_C\text{-comod}^{\bullet}_{\mathrm{sp}}, \tag{1.3}$$

where $H_C$-comod$^{\bullet}_{\mathrm{sp}}$ is the full subcategory of $H_C$-comod whose objects are the finite-dimensional $H_C$-comodules with projective socle having no injective summands of the form $\left[\begin{smallmatrix} S(i') \\ 0 \end{smallmatrix}\right]$ (see Theorem 5.11). It follows that $C$ is left pure semisimple if and only if $H_C$ is. Hence, by applying [14], [20] and [29], we get in Section 6 a diagrammatic characterisation of coradical square complete coalgebras $C$ that are left pure semisimple.

Following an idea of trivial extension algebra (see [2] and [13]), and in connection with the reduction functor (1.2), we study in Section 4 the trivial extension coalgebra $D \ltimes {}_DU_D$ (see (4.8)) of a given coalgebra $D$ by a $D$-$D$-bicomodule ${}_DU_D$, the repetitive coalgebra $\Re(D, {}_DU_D)$ (see (4.15)), and the covering functor (see (4.17))

$$f^{\blacktriangledown} : \Re(D, {}_DU_D)\text{-Comod} \to (D \ltimes {}_DU_D)\text{-Comod}$$

induced by the canonical coalgebra surjection

$$f : \Re(D, {}_DU_D) \to D \ltimes {}_DU_D.$$

Also we complete the results given in [3], [14], [17], [32], and [41] by presenting three alternative descriptions of the left valued Gabriel quiver of a given basic coalgebra

$$C = \bigoplus_{a \in I_C} E(a),$$

with indecomposable left coideals $E(a)$, $a \in I_C$. The descriptions are given by the $F_a$-$F_b$-bimodule isomorphisms (see (3.6)),

$$\text{(1.4)} \qquad \mathrm{Hom}_{F_a}(\mathrm{Ext}^1_C(S(a), S(b)), F_a) \underset{\simeq}{\to} \mathrm{Irr}_C(E(b), E(a)) \underset{\simeq}{\to} {}_a(C_1/C_0)_b,$$

where $S(j) = \mathrm{soc}\, E(j)$ and $F_j = \mathrm{End}_C S(j)$ for $j \in I_C$.

Throughout this paper, by a *quiver* we mean a pair $Q = (Q_0, Q_1)$, where $Q_0$ is the set of vertices of $Q$ and $Q_1$ is the set of arrows of $Q$. By a *valued quiver* we mean a pair $(Q, \mathbf{d})$, where $Q$ is a quiver such that each arrow $\beta \in Q_1$ is equipped with a pair $(d'_\beta, d''_\beta)$ of positive integers; we visualise $\beta$ as the valued arrow

$$a \xrightarrow{(d'_\beta, d''_\beta)} b.$$

If $d'_\beta = d''_\beta = 1$, then we simply write $a \to b$ instead of $a \xrightarrow{(d'_\beta, d''_\beta)} b$.

By a valued quiver *dual* to $(Q, \mathbf{d})$ we mean the valued quiver $(Q^\circ, \mathbf{d}^\circ)$, where $Q^\circ_0 = Q_0$ and, for each valued arrow $a \xrightarrow{(d'_\beta, d''_\beta)} b$ in $(Q, \mathbf{d})$, we define the unique valued arrow $\beta^\circ$ in $(Q^\circ, \mathbf{d}^\circ)$ to be $b \xrightarrow{(d''_\beta, d'_\beta)} a$.

Let $X$ be a right $C$-comodule and $Y$ be a left $C$-comodule. We recall from [9] that the *cotensor product* $X \,\square\, Y$ is the $K$-vector space

$$\text{(1.5)} \qquad X \,\square\, Y = \mathrm{Ker}(X \otimes Y \xrightarrow{\delta_X \otimes \mathrm{id}_Y - \mathrm{id}_X \otimes \delta_Y} X \otimes C \otimes Y).$$

It is known that $X \,\square\, C \cong X$, $C \,\square\, Y \cong Y$, the functors

$$X \,\square\, - : C\text{-Comod} \to \mathrm{mod}\, K \quad \text{and} \quad - \,\square\, Y : \text{Comod-}C \to \mathrm{mod}\, K$$

are left exact, commute with arbitrary direct sums, and there is a functorial isomorphism

$$X \,\square\, Y \cong \mathrm{Hom}_C(Y^*, X)$$

for any $X$ in Comod-$C$ and any $Y$ in $C$-comod, where $Y^* = \mathrm{Hom}_K(Y, K)$ is equipped with the $K$-dual right $C$-comodule structure (see [8] and [39]).

**2. Bipartite coalgebras and representations of bicomodules.** In this section we introduce a concept of a bipartite coalgebra (see (2.1)) in an analogy with the notion of a (generalised) triangular matrix algebra (see [1, Appendix 2.7], [27], and [38, Section VX.1]). We prove that, for a bipartite coalgebra $H$, the category $H$-Comod is equivalent to the category of (co)representations of the bicomodule defining $H$.

*Bipartite coalgebras.* In analogy with [1, Appendix 2.7], [27, Section 17.4], and [38, Section VX.1], we introduce the following definition.

Definition 2.0. Let $H'$ and $H''$ be $K$-coalgebras, and let ${}_{H'}U_{H''}$ be a non-zero $H'$-$H''$-bicomodule. We associate with ${}_{H'}U_{H''}$ the *bipartite $K$-coalgebra*

$$H = \begin{bmatrix} H' & {}_{H'}U_{H''} \\ 0 & H'' \end{bmatrix} \tag{2.1}$$

consisting of all formal matrices $h = \left[\begin{smallmatrix} h' & u \\ 0 & h'' \end{smallmatrix}\right]$, where $h' \in H'$, $h'' \in H''$ and $u \in U$. We make the following identification:

$$H \otimes H \equiv \begin{bmatrix} H' \otimes H' & H' \otimes U & H' \otimes H'' \\ U \otimes H' & U \otimes U & U \otimes H'' \\ H'' \otimes H' & H'' \otimes U & H'' \otimes H'' \end{bmatrix} \tag{2.2}$$

The comultiplication $\Delta : H \to H \otimes H$ of $H$ and the counit $\varepsilon : H \to K$ of $H$ are defined by the following formulae:

$$\begin{aligned} \Delta(h) &= \Delta'(h') + \Delta''(h'') + \delta'_U(u) + \delta''_U(u) \\ &= \begin{bmatrix} \Delta'(h') & \delta'_U(u) & 0 \\ 0 & 0 & \delta''_U(u) \\ 0 & 0 & \Delta''(h'') \end{bmatrix}, \\ \varepsilon(h) &= \varepsilon'(h') + \varepsilon''(h''). \end{aligned} \tag{2.3}$$

It is easy to check that $H$ is a $K$-coalgebra, the $K$-subspaces

$$\begin{bmatrix} H' \\ 0 \end{bmatrix} \equiv \begin{bmatrix} H' & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} U \\ H'' \end{bmatrix} \equiv \begin{bmatrix} 0 & {}_{H'}U_{H''} \\ 0 & H'' \end{bmatrix} \tag{2.4}$$

of $H$ are left coideals and, under the above identification, the left $H$-comodule ${}_H H$ has a direct sum decomposition

$$H = \begin{bmatrix} H' & {}_{H'}U_{H''} \\ 0 & H'' \end{bmatrix} = \begin{bmatrix} H' \\ 0 \end{bmatrix} \oplus \begin{bmatrix} U \\ H'' \end{bmatrix}. \tag{2.5}$$

Moreover, the canonical projection $\pi : H \to H' \oplus H''$, defined by the formula $\pi\left[\begin{smallmatrix} h' & u \\ 0 & h'' \end{smallmatrix}\right] = (h', h'')$, is a $K$-coalgebra homomorphism and induces a faithful $K$-linear embedding

$$\pi^\circ : H\text{-Comod} \to (H' \oplus H'')\text{-Comod} \tag{2.6}$$

associating to each left $H$-comodule $(X, \delta_X)$ the left $(H' \oplus H'')$-comodule $(X, \widehat{\delta}_X)$ with comultiplication $\widehat{\delta}_X = (\pi \otimes \mathrm{id}_X) \circ \delta_X : X \to (H' \oplus H'') \otimes X$. Denote by $\pi_{H'} : H \to H'$ and $\pi_{H''} : H \to H''$ the obvious projections.

*Representations of bicomodules.* In analogy with [1, Appendix 2.7] and [38, Section VX.1], we introduce the following definition.

Definition 2.7. Let $H'$ and $H''$ be $K$-coalgebras. Given an $H'$-$H''$-bicomodule ${}_{H'}U_{H''}$, we define the category $\mathrm{Rep}_{\Box}({}_{H'}U_{H''})$ of *left (co)representations of* ${}_{H'}U_{H''}$ as follows.

(a) The objects of $\mathrm{Rep}_{\Box}({}_{H'}U_{H''})$ are triples $(X', X'', \varphi)$, where $X'$ is a left $H'$-comodule, $X''$ is a left $H''$-comodule and $\varphi : X' \to U \,\Box\, X''$ is a homomorphism of left $H'$-comodules.

(b) A morphism from $(X', X'', \varphi)$ to $(Y', Y'', \psi)$ in $\mathrm{Rep}_{\Box}({}_{H'}U_{H''})$ is a pair $(f', f'')$, where $f' \in \mathrm{Hom}_{H'}(X', Y')$, $f'' \in \mathrm{Hom}_{H''}(X'', Y'')$ and $(\mathrm{id}_U \,\Box\, f'')\varphi = \psi f'$. The composition of morphisms in $\mathrm{Rep}_{\Box}({}_{H'}U_{H''})$ is componentwise.

(c) The representation $(X', X'', \varphi)$ is called *finite-dimensional* if the comodules $X'$ and $X''$ are of finite $K$-dimension.

(d) We denote by $\mathrm{rep}_{\Box}({}_{H'}U_{H''})$ the full subcategory of $\mathrm{Rep}_{\Box}({}_{H'}U_{H''})$ formed by the finite-dimensional representations.

It is clear that $\mathrm{Rep}_{\Box}({}_{H'}U_{H''})$ and $\mathrm{rep}_{\Box}({}_{H'}U_{H''})$ are abelian $K$-categories. We show below that there is an equivalence of categories $H\text{-Comod} \cong \mathrm{Rep}_{\Box}({}_{H'}U_{H''})$. For this, we define a pair of $K$-linear functors

$$H\text{-Comod} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \mathrm{Rep}_{\Box}({}_{H'}U_{H''}) \tag{2.8}$$

as follows.

*The functor $\Phi$.* Before we define the functor $\Phi$ (see (2.11)), we need a preparation. Given a left $H$-comodule $(X, \delta_X)$, we decompose the $K$-vector space $X$ as $X = X' \oplus X''$, where

$$X' = \widehat{\delta}_X^{-1}(H' \otimes X) \quad \text{and} \quad X'' = \widehat{\delta}_X^{-1}(H'' \otimes X). \tag{2.9}$$

It is easy to see that $X' = (X', \widehat{\delta}_{X'} = (\widehat{\delta}_X)_{|X'})$ and $X'' = (X'', \widehat{\delta}_{X''} = (\widehat{\delta}_X)_{|X''})$ are a left $H'$-comodule and a left $H''$-comodule, respectively. We denote by $\widetilde{\varphi} : X \to U \otimes X''$ the composite $K$-linear map

$$X \xrightarrow{\delta_X} H \otimes X \xrightarrow{\pi_U \otimes \pi_{X''}} U \otimes X'',$$

where $\pi_U : H \to U$ is the canonical projection defined by $\pi_U\left[\begin{smallmatrix} h' & u \\ 0 & h'' \end{smallmatrix}\right] = u$, and $\pi_{X''} : X \to X''$ is the obvious projection.

Lemma 2.10. *If $\widetilde{\varphi} : X \to U \otimes X''$ is the map defined above then* $\mathrm{Im}\,\widetilde{\varphi} \subseteq U \,\Box\, X''$.

*Proof.* Note that the diagram

$$\begin{array}{ccccccc}
X & \xrightarrow{\delta_X} & H \otimes X & \xrightarrow{\pi_U \otimes \mathrm{id}} & U \otimes X & \xrightarrow{\mathrm{id} \otimes \pi_{X''}} & U \otimes X'' \\
\downarrow{\scriptstyle \delta_X} & & \downarrow{\scriptstyle \mathrm{id} \otimes \delta_X} & & \downarrow{\scriptstyle \mathrm{id} \otimes \delta_X} & & \downarrow{\scriptstyle \mathrm{id} \otimes \widehat{\delta}_{X''}} \\
H \otimes X & \xrightarrow{\Delta \otimes \mathrm{id}} & H \otimes H \otimes X & \xrightarrow{\pi_U \otimes \mathrm{id} \otimes \mathrm{id}} & U \otimes H \otimes X & \xrightarrow{\mathrm{id} \otimes \pi_{H''} \otimes \pi_{X''}} & U \otimes H'' \otimes X''
\end{array}$$

is commutative. Indeed, by the definition of $\widehat{\delta}_X$, the right square commutes. Moreover, $(\mathrm{id} \otimes \delta_X)\delta_X = (\Delta \otimes \mathrm{id})\delta_X$, because $X$ is a left $H$-comodule.

The commutativity of this diagram yields

$$(\mathrm{id} \otimes \widehat{\delta}_{X''})\widetilde{\varphi} = (\pi_U \otimes \pi_{H''} \otimes \pi_{X''})(\Delta \otimes \mathrm{id})\delta_X.$$

Since the definition (2.3) of $\Delta$ yields $(\pi_U \otimes \pi_{H''})\Delta = \delta''_U \pi_U$, we obtain

$$\begin{aligned}(\mathrm{id} \otimes \widehat{\delta}_{X''})\widetilde{\varphi} &= (\pi_U \otimes \pi_{H''} \otimes \pi_{X''})(\Delta \otimes \mathrm{id})\delta_X = ((\pi_U \otimes \pi_{H''})\Delta \otimes \pi_{X''})\delta_X \\ &= (\delta''_U \pi_U \otimes \pi_{X''})\delta_X = (\delta''_U \otimes \mathrm{id})(\pi_U \otimes \pi_{X''})\delta_X = (\delta''_U \otimes \mathrm{id})\widetilde{\varphi}.\end{aligned}$$

Hence, the required inclusion $\mathrm{Im}\,\widetilde{\varphi} \subseteq U \,\square\, X''$ follows. ∎

Denote by $\varphi : X' \to U \otimes X''$ the composite $K$-linear map

$$X' \hookrightarrow X \xrightarrow{\delta_X} H \otimes X \xrightarrow{\pi_U \otimes \pi_{X''}} U \otimes X''.$$

By Lemma 2.10, we have $\mathrm{Im}\,\varphi \subseteq U \,\square\, X'' \subseteq U \otimes X''$. Now we show that $\varphi$ is a homomorphism of left $H'$-comodules. Put $i_{X'} : X' \hookrightarrow X$ and note that

$$\begin{aligned}(\delta'_U \otimes \mathrm{id})\varphi &= (\delta'_U \otimes \mathrm{id})(\pi_U \otimes \pi_{X''})\delta_X i_{X'} = (\delta'_U \pi_U \otimes \mathrm{id})(\mathrm{id} \otimes \pi_{X''})\delta_X i_{X'} \\ &= ((\pi_{H'} \otimes \pi_U)\Delta \otimes \mathrm{id})(\mathrm{id} \otimes \pi_{X''})\delta_X i_{X'} \\ &= ((\pi_{H'} \otimes \pi_U) \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \pi_{X''})(\Delta \otimes \mathrm{id})\delta_X i_{X'} \\ &= ((\pi_{H'} \otimes \pi_U) \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{id} \otimes \pi_{X''})(\mathrm{id} \otimes \delta_X)\delta_X i_{X'} \\ &= (\pi_{H'} \otimes \widetilde{\varphi})\delta_X i_{X'} = (\mathrm{id} \otimes \widetilde{\varphi})(\pi_{H'} \otimes \mathrm{id})\delta_X i_{X'} = (\mathrm{id} \otimes \varphi)\widehat{\delta}_{X'},\end{aligned}$$

that is, $\varphi$ is a homomorphism of left $H'$-comodules.

To define the functor $\Phi$, we denote by $\varphi_X : X' \to U \,\square\, X''$ the unique factorisation of $\varphi$ through the embedding $U \,\square\, X'' \subseteq U \otimes X''$. It follows that $\varphi_X$ is a homomorphism of left $H'$-comodules and therefore $(X', X'', \varphi_X)$ is an object of the category $\mathrm{Rep}_\square({}_{H'}U_{H''})$. We set

$$\Phi(X) = (X', X'', \varphi_X). \tag{2.11}$$

Let $f : X \to Y$ be a homomorphism of left $H$-comodules, and let $X = X' \oplus X''$, $Y = Y' \oplus Y''$ be the decompositions defined by (2.9), where $X'$, $Y'$ are left $H'$-comodules and $X''$, $Y''$ are left $H''$-comodules. It is easy to see that $f(X') \subseteq Y'$ and $f(X'') \subseteq Y''$. Then the restrictions $f_{|X'}$ and $f_{|X''}$ induce $K$-linear maps $f' : X' \to Y'$ and $f'' : X'' \to Y''$, respectively. A straightforward calculation shows that $f'$ and $f''$ are homomorphisms of left $H'$-comodules and $H''$-comodules, respectively, such that the diagram

$$\begin{array}{ccc} X' & \xrightarrow{\varphi_X} & U \,\square\, X'' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_U \otimes f''} \\ Y' & \xrightarrow{\varphi_Y} & U \,\square\, Y'' \end{array}$$

in $H'$-Comod is commutative, that is, $(f', f'') : (X', X'', \varphi_X) \to (Y', Y'', \varphi_Y)$ is a morphism in the category $\mathrm{Rep}_\square({}_{H'}U_{H''})$. We define $\Phi(f) : \Phi(X) \to \Phi(Y)$

by setting $\Phi(f) = (f', f'')$. It is clear that we have defined a $K$-linear, faithful and exact functor $\Phi : H\text{-Comod} \to \mathrm{Rep}_\square({}_{H'}U_{H''})$.

EXAMPLE 2.12. Let $H$ be a bipartite algebra of the form (2.1). Consider the left $H$-comodules $\left[\begin{smallmatrix} H' \\ 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} U \\ H'' \end{smallmatrix}\right]$. To illustrate the definition of $\Phi$, we compute the representations $\Phi\big(\left[\begin{smallmatrix} H' \\ 0 \end{smallmatrix}\right]\big)$ and $\Phi\big(\left[\begin{smallmatrix} U \\ H'' \end{smallmatrix}\right]\big)$. By (2.3) and (2.9), we get $\Phi\big(\left[\begin{smallmatrix} H' \\ 0 \end{smallmatrix}\right]\big) = (H', 0, 0)$ and $\Phi\big(\left[\begin{smallmatrix} U \\ H'' \end{smallmatrix}\right]\big) = (U, H'', \varphi)$. By the above considerations and the definition of $\Phi$, $\varphi = \delta''_U$ defines the right $H''$-comodule structure on $U$.

*The functor $\Psi$.* The functor $\Psi$ in (2.8) is defined by setting, for each object $(X', X'', \varphi)$ in $\mathrm{Rep}_\square({}_{H'}U_{H''})$,

$$\Psi(X', X'', \varphi) = (X, \delta_X), \tag{2.13}$$

where $X = X' \oplus X''$ and $\delta_X : X \to H \otimes X$ is the $K$-linear map defined by

$$\delta_X(x', x'') = \begin{bmatrix} \delta'_{X'}(x') & \varphi(x') \\ 0 & \delta''_{X''}(x'') \end{bmatrix} \in \begin{bmatrix} H' \otimes X' & {}_{H'}U_{H''} \otimes X'' \\ 0 & H'' \otimes X'' \end{bmatrix} \subseteq H \otimes X.$$

Here we make the following identification of $K$-vector spaces:

$$\begin{aligned} H \otimes X &= \begin{bmatrix} H' & {}_{H'}U_{H''} \\ 0 & H'' \end{bmatrix} \otimes (X' \oplus X'') \\ &\equiv \begin{bmatrix} H' \otimes (X' \oplus X'') & {}_{H'}U_{H''} \otimes (X' \oplus X'') \\ 0 & H'' \otimes (X' \oplus X'') \end{bmatrix}. \end{aligned}$$

Now, we show that $(X, \delta_X)$ is a left $H$-comodule. The definition of $\delta_X$ yields

$$\begin{aligned} (\mathrm{id}_H \otimes \delta_X) \circ \delta_X(x', x'') &= (\mathrm{id}_H \otimes \delta_X) \circ \begin{bmatrix} \delta'_{X'}(x') & \varphi(x') \\ 0 & \delta''_{X''}(x'') \end{bmatrix} \\ &= \begin{bmatrix} (\mathrm{id}_H \otimes \delta_X)\delta'_{X'}(x') & (\mathrm{id}_H \otimes \delta_X)\varphi(x') \\ 0 & (\mathrm{id}_H \otimes \delta_X)\delta''_{X''}(x'') \end{bmatrix} \\ &= \begin{bmatrix} ((\mathrm{id}_{H'} \otimes \delta'_{X'})\delta'_{X'}(x'), (\mathrm{id}_{H'} \otimes \varphi)\delta'_{X'}(x')) & (\mathrm{id}_U \otimes \delta''_{X''})\varphi(x') \\ 0 & (\mathrm{id}_{H''} \otimes \delta''_{X''})\delta''_{X''}(x'') \end{bmatrix} = a. \end{aligned}$$

Since $X'$ is a left $H'$-comodule and $X''$ is a left $H''$-comodule, and $\varphi$ is a homomorphism of $H'$-comodules with $\operatorname{Im} \varphi \subseteq U \,\square\, X''$, it follows that

$$\begin{aligned} a &= \begin{bmatrix} ((\Delta_{H'} \otimes \mathrm{id}_{X'})\delta'_{X'}(x'), (\delta'_U \otimes \mathrm{id}_{X''})\varphi(x')) & (\delta''_U \otimes \mathrm{id}_{X''})\varphi(x') \\ 0 & (\Delta_{H''} \otimes \mathrm{id}_{X''})\delta''_{X''}(x'') \end{bmatrix} \\ &= (\Delta_H \otimes \mathrm{id}_X) \circ \begin{bmatrix} \delta'_{X'}(x') & \varphi(x') \\ 0 & \delta''_{X''}(x'') \end{bmatrix} = (\Delta_H \otimes \mathrm{id}_X) \circ \delta_X(x', x''), \end{aligned}$$

and our claim is proved.

We define $\Psi(f', f'') : \Psi(X', X'', \varphi) \to \Psi(Y', Y'', \psi)$ to be the homomorphism of left $H$-comodules given by $f = f' \oplus f'' : X' \oplus X'' \to Y' \oplus Y''$. We show that if $(f', f'') : (X', X'', \varphi) \to (Y', Y'', \psi)$ is a morphism in $\mathrm{Rep}_\square({}_{H'}U_{H''})$ then $f = f' \oplus f'' : X' \oplus X'' \to Y' \oplus Y''$ defines a homomorphism of left $H$-comodules between $\Psi(X', X'', \varphi) = (X, \delta_X)$ and $\Psi(Y', Y'', \psi) = (Y, \delta_Y)$. Indeed, given $x' \in X'$ and $x'' \in X''$, we get

$$\begin{aligned}\delta_Y \circ f(x', x'') = \delta_Y \circ (f'(x'), f''(x'')) &= \begin{bmatrix} \delta'_{Y'} f'(x') & \psi(f'(x')) \\ 0 & \delta''_{Y''} f''(x'') \end{bmatrix} \\ &= \begin{bmatrix} (\mathrm{id}_{H'} \otimes f')\delta'_{X'}(x') & (\mathrm{id}_U \otimes f'')\varphi(x') \\ 0 & (\mathrm{id}_{H''} \otimes f'')\delta''_{X''}(x'') \end{bmatrix} \\ &= (\mathrm{id}_H \otimes f) \circ \delta_X(x', x''),\end{aligned}$$

and therefore $f$ is a homomorphism of left $H$-comodules.

It is clear that we have defined a $K$-linear, faithful and exact functor

$$\Psi : \mathrm{Rep}_\square({}_{H'}U_{H''}) \to H\text{-Comod}.$$

A straightforward computation shows that $\Psi$ is quasi-inverse to $\Phi$ and vice versa. Consequently, we get the following useful result.

THEOREM 2.14. *Let $H'$ and $H''$ be $K$-coalgebras, ${}_{H'}U_{H''}$ a non-zero $H'$-$H''$-bicomodule, and $H$ the bipartite $K$-coalgebra* (2.1). *The $K$-linear functors $\Phi$ and $\Psi$ in* (2.8) *are $K$-linear equivalences of categories quasi-inverse to each other and they restrict to $K$-linear equivalences of categories*

$$H\text{-comod} \underset{\Psi'}{\overset{\Phi'}{\rightleftarrows}} \mathrm{rep}_\square({}_{H'}U_{H''}). \tag{2.15}$$

By applying the equivalences (2.8) and (2.15), we are able to prove the following properties of the bipartite coalgebra $H$.

THEOREM 2.16. *Let $H'$ and $H''$ be basic $K$-coalgebras with the decompositions $\operatorname{soc} H' = \bigoplus_{j' \in I_{H'}} S'(j')$ and $\operatorname{soc} H'' = \bigoplus_{j'' \in I_{H''}} S''(j'')$ into direct sums of simple left comodules* (*and simple coalgebras*). *Let ${}_{H'}U_{H''}$ be a non-zero $H'$-$H''$-bicomodule and $H$ the bipartite $K$-coalgebra* (2.1).

(a) *The coalgebra $H$ is basic and*

$$\begin{aligned}\operatorname{soc}{}_H H = \begin{bmatrix} \operatorname{soc} H' & 0 \\ 0 & \operatorname{soc} H'' \end{bmatrix} &= \begin{bmatrix} \operatorname{soc} H' \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 0 \\ \operatorname{soc} H'' \end{bmatrix} \\ &= \bigoplus_{j' \in I_{H'}} S(j') \oplus \bigoplus_{j'' \in I_{H''}} S(j''),\end{aligned}$$

*where $S(j') = \left[\begin{smallmatrix} S'(j') \\ 0 \end{smallmatrix}\right]$ if $j' \in I_{H'}$, and $S(j'') = \left[\begin{smallmatrix} 0 \\ S''(j'') \end{smallmatrix}\right]$ if $j'' \in I_{H''}$, in the notation* (2.4) *and* (2.5).

(b) *For each* $j' \in I_{H'}$, *the left* $H$*-comodule* $E(j') = \left[\begin{smallmatrix} E'(j') \\ 0 \end{smallmatrix}\right]$ *is the* $H$*-injective envelope of* $S(j')$, *where* $E'(j')$ *is the* $H'$*-injective envelope of* $S'(j')$.

(c) *The left* $H$*-comodule* $\left[\begin{smallmatrix} U \\ H'' \end{smallmatrix}\right]$ *in* (2.5) *is injective and has a decomposition*

$$\begin{bmatrix} U \\ H'' \end{bmatrix} = \bigoplus_{t'' \in I_{H''}} \begin{bmatrix} {}_{H'}U_{t''} \\ E''(t'') \end{bmatrix} = \bigoplus_{t'' \in I_{H''}} E(t''),$$

*where* $E''(t'')$ *is the* $H''$*-injective envelope of* $S''(t'')$, ${}_{H'}U_{t''} = U \,\Box\, E''(t'')$ *is viewed as a left* $H'$*-subcomodule of* ${}_{H'}U_{H''}$ *and*

$$E(t'') = \begin{bmatrix} {}_{H'}U_{t''} \\ E''(t'') \end{bmatrix} \subseteq \begin{bmatrix} {}_{H'}U \\ H'' \end{bmatrix}$$

*is the* $H$*-injective envelope of* $S(t'')$.

(d) $\max\{\mathrm{gl.dim}\, H', \mathrm{gl.dim}\, H''\} \leq \mathrm{gl.dim}\, H \leq \mathrm{gl.dim}\, H' + \mathrm{gl.dim}\, H'' + 1$.

(e) *If* $H'$ *and* $H''$ *are semisimple then*

(e1) $H' = \bigoplus_{j' \in I_{H'}} S'(j')$ *and* $H'' = \bigoplus_{j'' \in I_{H''}} S''(j'')$ *are direct sums of coalgebras and the* $H'$*-*$H''$*-bicomodule* ${}_{H'}U_{H''}$ *has a* $K$*-vector space decomposition*

$$(2.16) \qquad {}_{H'}U_{H''} = \bigoplus_{s' \in I_{H'}} \bigoplus_{t'' \in I_{H''}} {}_{s'}U_{t''},$$

*where* ${}_{s'}U_{t''} = S'(s') \,\Box_{H'}\, U_{H''} \,\Box\, S''(t'')$ *is viewed as an* $S'(s')$*-*$S''(t'')$*-bicomodule* (*and* $H'$*-*$H''$*-bicomodule*, *in a natural way*).

(e2) $H$ *is coradical square complete and every simple left* $H$*-comodule* $S$ *is projective or injective.*

(e3) $\mathrm{gl.dim}\, H = 1$.

*Proof.* (a) Since $H'$ and $H''$ are basic, by the definition (2.3) of the comultiplication in $H$, $S(j')$ and $S(j'')$ are simple subcoalgebras of $H$ for all $j' \in I_{H'}$ and $j'' \in I_{H''}$, and $\left[\begin{smallmatrix} \mathrm{soc}\, H' & 0 \\ 0 & \mathrm{soc}\, H'' \end{smallmatrix}\right] \subseteq \mathrm{soc}\, H$.

To prove the opposite inclusion, we take a simple left subcomodule $S$ of $H$. In view of Theorem 2.14, we identify the category $H$-Comod with $\mathrm{Rep}_{\Box}({}_{H'}U_{H''})$ via the functor $\Phi$ in (2.11). Then $S$ has the form $S = (S', S'', \varphi)$ and $(0, S'', 0)$ is a left subcomodule of $S$. Hence, if $S' \neq 0$, then $S'' = 0$ and $S = (S', 0, 0)$ is a simple left $H'$-comodule, and we are done; otherwise, $S' = 0$, $S'' \neq 0$, and $S = (0, S'', 0)$ is a simple left $H''$-comodule. This proves the required equality $\left[\begin{smallmatrix} \mathrm{soc}\, H' & 0 \\ 0 & \mathrm{soc}\, H'' \end{smallmatrix}\right] = \mathrm{soc}\, H$.

(b) Since $E'(j')$ is the $H'$-injective envelope of $S'(j')$, it follows that $E'(j')$ is a direct summand of $H'$ and $\mathrm{soc}\, E'(j') = S'(j')$. Hence, $E(j') = \left[\begin{smallmatrix} E'(j') \\ 0 \end{smallmatrix}\right]$ is a direct summand of $\left[\begin{smallmatrix} H' \\ 0 \end{smallmatrix}\right] \subseteq H$ (and of $H$), and $\mathrm{soc}\, E(j') = S(j')$. This means that $E(j')$ is the $H$-injective envelope of $S(j')$.

(c) We have the decompositions

$$ {}_{H'}H' = \bigoplus_{s' \in I_{H'}} E'(s') \quad \text{and} \quad {}_{H''}H'' = \bigoplus_{t'' \in I_{H''}} E''(t'') $$

into direct sums of indecomposable injective left comodules. The decomposition of $H''$ yields the decomposition

$$ {}_{H'}U \cong {}_{H'}U \,\square\, H'' = {}_{H'}U \,\square \bigoplus_{t'' \in I_{H''}} E''(t'') = \bigoplus_{t'' \in I_{H''}} {}_{H'}U \,\square\, E''(t'') = \bigoplus_{t'' \in I_{H''}} {}_{H'}U_{t''} $$

of $U$, viewed as a left $H'$-comodule, where ${}_{H'}U_{t''} = {}_{H'}U \,\square\, E''(t'')$ is viewed as a left $H'$-comodule. We set $E(t'') = ({}_{H'}U_{t''}, E''(t''), \mathrm{id})$. It is clear that $\bigoplus_{t'' \in I_{H''}} E(t'') \cong \left[\begin{smallmatrix} U \\ H'' \end{smallmatrix}\right] \subseteq H$, and hence $E(t'')$ is an injective left $H$-comodule, as a direct summand of ${}_H H$. Since $\operatorname{soc} E(t'') = S(t'')$ we conclude that $E(t'')$ is the $H$-injective envelope of $S(t'')$.

(d) Each left $H$-comodule $X$ is a triple $X = (X', X'', \varphi_X)$ (see (2.11)). In particular, we get (cf. Example 2.12):

- $\left[\begin{smallmatrix} U \\ H'' \end{smallmatrix}\right] = (U, H'', \delta''_U)$, where $\delta''_U : {}_{H'}U \to {}_{H'}U \,\square\, H''$ is the canonical isomorphism,
- $S(i') = (S'(i'), 0, 0)$ for $i' \in I_{H'}$,
- $E(i') = (E'(i'), 0, 0)$ for $i' \in I_{H'}$,
- $S(t'') = (0, S''(t''), 0)$ for $t'' \in I_{H''}$,
- $E(t'') = ({}_{H'}U_{t''}, E''(t''), \mathrm{id})$ for $t'' \in I_{H''}$, where $\mathrm{id} : {}_{H'}U_{t''} \to {}_{H'}U \,\square\, E''(t'')$ is the identity map.

We recall that $\mathrm{gl.dim}\, H \le n$ if and only if $\mathrm{inj.dim}\, {}_H S \le n$ for each simple left $H$-comodule $S$ (see [18]). By (a), the comodules $S(i')$ with $i' \in I_{H'}$, and $S(j'')$ with $j'' \in I_{H''}$, form a complete set of pairwise non-isomorphic simple left $H$-comodules.

Given $i' \in I_{H'}$, we fix a minimal injective resolution

$$ 0 \to S'(i') \to {}_0E' \to {}_1E' \to \cdots \to {}_mE' \to \cdots $$

in $H'$-Comod of the simple left $H'$-comodule $S'(i')$. Then the induced sequence

$$ 0 \to S(i') \to ({}_0E', 0, 0) \to ({}_1E', 0, 0) \to \cdots \to ({}_mE', 0, 0) \to \cdots $$

in $H$-Comod $= \mathrm{Rep}_\square({}_{H'}U_{H''})$ is a minimal injective resolution of the left $H$-comodule $(S(i'), 0, 0)$. It follows that $\mathrm{inj.dim}\, {}_H S(i') = \mathrm{inj.dim}\, {}_{H'} S'(i')$ for each $i' \in I_{H'}$, and so $\mathrm{gl.dim}\, H \ge \mathrm{gl.dim}\, H'$.

Now fix $t'' \in I_{H''}$. By (c), there is a non-split exact sequence

$$ 0 \to S(t'') \to E(t'') \to L_0(t'') \to 0 $$

in $H$-Comod $= \mathrm{Rep}_\square({}_{H'}U_{H''})$, where

$$ L_0(t'') = ({}_{H'}U_{t''}, L''_0(t''), \overline{\varphi}_{t''}) \quad \text{and} \quad L''_0(t'') = E''(t'')/S''(t''). $$

Let

$$0 \to L''_0(t'') \to {}_1E'' \to {}_2E'' \to \cdots \to {}_mE'' \to \cdots$$

be a minimal injective resolution of $L''_0(t'')$ in $H''$-Comod. If ${}_mE'' \neq 0$ for all $m \geq 1$, then $\mathrm{gl.dim}\, H'' = \infty$ and the induced exact sequence

$$0 \to L_0(t'') \to (U \,\square\, {}_1E'', {}_1E'', {}_1h) \to \cdots \to (U \,\square\, {}_mE'', {}_mE'', {}_mh) \to \cdots$$

in $H$-Comod $= \mathrm{Rep}_\square({}_{H'}U_{H''})$, with ${}_mh = \mathrm{id} : U \,\square\, {}_mE'' \to U \,\square\, {}_mE''$ for $m \geq 1$, is a minimal injective resolution of $L_0(t'')$. Hence $\mathrm{inj.dim}\, {}_HS(t'') = \infty$, and we are done.

Assume that ${}_{m-1}E'' \neq 0$ and ${}_mE'' = 0$ for some $m \geq 1$. Then the induced sequence

$$0 \to L_0(t'') \to (U \,\square\, {}_1E'', {}_1E'', {}_1h) \to \cdots \to (U \,\square\, {}_{m-1}E'', {}_{m-1}E'', {}_{m-1}h) \to ({}_mN, 0, 0) \to 0,$$

with ${}_jh = \mathrm{id} : U \,\square\, {}_jE'' \to U \,\square\, {}_jE''$ for $j \geq 1$, is exact. If ${}_mN = 0$ then

$$\mathrm{inj.dim}\, {}_HS(t'') = m - 1 = 1 + \mathrm{inj.dim}\, {}_{H''}L''_0(t'') = \mathrm{inj.dim}\, {}_{H''}S''(t'').$$

Assume that ${}_mN \neq 0$. Let

$$0 \to {}_mN \to {}_mE' \to {}_{m+1}E' \to \cdots \to {}_{m+r}E' \to \cdots$$

be a minimal injective resolution of ${}_mN$ in $H'$-Comod. Then the induced sequence

$$0 \to ({}_mN, 0, 0) \to ({}_mE', 0, 0) \to \cdots \to ({}_{m+r}E', 0, 0) \to \cdots$$

is a minimal injective resolution of $({}_mN, 0, 0)$ in $H$-Comod. Therefore

$$\mathrm{inj.dim}\, {}_{H''}S''(t'') + \mathrm{gl.dim}\, H' + 1 \geq \mathrm{inj.dim}\, {}_HS(t'') \geq \mathrm{inj.dim}\, {}_{H''}S''(t'')$$

and (d) follows.

(e) Assume that the basic coalgebras $H'$ and $H''$ are semisimple. Then we have decompositions $H' = \bigoplus_{s' \in I_{H'}} S'(s')$ and $H'' = \bigoplus_{t'' \in I_{H''}} S''(t'')$ into direct sums of simple coalgebras. By (c), the semisimple decomposition of $H''$ yields the decomposition

$${}_{H'}U \cong {}_{H'}U \,\square\, H'' = \bigoplus_{t'' \in I_{H''}} {}_{H'}U_{t''}$$

of $U$, viewed as a left $H'$-comodule, where ${}_{H'}U_{t''} = {}_{H'}U \,\square\, S''(t'')$ is viewed as an $H'$-$S''(t'')$-bicomodule. We note that $E''(t'') = S''(t'')$ is a subcoalgebra of $H''$. Similarly, the semisimple decomposition of $H'$ yields the $H'$-$H''$-bicomodule decomposition

$${}_{H'}U_{H''} \cong {}_{H'}H' \,\square\, U_{H''} = \bigoplus_{s' \in I_{H'}} S'(s') \,\square\, U_{H''} = \bigoplus_{s' \in I_{H'}} \bigoplus_{t'' \in I_{H''}} {}_{s'}U_{t''},$$

where ${}_{s'}U_{t''} = S'(s') \,\square\, U_{t''} = S'(s') \,\square\, U \,\square\, S''(t'')$ is viewed as an $S'(s')$-$S''(t'')$-bicomodule, and hence as an $H'$-$H''$-bicomodule. This proves (e1).

By (c), the left $H$-comodule $\left[\begin{smallmatrix} U \\ H'' \end{smallmatrix}\right]$ is injective and has the decomposition

$$\begin{bmatrix} U \\ H'' \end{bmatrix} = \bigoplus_{t''\in I_{H''}} \begin{bmatrix} {}_{H'}U_{t''} \\ S''(t'') \end{bmatrix} = \bigoplus_{t''\in I_{H''}} E(t''),$$

where ${}_{H'}U_{t''} = {}_{H'}U \,\square\, S''(t'')$ is viewed as a left $H'$-subcomodule of ${}_{H'}U_{H''}$ and

$$E(t'') = \begin{bmatrix} {}_{H'}U_{t''} \\ S''(t'') \end{bmatrix} \subseteq \begin{bmatrix} {}_{H'}U \\ H'' \end{bmatrix}$$

is the injective envelope of $S(t'')$. Because (a) yields $\operatorname{soc} H = \operatorname{soc} H' \oplus \operatorname{soc} H''$, the above considerations imply that $(\operatorname{soc} H) \wedge (\operatorname{soc} H) = H$, that is, $H$ is coradical square complete. The remaining statement of (e2) is easily seen by applying the identification $H\text{-Comod} = \operatorname{Rep}_{\square}({}_{H'}U_{H''})$.

By (d), gl.dim $H \le 1$, because the coalgebras $H'$ and $H''$ are semisimple. Since $U \ne 0$, we have $\operatorname{soc} H = \operatorname{soc} H' \oplus \operatorname{soc} H'' \subsetneq H$ and hence $\operatorname{gl.dim} H \ge 1$. This completes the proof of (e3) and of the theorem. ∎

**3. The valued Gabriel quiver of a bipartite coalgebra and of a coradical square complete coalgebra.** Let $C$ be a basic coalgebra with a fixed left comodule decomposition

$$\operatorname{soc}{}_C C = \bigoplus_{i\in I_C} S(i),$$

of the left socle where $S(i)$, for $i \in I_C$, are pairwise non-isomorphic simple left $C$-comodules (and simple subcoalgebras).

We recall that the *left valued* (*Gabriel*) *quiver* of $C$ is the valued quiver $({}_CQ, {}_C\mathbf{d})$, where ${}_CQ_0 = I_C$ and, given two vertices $i, j \in {}_CQ_0$, there exists a unique valued arrow

$$i \xrightarrow{({}_Cd'_{ij},\, {}_Cd''_{ij})} j$$

in ${}_CQ_1$ if and only if $\operatorname{Ext}^1_C(S(i), S(j)) \ne 0$ and

$${}_Cd'_{ij} = \dim \operatorname{Ext}^1_C(S(i), S(j))_{F_i}, \qquad {}_Cd''_{ij} = \dim{}_{F_j} \operatorname{Ext}^1_C(S(i), S(j)),$$

where $F_a = \operatorname{End}_C S(a)$ for any $a \in I_C$ (see [14, Definition 4.3]).

Now we recall from [14, Proposition 4.10] and [32] an equivalent definition of the left valued Gabriel quiver $({}_CQ, {}_C\mathbf{d})$ of a basic coalgebra $C$ by means of irreducible morphisms.

Assume that $C$ is a basic coalgebra with a fixed left comodule decomposition of $\operatorname{soc}{}_C C$ as above. Given $a \in I_C$, we denote by $E(a)$ the injective envelope of $S(a)$. Denote by $C$-inj the full subcategory of $C$-Comod formed by socle-finite injective $C$-comodules, that is, a comodule $E$ lies in $C$-inj if and only if $E$ is isomorphic to a finite direct sum of indecomposable injective $C$-comodules. Given $E'$ and $E''$ in $C$-inj, we define the *radical* of $\mathrm{Hom}_C(E', E'')$ to be the $K$-subspace $\mathrm{rad}(E', E'') = \mathrm{rad}_{C\text{-inj}}(E', E'')$ of $\mathrm{Hom}_C(E', E'')$ generated by all non-isomorphisms $\varphi : E(i) \to E(j)$ between indecomposable summands $E(i)$ of $E'$ and $E(j)$ of $E''$, respectively. The square $\mathrm{rad}^2(E', E'')$ is defined to be the $K$-subspace of $\mathrm{rad}(E', E'')$ generated by all composite homomorphisms of the form

$$E' \xrightarrow{f'_j} E(j) \xrightarrow{f''_j} E'',$$

where $j \in I_C$, $f'_j \in \mathrm{rad}(E', E(j))$ and $f''_j \in \mathrm{rad}(E(j), E'')$. For any $a, b \in I_C$, we set $F_a = \mathrm{End}_C S(a)$, $F_b = \mathrm{End}_C S(b)$ and we consider the $K$-vector space

$$\mathrm{Irr}_C(E(b), E(a)) = \mathrm{rad}(E(b), E(a)) / \mathrm{rad}^2(E(b), E(a)) \tag{3.1}$$

as an $F_a$-$F_b$-bimodule. We call it the *bimodule of irreducible morphisms* (see [14], [30] and [32]).

By [14, Proposition 4.7] and [32, Theorem 2.3], there exists a unique valued arrow $a \xrightarrow{(d'_{ab}, d''_{ab})} b$ in $({}_CQ, {}_C\mathbf{d})$ if and only if the $F_a$-$F_b$-bimodule $\mathrm{Irr}(E(b), E(a))$ is non-zero and

$$d'_{ab} = \dim \mathrm{Irr}_C(E(b), E(a))_{F_b}, \qquad d''_{ab} = \dim {}_{F_a}\mathrm{Irr}_C(E(b), E(a)). \tag{3.2}$$

The following proposition gives a description of the left valued Gabriel quiver of a coalgebra $C$ in terms of the $C_0$-$C_0$-bicomodule

$${}_{C_0}(C_1/C_0)_{C_0} = \bigoplus_{a,b \in I_C} {}_a(C_1/C_0)_b, \tag{3.3}$$

where the $S(a)$-$S(b)$-bicomodule ${}_a(C_1/C_0)_b = S(a) \square (C_1/C_0) \square S(b)$ is viewed as a rational $F_a$-$F_b$-bimodule. To see this we note that, in the notation of the proof of Proposition 3.5 below, there is an $F_a$-$F_b$-bimodule isomorphism ${}_a(C_1/C_0)_b \cong e_b(C_1/C_0)e_a$ (see (3.6″) and cf. [3], [17], and [41]).

To formulate the result, we assume that $C$ is a basic coalgebra with a decomposition of $\mathrm{soc}\,{}_CC$ as above. Given $a \in I_C$, we denote by $E(a) \supseteq E_1(a)$ the injective envelope of $S(a)$ in $C$-Comod and $C_1$-Comod, respectively. Now, for $a, b \in I_C$, we define an $F_a$-$F_b$-bimodule homomorphism

$$\mathrm{Irr}_C(E(b), E(a)) \xrightarrow{\mathrm{res}_{ab}} \mathrm{Irr}_{C_1}(E_1(b), E_1(a)) \tag{3.4}$$

by associating to any non-isomorphism $f : E(b) \to E(a)$ its restriction $\mathrm{res}_{ab}(f) : E_1(b) \to E_1(a)$ to $E_1(b)$.

Now we complete [3], [14, Proposition 4.10], [17, Theorem 1.7] and [32, Theorem 2.5] as follows.

PROPOSITION 3.5. *Let $C$ be a basic $K$-coalgebra with a left comodule decomposition* $\mathrm{soc}\,{}_CC = \bigoplus_{i \in I_C} S(i)$ *as above, and let* $C_1 = C_0 \wedge C_0$.

(a) *Given $a, b \in I_C$, the $F_a$-$F_b$-bimodule homomorphism* $\mathrm{res}_{ab}$ *in* (3.4) *is an isomorphism.*

(b) *For any $a, b \in I_C$, there exist $F_a$-$F_b$-bimodule isomorphisms*

$$\mathrm{Hom}_{F_a}(\mathrm{Ext}^1_C(S(a), S(b)), F_a) \underset{\simeq}{\rightarrow} \mathrm{Irr}_C(E(b), E(a)) \underset{\simeq}{\rightarrow} {}_a(C_1/C_0)_b. \tag{3.6}$$

(c) *There exists a unique valued arrow $a \xrightarrow{(d'_{ab}, d''_{ab})} b$ in the left valued Gabriel quiver $({}_CQ, {}_C\mathbf{d})$ of $C$ if and only if the $F_a$-$F_b$-bimodule ${}_a(C_1/C_0)_b = S(a) \,\square\, (C_1/C_0) \,\square\, S(b)$ is non-zero and*

$$d'_{ab} = \dim({}_a(C_1/C_0)_b)_{F_a}, \qquad d''_{ab} = \dim{}_{F_b}({}_a(C_1/C_0)_b). \tag{3.7}$$

(d) *The left Gabriel quiver ${}_{C_1}Q$ coincides with ${}_CQ$.*

*Proof.* (a) To show that $\mathrm{res}_{ab}$ is bijective, we note that, given a non-isomorphism $f : E(b) \to E(a)$, the restriction $\mathrm{res}_{ab}(f) : E_1(b) \to E_1(a)$ is obviously a non-isomorphism. Conversely, if $g : E_1(b) \to E_1(a)$ is a non-isomorphism of $C_1$-comodules then, by the injectivity of $E(a)$, $g$ uniquely extends to a non-isomorphism $f : E(b) \to E(a)$ such that $\mathrm{res}_{ab}(f) = g$. This shows that (3.4) is bijective.

(b) The left-hand isomorphism in (3.6) is established in [14, Proposition 4.10]. To prove the right-hand one, we keep the notation of the proof of [14, Proposition 4.10]. Fix $a, b \in I_C$ and denote by $e_a$, $e_b$ the primitive idempotents in the pseudocompact $K$-algebra $C^* = \mathrm{Hom}_K(C, K)$ that correspond to the direct summands $E(a)^*$ and $E(b)^*$ of $C^*$. Let $J(C^*)$ be the Jacobson radical of $C^*$. We recall that the functor $M \mapsto M^*$ defines a $K$-linear duality $C$-Comod $\cong C^*$-PC, where $C^*$-PC is the category of pseudocompact left $C^*$-modules (see [29, 4.5]). Moreover, by [16, Proposition 5.2.9] there are isomorphisms $J(C^*)/J(C^*)^2 \cong C_0^{\perp}/C_1^{\perp} \cong (C_1/C_0)^*$ of pseudocompact $C^*$-bimodules.

By [14, p. 480], the equivalence $C$-Comod $\cong (C^*\text{-PC})^{\mathrm{op}}$, $M \mapsto M^*$, induces isomorphisms

$$\begin{aligned} \mathrm{Irr}_C(E_1(b), E_1(a)) &\cong (e_a[J(C^*)/J(C^*)^2]e_b)^\circ \cong (e_a[(C_1/C_0)^*]e_b)^\circ \\ &\cong e_b((C_1/C_0)^*)^\circ e_a \cong e_b(C_1/C_0)e_a \cong {}_a(C_1/C_0)_b \end{aligned} \tag{3.6$'$}$$

of $F_a$-$F_b$-bimodules. The final isomorphism is the inverse of the following composite one:

$$\begin{aligned} {}_a(C_1/C_0)_b &= S(a) \,\square\, (C_1/C_0) \,\square\, S(b) \\ &\cong \mathrm{Hom}_{C_0}(S(a)^*, (C_1/C_0) \,\square\, S(b)) \\ &\cong \mathrm{Hom}_{C_0}(S(a)^*, \mathrm{Hom}_{C_0}(S(b)^*, C_1/C_0)) \\ &\cong \mathrm{Hom}_{C_0}(S(a)^*, e_b(C_1/C_0)) \cong e_b(C_1/C_0)e_a. \end{aligned} \tag{3.6$''$}$$

Note also that, since the pseudocompact left $C^*$-modules $S(a)^* \cong (C_0)^*e_a$ and $S(b)^* \cong (C_0)^*e_b$ are finite-dimensional, they are discrete (= rational), and therefore they are viewed as left $C$-comodules. Moreover, there are algebra isomorphisms $S(a)^* \cong e_a(C_0)^*e_a \cong F_a^{\mathrm{op}}$, $S(b)^* \cong e_b(C_0)^*e_b \cong F_b^{\mathrm{op}}$, and $F_a$-$F_b$-bimodule isomorphisms

$$_a(C_1/C_0)_b = S(a) \,\square\, (C_1/C_0) \,\square\, S(b) \cong C_0e_a \,\square\, (C_1/C_0) \,\square\, e_bC_0 \cong e_b(C_1/C_0)e_a.$$

(c) Apply (a), (b) and (3.2).

(d) Apply (a) and (3.4). ∎

COROLLARY 3.8. *Let $C$ be a basic $K$-coalgebra. Then the left valued and right valued Gabriel quivers of $C$ are dual to each other.*

*Proof.* It is well-known that there is a $K$-duality $D : C\text{-inj} \to \text{inj-}C$ between the categories of socle finite injective left $C$-comodules and socle finite injective right $C$-comodules (see [5, Proposition 3.1(c)]). Given an indecomposable $E(a)$ in $C$-inj, we denote by $E'(a)$ the indecomposable $DE(a)$ in inj-$C$. Obviously, the socle $S'(a)$ of $E'(a)$ is isomorphic to the right $C$-comodule $S(a)^*$. Since, for any $a, b \in I_C$, there are division ring isomorphisms

$$F'_a = \mathrm{End}_C S'(a) \cong (\mathrm{End}_C S(a))^{\mathrm{op}} \cong F_a^{\mathrm{op}},$$
$$F'_b = \mathrm{End}_C S'(b) \cong (\mathrm{End}_C S(b))^{\mathrm{op}} \cong F_b^{\mathrm{op}},$$

the $F'_b$-$F'_a$-bimodule $\mathrm{Irr}(E'(a), E'(b))$ is viewed as an $F_a$-$F_b$-bimodule in a standard way. Moreover, the functor $D$ induces an isomorphism $\mathrm{Irr}(E(b), E(a)) \cong \mathrm{Irr}(E'(a), E'(b))$ of $F_a$-$F_b$-bimodules. Hence, in view of Proposition 3.5 and [32, Theorem 2.3], the corollary follows. ∎

We end this section by a description of the Gabriel quiver of an arbitrary bipartite coalgebra.

COROLLARY 3.9. *Let $H'$ and $H''$ be basic $K$-coalgebras, ${}_{H'}U_{H''}$ a nonzero $H'$-$H''$-bicomodule, and $H$ the bipartite $K$-coalgebra* (2.1). *In the notation of Theorem* 2.16 *we have*:

(a) *$H$ is basic and the Gabriel quiver $({}_HQ, {}_H\mathbf{d})$ has the form* [15]

$$({}_HQ, {}_H\mathbf{d}) = ({}_{H'}Q, {}_{H'}\mathbf{d}) \blacksquare_U ({}_{H''}Q, {}_{H''}\mathbf{d}), \tag{3.10}$$

*that is, $({}_HQ, {}_H\mathbf{d})$ is obtained from the disjoint union of $({}_{H'}Q, {}_{H'}\mathbf{d})$ and $({}_{H''}Q, {}_{H''}\mathbf{d})$ by adding, for each $s' \in {}_{H'}Q_0 = I_{H'}$ and each $t'' \in {}_{H''}Q_0 = I_{H''}$, the valued arrow*

$$s' \xrightarrow{(d'_{s't''}, d''_{s't''})} t'' \tag{3.11}$$

*from $s'$ to $t''$, provided that ${}_{s'}U_{t''} \neq 0$, and $d'_{s't''} = \dim({}_{s'}U_{t''})_{F_{s'}}$, $d''_{s't''} = \dim{}_{F_{t''}}({}_{s'}U_{t''})$. Here the $S'(s')$-$S''(t'')$-bicomodule ${}_{s'}U_{t''} = S'(s') \,\square\, U \,\square\, S''(t'')$ is viewed as a (rational) $F_{s'}$-$F_{t''}$-bimodule, in view of the division algebra isomorphisms $\mathrm{End}_H S''(t'') \cong F_{t''}$ and $\mathrm{End}_H S'(s') \cong F_{s'}$.*

(b) *If $H'$ and $H''$ are semisimple then $({}_{H'}Q, {}_{H'}\mathbf{d})$ and $({}_{H''}Q, {}_{H''}\mathbf{d})$ have no arrow, and the only arrows in $({}_{H}Q, {}_{H}\mathbf{d})$ are of the form* (3.11), *where $s' \in I_{H'}$ and $t'' \in I_{H''}$. If $H'$ and $H''$ are simple and ${}_{H'}U_{H''} \neq 0$, then $H$ is indecomposable and $({}_{H}Q, {}_{H}\mathbf{d})$ has the form $\bullet \xrightarrow{(d',d'')} \bullet$ for some natural numbers $d'$ and $d''$.*

*Proof.* Given $b \in I_H = I_{H'} \cup I_{H''}$, we set $\overline{E}(b) = E(b)/S(b)$. Since $E(b)$ is an injective $H$-comodule, there is an isomorphism

$$\mathrm{Ext}^1_H(S(a), S(b)) \cong \mathrm{Hom}_H(S(a), \overline{E}(b))$$

of right $\mathrm{End}_H S(a)$-modules for each $a \in I_H = I_{H'} \cup I_{H''}$ (see [14, p. 477]).

Since $H'$ and $H''$ are basic, so is $H$, by Theorem 2.16(a). We recall from Theorem 2.16 that, given $j' \in I_{H'}$ and $j'' \in I_{H''}$, we have

$$S(j') = \begin{bmatrix} S'(j') \\ 0 \end{bmatrix}, \qquad E(j') = \begin{bmatrix} E'(j') \\ 0 \end{bmatrix},$$

$$S(j'') = \begin{bmatrix} 0 \\ S''(j'') \end{bmatrix}, \qquad E(t'') = \begin{bmatrix} {}_{H'}U_{t''} \\ E''(t'') \end{bmatrix},$$

in the notation of Theorem 2.16 and (2.5). Hence, for $s' \in I_{H'}$ and $t'' \in I_{H''}$,

$$\overline{E}(t'') \cong \begin{bmatrix} {}_{H'}U_{t''} \\ \overline{E}''(t'') \end{bmatrix} \quad \text{and} \quad \overline{E}(s') \cong \begin{bmatrix} \overline{E}'(s') \\ 0 \end{bmatrix}.$$

It follows that $\mathrm{Ext}^1_H(S(a), S(b)) = 0$ if $a \in I_{H''}$ and $b \in I_{H'}$. Moreover, there are isomorphisms of $\mathrm{End}_H S(b)$-$\mathrm{End}_H S(a)$-bimodules

$$\mathrm{Ext}^1_H(S(a), S(b)) \cong \begin{cases} \mathrm{Hom}_{H'}(S'(a), \overline{E}'(b)) \cong \mathrm{Ext}^1_{H'}(S'(a), S'(b)) & \text{if } a, b \in I_{H'}, \\ \mathrm{Hom}_{H''}(S''(a), \overline{E}''(b)) \cong \mathrm{Ext}^1_{H''}(S''(a), S''(b)) & \text{if } a, b \in I_{H''}, \\ \mathrm{Hom}_{H'}(S'(a), {}_{H'}U_b) \cong {}_aU_b & \text{if } a \in I_{H'},\, b \in I_{H''} \end{cases}$$

(see [14, p. 480] and [41, Proposition 4.9]). Hence, (a) follows. Since (b) easily follows from (a), the proof is complete.

Following a suggestion of the referee we include another proof of (a). Let $H$ be a bipartite coalgebra as in the corollary. We consider $\check{U} = {}_{H'}(\mathrm{soc}\, {}_{H'}U) \cap (\mathrm{soc}\, U_{H''})_{H''}$ and we view it as an $H'$-$H''$-bicomodule. Note that, for all $a \in I_{H'}$ and $b \in I_{H''}$, there are isomorphisms of $S(a)$-$S(b)$-bicomodules

$$S(a) \mathbin{\square}_{H'} U \mathbin{\square}_{H''} S(b) \cong S(a) \mathbin{\square}_{H'_0} U \mathbin{\square}_{H''_0} S(b) \cong S(a) \mathbin{\square}_{H'_0} \check{U} \mathbin{\square}_{H''_0} S(b) = {}_a\check{U}_b.$$

By a straightforward calculation we show that $H_1 = H_0 \wedge H_0 = H'_1 \oplus \check{U} \oplus H''_1$, and hence $H_1/H_0 = H'_1/H'_0 \oplus \check{U} \oplus H''_1/H''_0$. Note also that $H^* = H'^* \oplus U^* \oplus H''^*$ is the upper triangular matrix algebra with the identity element $\varepsilon_H = \sum_{a \in I_{H'}} e'_a + \sum_{b \in I_{H''}} e''_b$, where $e'_a \cdot \left[\begin{smallmatrix} h' & u \\ 0 & h'' \end{smallmatrix}\right] = e'_a(h')$ and $e''_a \cdot \left[\begin{smallmatrix} h' & u \\ 0 & h'' \end{smallmatrix}\right] = e''_a(h'')$.

We also recall from [16] that

$$e \rightharpoonup h = eh = (1 \otimes e) \circ \Delta_H(h) \quad \text{and} \quad h \leftharpoonup e = he = (e \otimes 1) \circ \Delta_H(h).$$

Hence, for $a, \overline{a} \in I_{H'}$ and $b, \bar{b} \in I_{H''}$ we get

- ${}_a(H_1/H_0)_{\bar{a}} = e'_{\bar{a}}(H_1/H_0)e'_a = e'_{\bar{a}}(H'_1/H'_0)e'_a = {}_a(H'_1/H'_0)_{\bar{a}}$,
- ${}_a(H_1/H_0)_b = e''_b(H_1/H_0)e'_a = e''_b(H'_1/H'_0)e'_a = {}_a(H'_1/H'_0)_b$,
- ${}_b(H_1/H_0)_a = e'_a(H_1/H_0)e''_b = 0$,
- ${}_b(H_1/H_0)_{\bar{b}} = e''_{\bar{b}}(H_1/H_0)e''_b = e''_{\bar{b}}(H''_1/H''_0)e''_b = {}_b(H''_1/H''_0)_{\bar{b}}$.

Now (a) follows by applying Proposition 3.5. ∎

**4. Loop representations and trivial extensions of coalgebras.** Let $D$ be a $K$-coalgebra and ${}_DU_D$ be a $D$-$D$-bicomodule. We recall that the *cotensor $D$-coalgebra on $U$* is the positively graded $K$-vector space

$$T_D^{\square}(U) = \bigoplus_{n=0}^{\infty} U^{\square^n} = D \oplus U \oplus U \,\square\, U \oplus \cdots \oplus U^{\square^n} \oplus \ldots, \tag{4.1}$$

where $U^{\square^0} = D$, $U^{\square^1} = U$ and $U^{\square^n} = U \,\square \cdots \square\, U$ ($n$ times) for $n \geq 2$, equipped with the $K$-coalgebra structure defined as follows (see [10], [19] and [41] for details).

The counit $\varepsilon : T_D^{\square}(U) \to K$ of $T_D^{\square}(U)$ vanishes on $U^{\square^n}$ for all $n \geq 1$, and $\varepsilon|_D : D \to K$ is the counit of $D$. Under the identification

$$T_D^{\square}(U) \otimes T_D^{\square}(U) = \bigoplus_{n,m \geq 0} U^{\square^n} \otimes U^{\square^m},$$

for each $n \geq 0$ the component $\Delta_{n,i,j} : U^{\square^n} \to U^{\square^i} \otimes U^{\square^j}$ of the comultiplication of $T_D^{\square}(U)$ is zero if $i + j \neq n$. If $i + j = n$ and $i, j \geq 1$, then $\Delta_{n,i,j}$ is the inclusion; if either $i = 0$ or $j = 0$, then $\Delta_{n,i,j}$ is induced by the comultiplication on $U$ (or on $D$ if $i = j = 0$).

Following [10] and [41], we define the category $\operatorname{Rep}_{\square}^{\circlearrowleft}({}_DU_D)$ of *locally nilpotent loop* (*co*)*representations* of the $D$-$D$-bicomodule ${}_DU_D$ to be the category of all pairs $(Y, \mu)$, where $Y$ is a left $D$-comodule and $\mu : Y \to U \,\square\, Y$ is a homomorphism of left $D$-comodules such that

$$Y = \bigcup_{n=1}^{\infty} \operatorname{Ker}(\mu^{(n)} : Y \to U^{\otimes^n} \otimes Y), \tag{4.2}$$

where $\mu^{(n)} : Y \to U^{\otimes^n} \otimes Y$ is the composite

$$Y \xrightarrow{\mu'} U \otimes Y \xrightarrow{\mathrm{id}_U \otimes \mu'} U^{\otimes^2} \otimes Y \to \cdots \to U^{\otimes^{n-1}} \otimes Y \xrightarrow{\mathrm{id}_{U^{n-1}} \otimes \mu'} U^{\otimes^n} \otimes Y \tag{4.3}$$

and $\mu' : Y \to U \otimes Y$ is the composite $Y \xrightarrow{\mu} U \,\square\, Y \hookrightarrow U \otimes Y$. The left $D$-comodule structure on $U \,\square\, Y$ is induced from that of $U$.

A morphism from $(Y, \mu)$ to $(Z, \nu)$ in $\mathrm{Rep}_{\Box}^{\circlearrowleft}({}_DU_D)$ is a homomorphism $f : Y \to Z$ of left $D$-comodules such that $\nu \circ f = (\mathrm{id}_U \,\Box\, f) \circ \mu$. It is clear that $\mathrm{Rep}_{\Box}^{\circlearrowleft}({}_DU_D)$ is a Grothendieck $K$-category and its full subcategory $\mathrm{rep}_{\Box}^{\circlearrowleft}({}_DU_D)$, consisting of all pairs $(Y, \mu)$ with $Y$ finite-dimensional, is abelian and consists of objects of finite length.

THEOREM 4.4. *Let $D$ be a $K$-coalgebra, ${}_DU_D$ a $D$-$D$-bicomodule, and $T_D^{\Box}(U)$ the cotensor $D$-coalgebra.*

(a) $\mathrm{soc}\, T_D^{\Box}(U) = \mathrm{soc}\, D$. *As a consequence, $T_D^{\Box}(U)$ is basic if and only if $D$ is basic.*

(b) *There is a $K$-linear equivalence of categories*

$$\Theta : T_D^{\Box}(U)\text{-Comod} \to \mathrm{Rep}_{\Box}^{\circlearrowleft}({}_DU_D), \tag{4.4}$$

*which restricts to an equivalence* $\Theta' : T_D^{\Box}(U)\text{-comod} \xrightarrow{\cong} \mathrm{rep}_{\Box}^{\circlearrowleft}({}_DU_D)$.

(c) *If $D$ is semisimple, then $T_D^{\Box}(U)$ is hereditary and, given $i \in I_D$, the vector subspace*

$$E(i) = S(i) \oplus (S(i) \,\Box\, U) \oplus (S(i) \,\Box\, U \,\Box\, U) \oplus \cdots$$

*of $T_D^{\Box}(U)$ is the injective envelope of $S(i)$.*

*Proof.* For the proof of (a) the reader is referred to [41, Lemma 4.4].

(b) The equivalence (4.5) is proved in [41, Lemma 4.3]. Here, for the convenience of the reader, we recall the definition of $\Theta$. Since the canonical projection $\pi : T_D^{\Box}(U) \to D$ is a coalgebra homomorphism, every left $T_D^{\Box}(U)$-comodule $Y$ is a $D$-comodule via $\pi$. The functor $\Theta$ is defined by associating with $(Y, \delta_Y)$ in $T_D^{\Box}(U)$-Comod the pair

$$\Theta(Y, \delta_Y) = (Y, \delta'), \tag{4.6}$$

where $Y$ is the underlying $D$-comodule and $\delta' : Y \to U \Box Y$ is the composition of $\delta_Y : Y \to T_D^{\Box}(U) \,\Box\, Y$ with the canonical $D$-comodule projection $T_D^{\Box}(U) \,\Box\, Y \to U \,\Box\, Y$. If $f : (Y, \delta_Y) \to (Z, \delta_Z)$ is a homomorphism in $T_D^{\Box}(U)$-Comod, we take for $\Theta(f) : (Y, \delta') \to (Z, \delta')$ the morphism defined by $f : Y \to Z$ in $D$-Comod. By [41, Lemma 4.3], the functor $\Theta$ is an equivalence of categories and obviously it restricts to an equivalence $\Theta' : T_D^{\Box}(U)\text{-comod} \xrightarrow{\cong} \mathrm{rep}_{\Box}^{\circlearrowleft}({}_DU_D)$.

(c) Assume that $D$ is semisimple. To prove the second part of (c), note that there is a decomposition ${}_DU = D \,\Box\, {}_DU = \bigoplus_{i \in I_D} (S(i) \,\Box\, {}_DU)$ and, for any $i \in I_D$, $E(i)$ is a left subcomodule direct summand of $T_D^{\Box}(U)$; hence $E(i)$ is injective. Since obviously $\mathrm{soc}\, E(i) = S(i)$, it follows that $E(i)$ is the injective envelope of $S(i)$.

To show that $T_D^{\Box}(U)$ is hereditary, it is enough to prove $\mathrm{inj.dim}_{T_D^{\Box}(U)} S \leq 1$ for each simple $T_D^{\Box}(U)$-comodule $S(i)$ (see [18]). Consider the exact sequence

$$0 \to S(i) \to E(i) \to \overline{E}(i) \to 0$$

of left $T_D^\square(U)$-comodules, where $\overline{E}(i) = E(i)/S(i)$. It follows that there are isomorphisms of left $T_D^\square(U)$-comodules

$$\begin{aligned}\overline{E}(i) &\cong (S(i) \square U) \oplus (S(i) \square U \square U) \oplus (S(i) \square U \square U \square U) \oplus \cdots \\ &\cong [S(i) \oplus (S(i) \square U) \oplus (S(i) \square U \square U) \oplus (S(i) \square U \square U \square U) \oplus \cdots] \square U \\ &\cong E(i) \square U.\end{aligned}$$

Since $E(i) \square U$ is injective (see [8, Proposition 1]), so is $\overline{E}(i)$. This shows that $T_D^\square(U)$ is hereditary. ∎

COROLLARY 4.7. *Assume that $H'$ and $H''$ are $K$-coalgebras and ${}_{H'}U_{H''}$ is an $H'$-$H''$-bicomodule. Let $H = \begin{bmatrix} H' & {}_{H'}U_{H''} \\ 0 & H'' \end{bmatrix}$ be the bipartite coalgebra* (2.1) *and let $D = H' \oplus H''$.*

- (a) *The $H'$-$H''$-bicomodule structure on ${}_{H'}U_{H''}$ defines a $D$-$D$-bicomodule structure on $U$ such that ${}_DU \square {}_DU_D = 0$, $T_D^\square(U) = D \oplus {}_DU_D$, and $\begin{bmatrix} h' & u \\ 0 & h'' \end{bmatrix} \mapsto (h', h'', u)$ defines an isomorphism $H \cong T_D^\square(U)$ of coalgebras.*
- (b) *There are $K$-linear equivalences of categories*

$$\begin{array}{ccccccc} H\text{-Comod} & \xrightarrow[\cong]{\Phi} & \mathrm{Rep}_\square({}_{H'}U_{H''}) & \xrightarrow{\cong} & \mathrm{Rep}_\square^{\circlearrowleft}({}_DU_D) & \xleftarrow[\cong]{\Theta} & T_D^\square(U)\text{-Comod} \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ H\text{-comod} & \xrightarrow[\cong]{\Phi} & \mathrm{rep}_\square({}_{H'}U_{H''}) & \xrightarrow{\cong} & \mathrm{rep}_\square^{\circlearrowleft}({}_DU_D) & \xleftarrow[\cong]{\Theta} & T_D^\square(U)\text{-comod} \end{array}$$

*where $\Phi$ and $\Theta$ are the equivalences* (2.11) *and* (4.5), *respectively.*

*Proof.* (a) The first part of (a) is obvious. The equality ${}_DU \square {}_DU_D = 0$ follows immediately from the definition of the cotensor product, because of the definition of the right coaction of $H'$ on ${}_DU_D$ and the left coaction of $H''$ on ${}_DU_D$. Now the remaining part of (a) easily follows.

(b) By (a), the coalgebras $H$ and $T_D^\square(U)$ are isomorphic. Hence we get $H\text{-Comod} \cong T_D^\square(U)\text{-Comod}$. Since, according to Theorems 2.14 and 4.4, the functors $\Phi$ and $\Theta$ are $K$-linear equivalences of categories, they imply the equivalence $\mathrm{Rep}_\square({}_{H'}U_{H''}) \xrightarrow{\cong} \mathrm{Rep}_\square^{\circlearrowleft}({}_DU_D)$ required in (b). ∎

Let us now introduce the notion of trivial extension of a coalgebra.

DEFINITION 4.8. Let $D$ be a $K$-coalgebra and ${}_DU_D$ a $D$-$D$-bicomodule. The *trivial extension* of $D$ by ${}_DU_D$ is the coalgebra $D \ltimes {}_DU_D = (D \oplus U, \Delta, \varepsilon)$, where $\Delta(d,u) = (\Delta_D(d), \delta'_U(u), \delta''_U(u), 0)$ and $\varepsilon(d,u) = (\varepsilon_D(d), 0)$ for all $d \in D$ and $u \in U$. Here we make the identification $(D \oplus U) \otimes (D \oplus U) \equiv (D \otimes D) \oplus (D \otimes U) \oplus (U \otimes D) \oplus (U \otimes U)$.

Note that the $K$-linear map $(d,u) \mapsto \left[\begin{smallmatrix} d & u \\ 0 & d \end{smallmatrix}\right]$ defines an isomorphism

$$D \ltimes {}_DU_D \cong \begin{bmatrix} D & U \\ & \backslash\backslash & \\ 0 & D \end{bmatrix} = \left\{ \begin{bmatrix} d & u \\ 0 & d \end{bmatrix};\, d \in D,\, u \in U \right\} \subseteq \begin{bmatrix} D & {}_DU_D \\ 0 & D \end{bmatrix}$$

of vector spaces. However, unless $U = 0$, $\left[\begin{smallmatrix} D & U \\ \backslash\backslash \\ 0 & D \end{smallmatrix}\right]$ is not a subcoalgebra of the bipartite coalgebra $\left[\begin{smallmatrix} D & {}_DU_D \\ 0 & D \end{smallmatrix}\right]$.

We denote by $\mathrm{Rep}^{(2)}_{\square}({}_DU_D)$ the full subcategory of $\mathrm{Rep}^{\circlearrowleft}_{\square}({}_DU_D)$ whose objects are the pairs $(Y, \mu)$ such that $\mu^{(2)} = 0$.

To describe the left valued Gabriel quiver of the trivial extension coalgebra $D \ltimes {}_DU_D$, we define

$$({}_DQ, {}_D\mathbf{d}) \blacklozenge_U ({}_DQ, {}_D\mathbf{d}) \tag{4.9}$$

to be the quiver obtained from the valued quiver $({}_DQ, {}_D\mathbf{d}) \blacksquare_U ({}_DQ, {}_D\mathbf{d})$ (see (3.10)) of the bipartite coalgebra $\left[\begin{smallmatrix} D & {}_DU_D \\ 0 & D \end{smallmatrix}\right]$ by the identification of the left copy of $({}_DQ, {}_D\mathbf{d})$ in $({}_DQ, {}_D\mathbf{d}) \blacksquare_U ({}_DQ, {}_D\mathbf{d})$ with the right one, via the identification of the vertex $s'$ with $s''$ and the arrow $s' \to t'$ with $s'' \to t''$, for all $s, t \in {}_DQ_0 = I_D$. This operation is illustrated in Example 4.13 below.

Now we list some of the main properties of the coalgebra $C = D \ltimes {}_DU_D$.

PROPOSITION 4.10. *Let $C = D \ltimes {}_DU_D$ be the trivial extension of a $K$-coalgebra $D$ by a $D$-$D$-bicomodule ${}_DU_D$.*

(a) *$C$ is isomorphic to the subcoalgebra $D \oplus {}_DU_D$ of $T_D(U)$, $D = D \ltimes 0$ is a subcoalgebra of $C = D \ltimes {}_DU_D$, $\mathrm{soc}\, C = \mathrm{soc}\, D$, and $C_1 = D_1 \oplus U_1$, where $U_1 = \mathrm{soc}\, {}_DU \cap \mathrm{soc}\, U_D$. If $D$ is semisimple then $C$ is coradical square complete.*

(b) *If $C$ is basic then the left valued Gabriel quiver $({}_CQ, {}_C\mathbf{d})$ has the form*

$$({}_CQ, {}_C\mathbf{d}) = ({}_DQ, {}_D\mathbf{d}) \blacklozenge_U ({}_DQ, {}_D\mathbf{d}).$$

(c) *The canonical coalgebra embedding $C \hookrightarrow T_D(U)$ induces an embedding $C$-Comod $\subseteq T_D(U)$-Comod and the equivalence $\Theta$ of (4.5) restricts to a $K$-linear equivalence of categories*

$$\Theta : C\text{-Comod} \xrightarrow{\cong} \mathrm{Rep}^{(2)}_{\square}({}_DU_D) \subseteq \mathrm{Rep}^{\circlearrowleft}_{\square}({}_DU_D). \tag{4.11}$$

(d) *The $K$-linear map $\theta : \left[\begin{smallmatrix} D & {}_DU_D \\ 0 & D \end{smallmatrix}\right] \to D \ltimes_D U_D$, given by the formula $\left[\begin{smallmatrix} d' & u \\ 0 & d'' \end{smallmatrix}\right] \mapsto (d' + d'', u)$, is a coalgebra surjection. If*

$$\Theta_+ : C\text{-Comod} \to \begin{bmatrix} D & {}_DU_D \\ 0 & D \end{bmatrix}\text{-Comod} \tag{4.12}$$

*is the composite $K$-linear functor*

$$C\text{-Comod} \xrightarrow[\cong]{\Theta} \operatorname{Rep}^{(2)}_{\square}({}_DU_D) \subseteq \operatorname{Rep}_{\square}({}_DU_D) \cong \begin{bmatrix} D & {}_DU_D \\ 0 & D \end{bmatrix}\text{-Comod}$$

*then $\Theta_+$ is a full, faithful, and exact embedding such that, for each $Y$ in $C$-Comod, $\Theta_+(Y) = (Y, \mu : Y \to U \,\square\, Y)$ and $\mu^{(2)} = 0$.*

*Proof.* (a) It is easy to see that the canonical inclusion $C = D \ltimes {}_DU_D \hookrightarrow T^{\square}_D(U)$ is a coalgebra embedding and defines a coalgebra isomorphism of $C$ with the $D$-subcoalgebra $D \oplus {}_DU_D$ of $T^{\square}_D(U)$ consisting of the sums of elements of degree 0 and 1 (see (4.1)). Hence the first part of (a) easily follows.

Now we show that $C_1 = D_1 \oplus U_1$, where $U_1 = \operatorname{soc}{}_DU \cap \operatorname{soc} U_D$. We recall that $C_1 = \Delta^{-1}(C_0 \otimes C \oplus C \otimes C_0)$ and $C_0 = D_0 \oplus 0$. Then Definition 4.8 yields

$$\Delta(d) = \Delta_D(d) \in D \otimes D \quad \text{ for } d \in D$$
$$\Delta(u) = (\delta'_U(u), \delta''_U(u)) \in D \otimes U \oplus U \otimes D \quad \text{ for } u \in U.$$

Hence $C_1 = D_1 \oplus U_1$. The final part of (a) follows from the previous one.

(b) We apply Proposition 3.5. By (a), $C_1/C_0 \cong (D_1/D_0) \oplus U_1$. Let $H = \left[\begin{smallmatrix} D & {}_DU_D \\ 0 & D \end{smallmatrix}\right]$ be the bipartite coalgebra and

$$H_0 = \begin{bmatrix} D_0 & 0 \\ 0 & D_0 \end{bmatrix} = \bigoplus_{j' \in I_D} S(j') \oplus \bigoplus_{j'' \in I_D} S(j''),$$

Note that $H_1 = \left[\begin{smallmatrix} D_1 & U_1 \\ 0 & D_1 \end{smallmatrix}\right]$ and $C_0 = \bigoplus_{a \in I_D} S(a) = \bigoplus_{a \in I_C} S(a)$. It follows from the definition that

$$\{a\,;\, a \in I_D\} \quad \text{and} \quad \{a';\, a' \in I_D\} \cup \{a'';\, a'' \in I_D\}$$

are the sets of vertices of the left valued Gabriel quivers of $C$ and $H$, respectively. To describe the set of arrows of the quiver $({}_CQ, {}_C\mathbf{d})$, given a pair $a, b \in I_D = I_C$, we consider the vector space

$$\begin{aligned} {}_a(C_1/C_0)_b &= S(a) \,\square\, (C_1/C_0) \,\square\, S(b) \\ &\cong (S(a) \,\square\, (D_1/D_0) \,\square\, S(b)) \oplus (S(a) \,\square\, U_1 \,\square\, S(b)). \end{aligned}$$

By the definition of comultiplication in $C$ and $H$, we have

$$\begin{aligned} {}_a(D_1/D_0)_b &= S(a) \,\square\, (D_1/D_0) \,\square\, S(b) \cong S(a') \,\square\, (D_1/D_0) \,\square\, S(b') \\ &\cong S(a'') \,\square\, (D_1/D_0) \,\square\, S(b'') = {}_{a''}(D_1/D_0)_{b''}, \end{aligned}$$

and

$$S(a) \,\square\, U_1 \,\square\, S(b) \cong S(a') \,\square\, U_1 \,\square\, S(b'').$$

Hence, by applying Proposition 3.5, we get (b).

(c) Note that the canonical coalgebra embedding

$$D \ltimes {}_DU_D = D \oplus {}_DU_D \hookrightarrow T_D^{\square}(U)$$

induces an embedding $D \ltimes {}_DU_D$-Comod $\subseteq T_D(U)$-Comod. By applying the definitions, it is easy to check that the equivalence

$$\Theta : (D \ltimes {}_DU_D)\text{-Comod} \underset{\cong}{\to} \mathrm{Rep}_{\square}^{\circlearrowleft}({}_DU_D)$$

(see (4.5)) restricts to the required $K$-linear equivalence of categories (4.11).

(d) The first statement follows by a direct calculation, and the second follows easily from the definitions. ∎

EXAMPLE 4.13. Let $C = K^{\square}Q$ be the hereditary path coalgebra of the infinite linear quiver

$$Q: \ 1 \to 2 \to \cdots \to s-1 \to s \to s+1 \to \cdots$$

and let $H = \left[\begin{smallmatrix} C & {}_CC_C \\ 0 & C \end{smallmatrix}\right]$ be the bipartite coalgebra (2.1), where we set $H' = H'' = C$ and ${}_CU_C = {}_CC_C$. Here ${}_CC_C$ is viewed as a $C$-$C$-bicomodule in the obvious way. It follows from Corollary 3.9 that the left Gabriel quiver of $H$ has the form

$$I_Q: \begin{array}{ccccccccccc} 1' & \longrightarrow & 2' & \longrightarrow \cdots \longrightarrow & (s-1)' & \longrightarrow & s' & \longrightarrow & (s+1)' & \longrightarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 1'' & \longrightarrow & 2'' & \longrightarrow \cdots \longrightarrow & (s-1)'' & \longrightarrow & s'' & \longrightarrow & (s+1)'' & \longrightarrow & \cdots \end{array}$$

By Proposition 4.10, the left Gabriel quiver of $D \ltimes {}_DU_D$ has the form

$$Q': \ \overset{\circlearrowright}{1} \longrightarrow \overset{\circlearrowright}{2} \longrightarrow \cdots \longrightarrow \overset{\circlearrowright}{s-1} \longrightarrow \overset{\circlearrowright}{s} \longrightarrow \overset{\circlearrowright}{s+1} \longrightarrow \cdots$$

By applying the results in [31] and [33], one can show that there is a coalgebra isomorphism $H \cong K^{\square}I_Q$, where $I_Q$ is viewed as a poset and $K^{\square}I_Q$ is its incidence coalgebra. Hence, $H$-comod $\cong K^{\square}I_Q$-comod is equivalent to the category $\mathrm{rep}_K(I_Q)$ of finite-dimensional $K$-linear representations of the poset $I_Q$.

Now, following [36] and [13], we define the repetitive coalgebra and its connection with the trivial extension coalgebra (4.8).

DEFINITION 4.14. Let $(D, \Delta_D, \varepsilon_D)$ be a coalgebra and $U = ({}_DU_D, \delta_U', \delta_U'')$ be a $D$-$D$-bicomodule.

(a) The *repetitive coalgebra* of the pair $(D, {}_DU_D)$ is the $\mathbb{Z}$-graded $K$-vector space

(4.15)
$$\Re(D, {}_DU_D) = \bigoplus_{m\in\mathbb{Z}} (D^{(m)} \oplus U^{(m)})$$
$$= \begin{bmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \dots & 0 & 0 & \dots & D & {}_DU_D & 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & \dots & 0 & D & {}_DU_D & 0 & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & \dots & 0 & 0 & D & {}_DU_D & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & \dots & 0 & 0 & 0 & D & {}_DU_D & 0 & 0 & \dots \\ & \vdots & \vdots & & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}$$

with $D^{(m)} = D$ and $U^{(m)} = {}_DU_D$ in the $m$th row, for all $m \in \mathbb{Z}$, equipped with the coalgebra structure maps

$\widehat{\Delta} : \Re(D, {}_DU_D) \to \Re(D, {}_DU_D) \otimes \Re(D, {}_DU_D)$ and $\widehat{\varepsilon} : \Re(D, {}_DU_D) \to K$

defined by:

- $\widehat{\Delta}(d) = \Delta_D(d) \in D^{(i)} \otimes D^{(i)}$, $\widehat{\varepsilon}(d) = \varepsilon_D(d)$, for $d \in D^{(i)}$, and
- $\widehat{\Delta}(u) = (\delta'_U(u), \delta''_U(u)) \in D^{(i)} \otimes U^{(i)} \oplus U^{(i)} \otimes D^{(i+1)}$, $\widehat{\varepsilon}(u) = 0$, for $u \in U^{(i)}$.

(b) The group $\mathbb{Z}$ of integers acts on $\Re(D, {}_DU_D)$ as a group of coalgebra automorphisms by the shift

$$\nu : \Re(D, {}_DU_D) \to \Re(D, {}_DU_D), \qquad D^{(m)} \oplus U^{(m)} \mapsto D^{(m+1)} \oplus U^{(m+1)},$$

called the *Nakayama automorphism* of $\Re(D, {}_DU_D)$.

It is easy to check that the $K$-linear map

(4.16)
$$f : \Re(D, {}_DU_D) \to D \ltimes {}_DU_D$$

defined by the formula

$$f(\dots, (d^{(-1)}, u^{(-1)}), (d^{(0)}, u^{(0)}), (d^{(1)}, u^{(1)}), \dots) = \Big(\sum_{m\in\mathbb{Z}} d^{(m)}, \sum_{m\in\mathbb{Z}} u^{(m)}\Big) \in D \ltimes {}_DU_D,$$

with $(d^{(m)}, u^{(m)}) \in D^{(m)} \oplus U^{(m)}$, is a coalgebra surjection, and induces a pair of $K$-linear functors

(4.17)
$$\Re(D, {}_DU_D)\text{-Comod} \underset{f_\bullet}{\overset{f^\blacktriangledown}{\rightleftarrows}} (D \ltimes {}_DU_D)\text{-Comod}$$

defined as follows. We define $f_\bullet$ by setting $f_\bullet(-) = \widehat{D} \,\square\, (-)$. Here the repetitive coalgebra $\widehat{D} = \Re(D, {}_DU_D)$ is viewed as a right $D \ltimes {}_DU_D$-comodule and as a left $D \ltimes {}_DU_D$-comodule with comultiplications

$$\widehat{\delta}_r = (\mathrm{id} \otimes f)\widehat{\Delta} : \widehat{D} \to \widehat{D} \otimes (D \ltimes {}_DU_D),$$
$$\widehat{\delta}_l = (f \otimes \mathrm{id})\widehat{\Delta} : \widehat{D} \to (D \ltimes {}_DU_D) \otimes \widehat{D},$$

respectively. The functor $f^{\blacktriangledown}$ associates to any left $\widehat{D}$-comodule $(X,\delta_X)$ the left $(D \ltimes {}_DU_D)$-comodule $f^{\blacktriangledown}(X,\delta_X) = (X,(f \otimes \mathrm{id})\delta_X)$. Given $h \in \mathrm{Hom}(X,Y)$, we set $f^{\blacktriangledown}(h) = h : f^{\blacktriangledown}(X) \to f^{\blacktriangledown}(Y)$.

Now we collect some of the main properties of the functors (4.17). In particular, $f$ is a Galois $\mathbb{Z}$-covering homomorphism and $f^{\blacktriangledown}$ plays the role of a covering functor for comodule categories (see [11] and [29, (10.7)]).

PROPOSITION 4.18. *Let $D$ be a coalgebra, $U = {}_DU_D$ a $D$-$D$-bicomodule, $D \ltimes U$ the trivial extension coalgebra* (4.8), *and $\Re(D, {}_DU_D)$ the $\mathbb{Z}$-graded repetitive coalgebra* (4.15) *with the $\mathbb{Z}$-action defined above.*

(a) *The $K$-linear space $\Re(D, {}_DU_D)/\mathbb{Z}$ of $\mathbb{Z}$-orbits has a canonical coalgebra structure such that the $\mathbb{Z}$-invariant coalgebra surjection* (4.16) *induces a coalgebra isomorphism $\widetilde{f} : \Re(D, {}_DU_D)/\mathbb{Z} \xrightarrow{\simeq} D \ltimes U$.*

(b) *The $K$-linear functor $f_{\bullet}$ in* (4.17) *is right adjoint to $f^{\blacktriangledown}$.*

(c) *The $K$-linear functor $f^{\blacktriangledown}$ in* (4.17) *is exact and faithful.*

*Proof.* For simplicity of notation, we set $\widehat{D} = \Re(D, {}_DU_D)$. The fact that (4.16) is a coalgebra surjection follows by a direct calculation, and we leave it to the reader.

(a) We define a coalgebra structure on $\widehat{D}/\mathbb{Z}$ by the linear maps $\overline{\Delta} : \widehat{D}/\mathbb{Z} \to \widehat{D}/\mathbb{Z} \otimes \widehat{D}/\mathbb{Z}$ and $\overline{\varepsilon} : \widehat{D}/\mathbb{Z} \to K$ given by $\overline{\varepsilon}(\mathbb{Z} * c) = \varepsilon(c)$ and $\overline{\Delta}(\mathbb{Z} * c) = \sum \mathbb{Z} * c_{(1)} \otimes \mathbb{Z} * c_{(2)}$, where $c \in \widehat{D}$ and $\widehat{\Delta}(c) = \sum c_{(1)} \otimes c_{(2)}$. It is straightforward to check that $\overline{\Delta}$ and $\overline{\varepsilon}$ are well-defined and define a coalgebra structure on $\widehat{D}/\mathbb{Z}$.

A direct check shows that the coalgebra surjection $f : \widehat{D} \to D \ltimes U$ is $\mathbb{Z}$-invariant. Hence it easily follows that $f$ induces the required coalgebra isomorphism $\widetilde{f}$.

(b) It follows from [40, Proposition 1.10] that $f_{\bullet}$ has a left adjoint functor. Given a left $\widehat{D}$-comodule $X$ and a left $(D \ltimes U)$-comodule $Z$, the $K$-linear map

$$\widehat{\varepsilon}_* : \mathrm{Hom}_{\widehat{D}}(X, \widehat{D} \,\Box\, Z) \to \mathrm{Hom}_{D \ltimes U}(f^{\blacktriangledown}(X), Z)$$

that associates to any $h \in \mathrm{Hom}_{\widehat{D}}(X, \widehat{D} \,\Box\, Z)$ the homomorphism

$$\widehat{\varepsilon}_*(h) = ((\varepsilon_{D \ltimes U} \circ f) \,\Box\, \mathrm{id}_Z) \circ h : f^{\blacktriangledown}(X) \to Z$$

of left $(D \ltimes U)$-comodules, is an isomorphism. The inverse $F$ of $\widehat{\varepsilon}_*$ is defined by the formula

$$F(h') = (\mathrm{id}_{\widehat{D}} \otimes h') \circ \delta_X^{\widehat{D}} : X \to \widehat{D} \,\Box\, Z$$

for $h' \in \mathrm{Hom}_{D \ltimes U}(f^{\blacktriangledown}(X), Z)$ (see [7, Theorem 1.5] for a proof). Since $\widehat{\varepsilon}_*$ is functorial with respect to comodule homomorphisms $X \to X'$ and $Z \to Z'$, the functor $f^{\blacktriangledown}$ is the right adjoint of $f_{\bullet}$, and (b) follows.

Since (c) follows from the definition of $f^{\blacktriangledown}$, the proof is complete. ■

**5. A reduction functor for coradical square complete coalgebras.** Assume that $C$ is a coradical square complete $K$-coalgebra, that is, $C = C_1 = C_0 \wedge C_0$, where $C_0 = \operatorname{soc} C$. Following an idea of Gabriel [10], we associate with $C$ the bipartite coalgebra

$$H_C = \begin{bmatrix} C_0 & \overline{C} \\ 0 & C_0 \end{bmatrix} \quad \text{with} \quad \overline{C} = C/C_0 \tag{5.1}$$

(see (2.1)) and a $K$-linear *reduction functor*

$$\mathbb{H}_C : C\text{-Comod} \to H_C\text{-Comod} \tag{5.2}$$

defined as follows. We view $\overline{C} = C/C_0$ as a $C_0$-$C_0$-bicomodule and we make the identification $H_C$-Comod $= \operatorname{Rep}_{\square}({}_{C_0}\overline{C}_{C_0})$ via the functor $\Phi$ (see (2.8) and (2.15)). Then each left $H_C$-comodule $X$ is a triple $X = (X', X'', \varphi_X)$ as in (2.11), where $X', X''$ are left $C_0$-comodules and $\varphi_X : X' \to \overline{C} \,\square\, X''$ is a homomorphism of left $C_0$-comodules. In particular, we make the identification

$$\begin{bmatrix} \overline{C} \\ C_0 \end{bmatrix} = (\overline{C}, C_0, j),$$

where $j : \overline{C} \to \overline{C} \,\square\, C_0$ is the canonical isomorphism.

Note that, given $(X, \delta_X)$ in $C$-Comod, $X_0 = \delta_X^{-1}(C_0 \otimes X)$ is the socle of $X$. If $\delta_0$ is the restriction of $\delta_X$ to $X_0$ and $\pi : X \to \overline{X} = X/X_0$ is the projection on the quotient $C$-comodule $(\overline{X}, \delta_{\bar{X}})$, then the diagram of left $C$-comodules

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_0 & \longrightarrow & X & \xrightarrow{\pi} & \overline{X} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \delta_0} & & \downarrow{\scriptstyle \delta_X} & & \downarrow{\scriptstyle \bar{\delta}_X} & & \\ 0 & \longrightarrow & C_0 \,\square\, X & \longrightarrow & C \,\square\, X & \xrightarrow{\pi_C \square \mathrm{id}} & \overline{C} \,\square\, X & & \end{array} \tag{5.3}$$

with exact rows is commutative, where $\pi_C$ is the canonical projection and $\overline{\delta}_X$ is induced by $\delta_X$. It follows that

$$\delta_0(X_0) \subseteq C_0 \,\square\, X_0 \subseteq C_0 \,\square\, X \quad \text{and} \quad X = \delta_X^{-1}((C_0 \otimes X) + (C \otimes X_0)),$$

because $C = C_0 \wedge C_0$. Consequently, $\overline{X}$ is a semisimple $C$-comodule and has a left $C_0$-comodule structure $\delta_{\bar{X}} : \overline{X} \to C_0 \,\square\, \overline{X}$. Hence, we also conclude that $(\pi_C \,\square\, \pi)\delta_X = 0$ and $(\mathrm{id} \,\square\, \pi)\overline{\delta}_X = 0$, because

$$X = \delta_X^{-1}((C_0 \otimes X) + (C \otimes X_0)), \qquad (\mathrm{id} \,\square\, \pi)\overline{\delta}_X \pi = (\pi_C \,\square\, \pi)\delta_X = 0$$

and $\pi$ is surjective. Since the row of the commutative diagram

$$\begin{array}{ccccccc} & & & & \overline{X} & & \\ & & & & \downarrow{\scriptstyle \bar{\delta}_X} & \searrow{\scriptstyle 0} & \\ 0 & \longrightarrow & \overline{C} \,\square\, X_0 & \xrightarrow{\mathrm{id} \square u} & \overline{C} \,\square\, X & \xrightarrow{\mathrm{id} \square \pi} & \overline{C} \,\square\, \overline{X} \end{array} \tag{5.4}$$

is exact and $(\mathrm{id} \,\Box\, \pi)\overline{\delta}_X = 0$, there is a unique map $\varphi_X : \overline{X} \to \overline{C} \,\Box\, X_0$ of left $C$-comodules such that $\overline{\delta}_X = (\mathrm{id} \,\Box\, u)\varphi_X$, where $u : X_0 \to X$ is the inclusion. The left $C$-comodules $\overline{C}$ and $\overline{X}$ are semisimple, so they are left $C_0$-comodules and therefore $\varphi_X$ is a map of left $C_0$-comodules. Note that $\overline{C} \,\Box_C X_0 = \overline{C} \,\Box_{C_0} X_0 = \overline{C} \,\Box\, X_0$ and there is a $K$-vector space decomposition $X \cong X_0 \oplus \overline{X}$ of $X$.

The following lemma is of importance.

LEMMA 5.5. *Let $C$ be a coradical square complete coalgebra and $(X, \delta_X)$ be a left $C$-comodule. Under the identification $X = X_0 \oplus \overline{X}$ and the notation above, the $C$-comodule structure map $\delta_X : X_0 \oplus \overline{X} \to (C \otimes X_0) \oplus (C \otimes \overline{X})$ of $X$ has the form*

$$\delta_X = \begin{bmatrix} \delta_0 & \overline{\varphi}_X \\ 0 & \delta_{\bar{X}} \end{bmatrix},$$

*where $\overline{\varphi}_X : \overline{X} \to C \otimes X_0$ is the composite $K$-linear map*

$$\overline{X} \hookrightarrow X_0 \oplus \overline{X} \xrightarrow{\delta_X} C \,\Box\, X \xrightarrow{\mathrm{id}\Box\pi_{X_0}} C \,\Box\, X_0 \hookrightarrow C \otimes X_0$$

*and $(\pi_C \otimes \mathrm{id})\overline{\varphi}_X = \varphi_X$. Moreover, $\mathrm{Im}\, \overline{\varphi}_X \cap (C_0 \otimes X) = (0)$.*

*Proof.* Consider the $K$-linear map

$$\delta_X = \begin{bmatrix} (\delta_X)_{1,1} & (\delta_X)_{1,2} \\ (\delta_X)_{2,1} & (\delta_X)_{2,2} \end{bmatrix} : X_0 \oplus \overline{X} \to (C \otimes X_0) \oplus (C \otimes \overline{X}).$$

Since $\delta_X(X_0) \subseteq C_0 \otimes X_0$, we have $(\delta_X)_{1,1} = \delta_0$ and $(\delta_X)_{2,1} = 0$. By the definition of $\overline{X}$, we have $\delta_{\bar{X}}\pi = (\mathrm{id} \otimes \pi)\delta_X$ and therefore $(\delta_X)_{2,2} = \delta_{\bar{X}}$. Finally, if $\overline{\varphi}_X = (\delta_X)_{1,2} : \overline{X} \to C \otimes X_0$ and $i : \overline{X} \to X$ is the inclusion, then the equality $X_0 = \delta_X^{-1}(C_0 \otimes X)$ and the commutativity of the diagrams (5.3) and (5.4) yield

$$\begin{aligned}(\pi_C \otimes \mathrm{id})\overline{\varphi}_X &= (\pi_C \otimes \mathrm{id})(\mathrm{id} \otimes \pi_{X_0})\delta_X i = (\mathrm{id} \otimes \pi_{X_0})(\pi_C \otimes \mathrm{id})\delta_X i \\ &= (\mathrm{id} \otimes \pi_{X_0})\overline{\delta}_X \pi i = (\mathrm{id} \otimes \pi_{X_0})(\mathrm{id} \otimes u)\varphi_X = \varphi_X. \ \blacksquare\end{aligned}$$

DEFINITION 5.6. We assume that $C = C_1$ and use the notation introduced above. We define the reduction functor (5.2) by associating with each left $C$-comodule $(X, \delta_X)$ the left $H_C$-comodule

$$\mathbb{H}_C(X) = (X', X'', \varphi_X), \tag{5.7}$$

where $X'' = X_0 = \delta_X^{-1}(C_0 \otimes X) = \mathrm{soc}\, X$ and $X' = \overline{X} = X/X_0$ are viewed as left $C_0$-comodules (see (5.3)), and $\delta_X = \left[\begin{smallmatrix} \delta_0 & \overline{\varphi}_X \\ 0 & \delta_{\bar{X}} \end{smallmatrix}\right]$ is as in Lemma 5.5.

Given $f \in \mathrm{Hom}_C(X, Y)$, we define $\mathbb{H}_C(f) : \mathbb{H}_C(X) \to \mathbb{H}_C(Y)$ to be the pair $\mathbb{H}_C(f) = (f', f'')$, where $f'' : X_0 \to Y_0$ is the restriction of $f$ and $f' : \overline{X} \to \overline{Y}$ is induced by $f$.

We show that $\mathbb{H}_C(f)$ is an $H_C$-comodule homomorphism, by proving that the pair $(f', f'')$ is a morphism in the category $\mathrm{Rep}_\square({}_{C_0}\overline{C}_{C_0})$. We make the identifications $X = X_0 \oplus \overline{X}$ and $Y = Y_0 \oplus \overline{Y}$. Since $f : X_0 \oplus \overline{X} \to Y_0 \oplus \overline{Y}$ is a $C$-comodule homomorphism and $f(X_0) \subseteq Y_0$, $f$ has the matrix form

$$f = \begin{bmatrix} f'' & f_{1,2} \\ 0 & f' \end{bmatrix}$$

and $\delta_Y f = (\mathrm{id} \otimes f)\delta_X$. By Lemma 5.5, we have $\delta_0 f'' = (\mathrm{id} \otimes f'')\delta_0$, $\delta_{\bar{Y}} f' = (\mathrm{id} \otimes f')\delta_{\bar{X}}$ and $\delta_0 f_{1,2} + \overline{\varphi}_Y f' = (\mathrm{id} \otimes f'')\overline{\varphi}_X + (\mathrm{id} \otimes f_{1,2})\delta_{\bar{X}}$, and therefore $f'$ and $f''$ are $C_0$-comodule homomorphisms. Since $\mathrm{Im}(\delta_0 f_{1,2}) \subseteq C_0 \otimes Y$, $\mathrm{Im}(\mathrm{id} \otimes f_{1,2})\delta_{\bar{X}} \subseteq C_0 \otimes Y$, $\mathrm{Im}(\overline{\varphi}_Y f') \cap (C_0 \otimes Y) = (0)$, and $\mathrm{Im}((\mathrm{id} \otimes f'')\overline{\varphi}_X) \cap (C_0 \otimes Y) = (0)$, the final equality yields $\overline{\varphi}_Y f' = (\mathrm{id} \otimes f'')\overline{\varphi}_X$ and our claim is proved.

The main properties of the functor $\mathbb{H}_C$ are collected in Theorem 5.11 below. To formulate it, we need the following definition (cf. Gabriel [10]).

Definition 5.8. Let $C$ be a basic coalgebra and let $({}_CQ, {}_C\mathbf{d})$ be the left valued Gabriel quiver of $C$. The *left separated valued quiver* $({}_C\overset{s}{Q}, {}_C\overset{s}{\mathbf{d}})$ of $C$ is defined as follows. The set ${}_C\overset{s}{Q}_0$ of vertices is the disjoint union ${}_CQ'_0 \cup {}_CQ''_0$ of two copies of ${}_CQ_0$, where ${}_CQ'_0 = \{i'; i \in I_C\}$ and ${}_CQ''_0 = \{j''; j \in I_C\}$. Given two vertices $a, b \in {}_C\overset{s}{Q}_0 = {}_CQ'_0 \cup {}_CQ''_0$, there exists a unique valued arrow

$$a \xrightarrow{({}_C\overset{s}{d}{}'_{ab}, {}_C\overset{s}{d}{}''_{ab})} b$$

if and only if $a = i'$ with $i' \in {}_CQ'_0$, $b = j''$ with $j'' \in {}_CQ''_0$, and there exists a valued arrow

$$i \xrightarrow{({}_Cd'_{ij}, {}_Cd''_{ij})} j$$

in $({}_CQ, {}_C\mathbf{d})$. We set ${}_C\overset{s}{d}{}'_{ab} = {}_Cd'_{ij}$ and ${}_C\overset{s}{d}{}''_{ab} = {}_Cd''_{ij}$.

It follows that the valued quiver $({}_C\overset{s}{Q}, {}_C\overset{s}{\mathbf{d}})$ has no loops, no valued arrows between the vertices in ${}_CQ'_0$, between the vertices in ${}_CQ''_0$, and no valued arrow from a vertex $a \in {}_CQ''_0$ to $b \in {}_CQ'_0$.

To formulate the next result, we define the *stable categories* of $C$-Comod and $C$-comod to be the quotient categories

$$(5.9) \qquad C\text{-}\overline{\mathrm{Comod}} = C\text{-Comod}/\mathcal{I} \quad \text{and} \quad C\text{-}\overline{\mathrm{comod}} = C\text{-comod}/\mathcal{I}$$

modulo the ideal $\mathcal{I}$ in $C$-Comod and $C$-comod, respectively, consisting of all $C$-comodule homomorphisms $f : X \to Y$ having a factorisation through an injective comodule $E$ in $C$-Comod. More precisely, the objects of $C$-$\overline{\mathrm{Comod}}$ and $C$-$\overline{\mathrm{comod}}$ are the same as in $C$-Comod and $C$-comod, respectively, and the space of morphisms from $X$ to $Y$ in the quotient category is the quotient $K$-vector space

$$(5.10) \qquad \overline{\mathrm{Hom}}_C(X, Y) = \mathrm{Hom}_C(X, Y)/\mathcal{I}(X, Y),$$

where $\mathcal{I}(X,Y)$ is formed by all $f : X \to Y$ that have a factorisation through an injective in $C$-Comod (see [2]).

We denote by $H_C$-$\mathrm{Comod}^{\bullet}_{\mathrm{sp}}$ the full subcategory of $H_C$-Comod whose objects are $H_C$-comodules $X$ such that $\operatorname{soc} X$ is projective and has no injective summands of the form $\left[\begin{smallmatrix} S(i') \\ 0 \end{smallmatrix}\right]$, where $S(i')$ is a simple $C_0$-comodule.

THEOREM 5.11. *Assume that $C$ is a basic coradical square complete $K$-coalgebra. Let*

$$H_C = \begin{bmatrix} C_0 & \overline{C} \\ 0 & C_0 \end{bmatrix}$$

*be the associated bipartite coalgebra* (5.1), *with $C_0 = \operatorname{soc} C$ and $\overline{C} = C/C_0$.*

(a) *$H_C$ is basic, hereditary, coradical square complete, and every simple $C$-comodule is projective or injective.*

(b) *The reduction functor $\mathbb{H}_C : C$-Comod $\to H_C$-Comod of* (5.2) *is $K$-linear, full, additive, commutes with arbitrary direct sums and has the following properties:*

(b1) *Given a $C$-comodule homomorphism $f : X \to Y$, we have $\mathbb{H}_C(f) = 0$ if and only if $f(\operatorname{soc} X) = 0$. In particular, the kernel of the algebra surjection $\operatorname{End}_C X \to \operatorname{End}_{H_C} \mathbb{H}_C(X)$, $f \mapsto \mathbb{H}_C(f)$, equals $\operatorname{Hom}_C(X/\operatorname{soc} X, X)$. If $X$, $Y$ have no injective direct summands then $\mathbb{H}_C(f) = 0$ if and only if $f \in \mathcal{I}(X,Y)$.*

(b2) *$\mathbb{H}_C$ does not vanish on non-zero comodules, carries ${}_C C$ to the left coideal $\left[\begin{smallmatrix} \overline{C} \\ C_0 \end{smallmatrix}\right]$ of $H_C = \left[\begin{smallmatrix} C_0 & \overline{C} \\ 0 & C_0 \end{smallmatrix}\right]$ and carries simple comodules to simple ones.*

(b3) *A comodule $X = (X', X'', \varphi)$ in $H_C$-comod lies in $\operatorname{Im} \mathbb{H}_C$ if and only if $\varphi : X' \to \overline{C} \,\square\, X''$ is a monomorphism.*

(b4) *An indecomposable comodule $X$ in $H_C$-comod does not belong to $\operatorname{Im} \mathbb{H}_C$ if and only if $X$ is simple injective of the form $\left[\begin{smallmatrix} S'(i') \\ 0 \end{smallmatrix}\right]$, where $S'(i')$ is a simple subcomodule of $C$.*

(b5) $\operatorname{Im} \mathbb{H}_C = H_C$-$\mathrm{Comod}^{\bullet}_{\mathrm{sp}}$.

(c) *The functor $\mathbb{H}_C$ defines a representation equivalence* (*see* [27], [38])

$$\mathbb{H}_C : C\text{-Comod} \to H_C\text{-Comod}^{\bullet}_{\mathrm{sp}} \subseteq H_C\text{-Comod}$$

*and carries indecomposable $C$-comodules to indecomposable ones.*

(d) *A $C$-comodule $E$ is injective if and only if $\mathbb{H}_C(E)$ is an injective $H_C$-comodule. Moreover, the functor $\mathbb{H}_C$ induces*

- *an isomorphism $F_a = \operatorname{End}_C S(a) \cong \operatorname{End}_{H_C} \mathbb{H}_C(S(a))$ of division rings for each $a \in I_C$,*
- *equivalences of stable categories*

$$C\text{-}\overline{\mathrm{Comod}} \cong H_C\text{-}\overline{\mathrm{Comod}} \quad \textit{and} \quad C\text{-}\overline{\mathrm{comod}} \cong H_C\text{-}\overline{\mathrm{comod}}.$$

(e) *The left valued Gabriel quiver of the hereditary coalgebra* $H_C$ *is the left separated valued quiver* $({}_C\overset{s}{Q}, {}_C\overset{s}{\mathbf{d}})$ *of* $C$.

*Proof.* Throughout the proof, we make the identification $H_C$-Comod $=$ $\mathrm{Rep}_\Box({}_{C_0}\overline{C}_{C_0})$ via the functor $\Phi$ of (2.8) and (2.15) (see Theorem 2.14).

(a) Apply Theorem 2.16.

(b) That $\mathbb{H}_C$ is additive and commutes with arbitrary direct sums follows immediately from its definition.

Now we prove that $\mathbb{H}_C$ is full. Let $X$, $Y$ be $C$-comodules and $\mathbb{H}_C(X) = (\overline{X}, X_0, \varphi_X)$, $\mathbb{H}_C(Y) = (\overline{Y}, Y_0, \varphi_Y)$. Given a homomorphism $(f', f'') : \mathbb{H}_C(X) \to \mathbb{H}_C(Y)$ of $H_C$-comodules, we define a $K$-linear map

$$f = \begin{bmatrix} f'' & 0 \\ 0 & f' \end{bmatrix} : X \cong X_0 \oplus \overline{X} \to Y \cong Y_0 \oplus \overline{Y}.$$

We claim that $f$ is a $C$-comodule homomorphism such that $\mathbb{H}_C(f) = (f', f'')$. Indeed,

$$\delta_Y \circ f = \begin{bmatrix} \delta_0 & \overline{\varphi}_Y \\ 0 & \delta_{\bar{Y}} \end{bmatrix} \circ \begin{bmatrix} f'' & 0 \\ 0 & f' \end{bmatrix} = \begin{bmatrix} \delta_0 f'' & \overline{\varphi}_Y f' \\ 0 & \delta_{\bar{Y}} f' \end{bmatrix}.$$

On the other hand,

$$(I \otimes f) \circ \delta_X = \begin{bmatrix} I \otimes f'' & 0 \\ 0 & I \otimes f' \end{bmatrix} \circ \begin{bmatrix} \delta_0 & \overline{\varphi}_X \\ 0 & \delta_{\bar{X}} \end{bmatrix} = \begin{bmatrix} (I \otimes f'')\delta_0 & (I \otimes f'')\overline{\varphi}_X \\ 0 & (I \otimes f')\delta_{\bar{X}} \end{bmatrix}.$$

Since $(f', f'')$ is an $H_C$-comodule homomorphism, $\delta_Y \circ f = (I \otimes f) \circ \delta_X$ and our claim follows, because the equality $\mathbb{H}_C(f) = (f', f'')$ is obvious. This shows that $\mathbb{H}_C$ is full.

(b1) If $X$ is a non-zero $C$-comodule, then $X_0 = \operatorname{soc} X \neq 0$ and therefore $\mathbb{H}_C(X) \neq 0$.

Let $f : X \to Y$ be a non-zero $C$-homomorphism such that $\mathbb{H}_C(f) = 0$. By the definition of $\mathbb{H}_C$, we get $X_0 \subseteq \operatorname{Ker} f$. Conversely, let $X_0 \subseteq \operatorname{Ker} f$; then $f_{|X_0} = 0$. Since $C = C_0 \wedge C_0$, the left $C$-comodule $X/X_0$ is semisimple. Therefore $\operatorname{Im} f \cong X/\operatorname{Ker} f$ is semisimple and $\operatorname{Im} f \subseteq Y_0$. Consequently, $\bar{f} = 0$ and $\mathbb{H}_C(f) = (\bar{f}, f_{|X_0}) = 0$.

To prove the second statement in (b1), assume that $X$ and $Y$ are $C$-comodules having no injective direct summands. Let $f \in \mathcal{I}(X, Y)$, that is, $f : X \to Y$ is a $C$-comodule homomorphism that factorises through an injective $C$-comodule $E$. Let $g : X \to E$ and $h : E \to Y$ be $C$-comodule homomorphisms such that $f = hg$. Assume, to the contrary, that $\mathbb{H}_C(f) \neq 0$. By the above considerations, $f(X_0) \neq 0$ and therefore $hg(X_0) \neq 0$. Since $g(X_0) \subseteq E_0$, we have $0 \neq h(E_0) \subseteq Y$. There exists an indecomposable direct summand $E'$ of $E$ such that $0 \neq h(E'_0) \subseteq Y$. If $\operatorname{Ker} h_{|E'} \neq 0$ then the simple $C$-comodule $E'_0$ is contained in $\operatorname{Ker} h_{|E'}$ and therefore $h(E'_0) = 0$, a contradiction. This proves that $h_{|E'} : E' \to Y$ is a monomorphism. Since $E'$ is injective, it is a direct summand of $Y$, contrary to our assumption. Consequently, $\mathbb{H}_C(f) = 0$.

Conversely, let $f : X \to Y$ be such that $\mathbb{H}_C(f) = 0$. Let $\pi : X \to X/X_0$ be the natural projection. By the first part of (b1) we have $f(X_0) = 0$. Therefore $f = g\pi$ for some homomorphism $g : X/X_0 \to Y$. Assume that $j : X \to E(X)$ is the injective envelope of $X$. Applying standard arguments we can construct commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X_0 & \longrightarrow & X & \xrightarrow{\pi} & \overline{X} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle h} & & \\ 0 & \longrightarrow & X_0 & \longrightarrow & E(X) & \xrightarrow{\pi_1} & E(X)/X_0 & \longrightarrow & 0 \end{array}$$

where $h$ is a monomorphism and the comodules $\overline{X} = X/X_0$, $E(X)/X_0$ are semisimple (because $C$ is coradical square complete). Therefore there exists a homomorphism $h_1 : E(X)/X_0 \to \overline{X}$ such that $h_1 h = \mathrm{id}_{\bar{X}}$, and hence $f = g\pi = gh_1h\pi = gh_1\pi_1 j \in \mathcal{I}(X, Y)$.

(b2) It was shown in the proof of (b1) that $\mathbb{H}_C(X) \neq 0$ if $X \neq 0$. By the definition of $\mathbb{H}_C$, we know that $\mathbb{H}_C(C) = \left[\begin{smallmatrix} \overline{C} \\ C_0 \end{smallmatrix}\right]$. Moreover, for any simple $C$-comodule $S$, $\mathbb{H}_C(S) = (0, S, 0) = \left[\begin{smallmatrix} 0 \\ S \end{smallmatrix}\right]$ is a simple $H_C$-comodule, by Theorem 2.16.

(b3) Take a $C$-comodule $X$ and consider $\mathbb{H}_C(X) = (\overline{X}, X_0, \varphi_X)$. Note that $\overline{\delta}_X$ (defined in (5.3)) is a monomorphism. Indeed, assume that $\overline{\delta}_X(x) = 0$ for some $x \in \overline{X}$. Then there exists $y \in X$ such that $\pi(y) = x$ and $(\pi_C \,\square\, \mathrm{id})\delta_X(y) = \overline{\delta}_X(y) = 0$. It follows that $\delta_X(y) \in C_0 \,\square\, X$ and $y \in X_0$. Finally, $0 = \pi(y) = x$ and $\overline{\delta}_X$ is a monomorphism. Therefore, by the definition, $\varphi_X$ is a monomorphism. Conversely, let $(X', X'', \varphi)$ be an $H_C$-comodule such that $\varphi$ is a monomorphism. Let $X$ be the $K$-vector space $X = X'' \oplus X'$. Note that there is an isomorphism of vector spaces $C \cong C_0 \oplus C/C_0$. It is easy to see that the $K$-linear map

$$\delta_X = \begin{bmatrix} \delta_{X''} & \varphi \\ 0 & \delta_{X'} \end{bmatrix} : X'' \oplus X' \to (C \otimes X'') \oplus (C \otimes X')$$

defines a $C$-comodule structure on $X$. Since $\varphi$ is a monomorphism, we have $\operatorname{soc} X = X''$ and therefore $\mathbb{H}_C(X) = (X', X'', \varphi)$ (see Lemma 5.5).

(b4) The proof above shows that the $H_C$-comodules of the form $(X', 0, 0)$, where $X' \neq 0$, are not in $\operatorname{Im} \mathbb{H}_C$. Conversely, let $(X', X'', \varphi)$ be an $H_C$-comodule such that $\varphi$ is not a monomorphism. Then there exists a non-zero direct summand of $(X', X'', \varphi)$ of the form $(Y', 0, 0)$, namely $(\operatorname{Ker} \varphi, 0, 0)$. Hence (b4) follows, because $C_0$ is a semisimple $K$-coalgebra.

(b5) follows from (b3), (b4), and Theorem 2.16.

(c) We recall that an additive functor is said to be a *representation equivalence* (or *epivalence*, see [12]) if it is full, dense, and respects isomorphisms (see [27], [28], and [38]). By (b), the functor $\mathbb{H}_C : C\text{-comod} \to H_C\text{-comod}^{\bullet}_{\rm sp}$ is full and dense. To show that $\mathbb{H}_C$ reflects isomorphisms, assume that $f : X \to Y$ is a $C$-homomorphism in $C$-Comod such that $\mathbb{H}_C(f) = (f', f'')$ is an isomorphism. It follows that $f'' : X_0 \to Y_0$ and $f' : \overline{X} \to \overline{Y}$ are isomorphisms. Hence, in view of the Snake Lemma, $f$ is an isomorphism and the first part of (c) follows.

To finish the proof of (c), assume that $X$ is an indecomposable $C$-comodule but $\mathbb{H}_C(X) \cong \overline{Y} \oplus \overline{Z}$ decomposes. By (b4), the $H_C$-comodules $\overline{Y}$ and $\overline{Z}$ lie in the image of $\mathbb{H}_C$. Therefore there exist $C$-comodules $Y$ and $Z$ such that $\overline{Y} \cong \mathbb{H}_C(Y)$ and $\overline{Z} \cong \mathbb{H}_C(Z)$. Hence $\mathbb{H}_C(X) \cong \mathbb{H}_C(Y \oplus Z)$, because $\mathbb{H}_C$ is additive. Since we have shown that $\mathbb{H}_C$ reflects isomorphisms, the $C$-comodule $X \cong Y \oplus Z$ decomposes, a contradiction.

(d) Let $E$ be an indecomposable injective $C$-comodule. There exists a $C$-comodule $E'$ such that $E \oplus E' \cong C$. Then $\mathbb{H}_C(C) \cong \mathbb{H}_C(E \oplus E') \cong \mathbb{H}_C(E) \oplus \mathbb{H}_C(E')$ and $\mathbb{H}_C(E)$ is a direct summand of $\mathbb{H}_C(C)$. By (b2) and (2.5) the $H_C$-comodule $\mathbb{H}_C(E)$ is injective.

Conversely, let $\mathbb{H}_C(E)$ be an indecomposable injective $H_C$-comodule. By (b4), there exists an $H_C$-comodule $\overline{X}$ such that $\mathbb{H}_C(E) \oplus \overline{X} \cong \left[\begin{smallmatrix} \overline{C} \\ C_0 \end{smallmatrix}\right]$ and there exists a $C$-comodule $X$ such that $\mathbb{H}_C(X) \cong \overline{X}$. Therefore $\mathbb{H}_C(C) \cong \mathbb{H}_C(E \oplus X)$. Since $\mathbb{H}_C$ reflects isomorphisms, we have $C \cong E \oplus X$, and hence $E$ is injective.

The first item in the final part of (d) follows from the first one and (b). To finish the proof of (d), we note that $\mathbb{H}_C : C\text{-Comod} \to H_C\text{-Comod}$ induces the functors

$$\overline{\mathbb{H}}_C : C\text{-}\overline{\text{Comod}} \longrightarrow H_C\text{-}\overline{\text{Comod}} \quad \text{and} \quad \overline{\mathbb{H}}_C : C\text{-}\overline{\text{comod}} \longrightarrow H_C\text{-}\overline{\text{comod}}$$

that are full (by (c)) and dense, because $\mathbb{H}_C$ carries injectives to injectives and all non-injective comodules in $H_C$-Comod are in $\operatorname{Im} \mathbb{H}_C$, by (b4). It remains to show that $\overline{\mathbb{H}}_C$ is faithful. Let $\overline{f} : \overline{X} \to \overline{Y}$ be a morphism in $C\text{-}\overline{\text{Comod}}$ with $f \in \operatorname{Hom}_C(X, Y)$ such that $\overline{\mathbb{H}}_C(\overline{f}) = 0$. We can assume that $X$ and $Y$ have no non-zero injective summands. Then $\mathbb{H}_C(f) : \mathbb{H}_C(X) \to \mathbb{H}_C(Y)$ has a factorisation $\mathbb{H}_C(X) \xrightarrow{g_1} Z \xrightarrow{g_2} \mathbb{H}_C(Y)$, where $Z$ is an injective $\mathbb{H}_C$-comodule. By (c) and the first part of (d), $Z \cong \mathbb{H}_C(E)$, where $E$ is an injective $C$-comodule, and there exist $C$-comodule homomorphisms $X \xrightarrow{f_1} E \xrightarrow{f_2} Y$ such that $\mathbb{H}_C(f_1) = g_1$ and $\mathbb{H}_C(f_2) = g_2$. It follows that $\mathbb{H}_C$ vanishes on $h = f - g_2 g_1 : X \to Y$ and, by (b1), $h \in \mathcal{I}(X, Y)$. Hence $f = h + g_2 g_1 \in \mathcal{I}(X, Y)$ and therefore $\overline{f}$ is zero in the quotient category $C\text{-}\overline{\text{Comod}}$. This shows that the functor $\overline{\mathbb{H}}_C$ is faithful, and consequently, it is an equivalence of categories.

(e) We apply Corollary 3.8 to $H = H_C$. In this case, we have

$$H' = C_0, \quad H'' = C_0, \quad U = \overline{C} = C/C_0, \quad I_{H'} = I_C, \quad I_{H''} = I_C.$$

In the notation of (3.11), given $s' = s \in I_{H'} = I_C$ and $s' = s \in I_{H'} = I_C$, we have ${}_{s'}U_{t''} = {}_s(C/C_0)_t$. Hence, (e) follows from Corollary 3.8, Proposition 3.5 and the definition of the separated Gabriel valued quiver of $C$. ■

Following [38, Remark XIX.1.13] and the proof of the previous theorem, we construct a functor

$$\mathbb{H}^{\bullet}_C : H_C\text{-comod}^{\bullet}_{\mathrm{sp}} \to C\text{-comod} \tag{5.12}$$

as follows. Given an $H_C$-comodule $(X', X'', \varphi)$ in $H_C\text{-comod}^{\bullet}_{\mathrm{sp}} = \operatorname{Im} \mathbb{H}_C$, we set

$$\mathbb{H}^{\bullet}_C(X', X'', \varphi) = \left( X'' \oplus X', \begin{bmatrix} \delta_{X''} & \varphi \\ 0 & \delta_{X'} \end{bmatrix} \right),$$

and given a homomorphism $(f', f'') : (X', X'', \varphi) \to (Y', Y'', \varphi)$ in the category $H_C\text{-comod}^{\bullet}_{\mathrm{sp}}$, we set $\mathbb{H}^{\bullet}_C(f', f'') = \left[\begin{smallmatrix} f'' & 0 \\ 0 & f' \end{smallmatrix}\right]$. It is clear that we have defined a covariant $K$-linear functor $\mathbb{H}^{\bullet}_C$. Now we collect its main properties.

Corollary 5.13. *Assume that $C$ is a basic coradical square complete $K$-coalgebra. Under the notation and assumptions of Theorem* 5.11, *the functor* $\mathbb{H}^{\bullet}_C : H_C\text{-comod}^{\bullet}_{\mathrm{sp}} \to C\text{-Comod}$ *has the following properties.*

(a) $\mathbb{H}_C \circ \mathbb{H}^{\bullet}_C$ *is isomorphic to the identity functor on* $H_C\text{-comod}^{\bullet}_{\mathrm{sp}}$.
(b) $\mathbb{H}^{\bullet}_C$ *is faithful, exact, carries indecomposables to indecomposables, and non-isomorphic comodules to non-isomorphic ones.*

*Proof.* (a) This follows from the proof of Theorem 5.11(b).

(b) Obviously, $\mathbb{H}^{\bullet}_C$ is faithful and exact. Let $(X', X'', \varphi)$ be an object in $H_C\text{-comod}^{\bullet}_{\mathrm{sp}} = \operatorname{Im} \mathbb{H}_C$ and assume that $X = \mathbb{H}^{\bullet}_C((X', X'', \varphi)) \cong Y \oplus Z$ for some non-zero $C$-comodules $Y$ and $Z$. By (a), we have

$$(X', X'', \varphi) \cong \mathbb{H}_C \circ \mathbb{H}^{\bullet}_C((X', X'', \varphi)) \cong \mathbb{H}_C(Y \oplus Z) \cong \mathbb{H}_C(Y) \oplus \mathbb{H}_C(Z).$$

It follows that $(X', X'', \varphi)$ is decomposable, because by Theorem 5.11(b2) the functor $\mathbb{H}_C$ does not vanish on non-zero objects. Since the final part of (b) is a consequence of (a), the proof is complete. ■

**6. Applications.** We recall from [20], [29] and [30] that a $K$-coalgebra $C$ is said to be *left pure semisimple* if every left $C$-comodule is a direct sum of finite-dimensional $C$-comodules (see also [23], [24], and [25]).

The following characterisation of left pure semisimple coalgebras is of importance.

THEOREM 6.1. *Assume that $C$ is a $K$-coalgebra. The following conditions are equivalent.*

(a) *$C$ is left pure semisimple.*
(b) *For every infinite sequence $N_1 \xrightarrow{f_1} N_2 \xrightarrow{f_2} \cdots$ of non-zero monomorphisms between indecomposables in $C$-comod there exists $m_0 \geq 1$ such that $f_j$ is an isomorphism for all $j \geq m_0$.*
(c) *For every infinite sequence $N_1 \xrightarrow{f_1} N_2 \xrightarrow{f_1} \cdots$ of non-zero non-isomorphisms between indecomposables in $C$-comod there exists $m_0 \geq 1$ such that $f_j \dots f_1 = 0$ for all $j \geq m_0$.*

*Proof.* Apply [21, Theorem 3.1] and [22, Theorem 6.3] to $\mathcal{A} = C$-Comod (see also [29, Theorem 7.2]). ■

The following result shows that the reduction functor $\mathbb{H}_C$ respects pure semisimplicity.

PROPOSITION 6.2. *Assume that $C$ is a basic coradical square complete $K$-coalgebra and let $H_C = \left[\begin{smallmatrix} C_0 & \overline{C} \\ 0 & C_0 \end{smallmatrix}\right]$ be the associated bipartite hereditary coalgebra, with $C_0 = \operatorname{soc} C$ and $\overline{C} = C/\operatorname{soc} C$. The following conditions are equivalent.*

(a) *$C$ is left pure semisimple.*
(b) *$H_C$ is left pure semisimple.*
(c) *$H_C$ is a direct sum of finite-dimensional coalgebras of finite comodule type.*
(d) *The left separated valued quiver $({}_C\overset{s}{Q}, {}_C\overset{s}{\mathbf{d}})$ is a disjoint union of Dynkin valued quivers, that is, finite valued quivers whose underlying graphs are Dynkin diagrams of one of the types $\mathbb{A}_n$ $(n \geq 1)$, $\mathbb{B}_n$ $(n \geq 2)$, $\mathbb{C}_n$ $(n \geq 3)$, $\mathbb{D}_n$ $(n \geq 4)$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$, $\mathbb{F}_4$ or $\mathbb{G}_2$ (see* [14, Table 2]*).*

*Proof.* We prove that (a) implies (b) by applying Theorem 6.1. Assume that $C$ is a basic left pure semisimple coalgebra and

$$Y_1 \xrightarrow{\bar{f}_1} Y_2 \xrightarrow{\bar{f}_2} \cdots$$

is a sequence of non-zero non-isomorphisms between finite-dimensional indecomposable left $H_C$-comodules. We may assume that no $Y_i$ is simple injective, because otherwise some $\overline{f}_i$ is zero or an isomorphism, contrary to assumption.

By Theorem 5.11(b), this sequence lies in $H_C\text{-comod}^{\bullet}_{\mathrm{sp}} = \operatorname{Im} \mathbb{H}_C$. By Theorem 5.11(c), for each $i \geq 1$, there exists an indecomposable $C$-comodule $X_i$ in $C$-comod and a non-zero non-isomorphism $f_i \in \operatorname{Hom}_C(X_i, X_{i+1})$ such that $\mathbb{H}_C(X_i) = Y_i$ and $\mathbb{H}_C(f_i) = \bar{f}_i$. Thus we have a sequence

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots$$

of non-zero non-isomorphisms between finite-dimensional indecomposable $C$-comodules. Since $C$ is left pure semisimple, there exists $m_0 \geq 1$ such that $f_j \dots f_1 = 0$ for all $j \geq m_0$; hence $\bar{f}_j \dots \bar{f}_1 = 0$ for all $j \geq m_0$. Then, in view of Theorem 6.1, $H_C$ is left pure semisimple.

To prove that (b) implies (a), assume that $H_C$ is left pure-semisimple. Let

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots$$

be a sequence of non-zero monomorphisms between finite-dimensional indecomposable $C$-comodules. It follows that $f_m \dots f_1(\operatorname{soc} X_1) \neq 0$ for each $m \geq 1$, and, according to Theorem 5.10, $\mathbb{H}_C(f_m \dots f_1) = \mathbb{H}_C(f_m) \dots \mathbb{H}_C(f_1) : \mathbb{H}_C(X_1) \to \mathbb{H}_C(X_m)$ is non-zero. By Theorem 5.10, the sequence

$$Y_1 \xrightarrow{\bar{f}_1} Y_2 \xrightarrow{\bar{f}_2} \cdots$$

with $Y_i = \mathbb{H}_C(X_i)$, $\bar{f}_i = \mathbb{H}_C(f_i)$ in $H_C\text{-comod}^{\bullet}_{\rm sp}$ consists of indecomposable comodules connected by non-zero homomorphisms. The observation made above yields $\bar{f}_n \dots \bar{f}_1 \neq 0$ for each $n \geq 1$. Since $H_C$ is pure semisimple, there exists $i_0$ such that $\bar{f}_n$ is an isomorphism for any $n \geq i_0$. Hence, $f_n$ is an isomorphism for any $n \geq i_0$, because $\mathbb{H}_C$ reflects isomorphisms by Theorem 5.10(c). Consequently, $C$ is left pure semisimple by Theorem 6.1, and therefore (a) and (b) are equivalent.

To prove (b)$\Leftrightarrow$(c), it is sufficient to show that the left pure semisimplicity of $H_C$ implies (c), because the converse follows from [29, Theorem 7.5].

Assume that $H_C$ is left pure semisimple and decompose it into a direct sum

$$H_C = \bigoplus_{\beta \in T} H_\beta$$

of indecomposable coalgebras $H_\beta$. It follows that, for each $\beta \in T$, the left valued Gabriel quiver $({}_{H_\beta}Q, {}_{H_\beta}\mathbf{d})$ is a connected component of $({}_{H_C}Q, {}_{H_C}\mathbf{d})$ (see [29, Corollary 8.7] and [32, Corollary 2.8]). Since $H_C$ is hereditary and left pure semisimple, so is $H_\beta$ for each $\beta \in T$. Then, according to [14, Theorem 4.14] (see also [20] and [29]), either the quiver $({}_{H_\beta}Q, {}_{H_\beta}\mathbf{d})$ is one of the infinite pure semisimple locally Dynkin valued quivers $\mathbb{A}_\infty^{(s)}$, ${}_\infty\mathbb{A}_\infty^{(s)}$, $\mathbb{B}_\infty^{(s)}$, $\mathbb{C}_\infty^{(s)}$ or $\mathbb{D}_\infty^{(s)}$, with $s \geq 0$, presented in [14, Table 1], or $({}_{H_\beta}Q, {}_{H_\beta}\mathbf{d})$ is finite and its underlying valued graph is one of the Dynkin valued diagrams $\mathbb{A}_n$ $(n \geq 1)$, $\mathbb{B}_n$ $(n \geq 2)$, $\mathbb{C}_n$ $(n \geq 3)$, $\mathbb{D}_n$ $(n \geq 4)$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$, $\mathbb{F}_4$ or $\mathbb{G}_2$ presented in [14, Table 2].

Since every infinite pure semisimple locally Dynkin valued quiver contains an infinite chain of the form $\bullet\!\to\!\bullet\!\to\!\bullet\!\to\!\cdots\!\to\!\bullet\!\to\!\bullet\!\to\!\cdots$, it follows that $({}_{H_\beta}Q, {}_{H_\beta}\mathbf{d})$ is not infinite, because $({}_{H_C}Q, {}_{H_C}\mathbf{d})$ is the separated valued quiver $({}_C\overset{s}{Q}, {}_C\overset{s}{\mathbf{d}})$ of $C$, by Theorem 5.11(d), the quiver $({}_{H_\beta}Q, {}_{H_\beta}\mathbf{d})$ is a connected subquiver of $({}_{H_C}Q, {}_{H_C}\mathbf{d}) = ({}_C\overset{s}{Q}, {}_C\overset{s}{\mathbf{d}})$, and it follows from the definition of separated valued quiver that it does not contain infinite chains of the above form. Consequently, $({}_{H_\beta}Q, {}_{H_\beta}\mathbf{d})$ is finite and the underlying valued graph of $({}_{H_\beta}Q, {}_{H_\beta}\mathbf{d})$ is one of the Dynkin valued diagrams. It follows that $\dim_K H_\beta$ is finite and, by [29, Theorem 7.5], the coalgebra $H_\beta$ is of finite comodule type for each $\beta \in T$. This finishes the proof of (b)$\Leftrightarrow$(c). Since this also shows that (c) and (d) are equivalent, the proposition is proved. ∎

Corollary 6.3. *Let $C = D \ltimes {}_DU_D$ be the trivial extension of a basic semisimple coalgebra $D$ by a $D$-$D$-bicomodule ${}_DU_D$.*

(a) *$C$ is coradical square complete, the associated bipartite coalgebra $H_C$ is the hereditary coalgebra $\left[\begin{smallmatrix} D & {}_DU_D \\ 0 & D \end{smallmatrix}\right]$ and the reduction functor $\mathbb{H}_C$ : $C$-comod $\to H_C$-comod$^\bullet_{\rm sp}$ is a representation equivalence.*

(b) *The left valued Gabriel quiver of $C$ has the form $({}_DQ, {}_D\mathbf{d}) \blacklozenge_U ({}_DQ, {}_D\mathbf{d})$ (see (4.9)), that is, it is obtained from the valued quiver $({}_DQ, {}_D\mathbf{d}) \blacksquare_U ({}_DQ, {}_D\mathbf{d})$ (see (3.10)) of the bipartite coalgebra $\left[\begin{smallmatrix} D & {}_DU_D \\ 0 & D \end{smallmatrix}\right]$ by the identification of the vertex $s'$ with the vertex $s''$ and the arrow $s' \to t'$ with the arrow $s'' \to t''$ in $({}_DQ, {}_D\mathbf{d}) \blacksquare_U ({}_DQ, {}_D\mathbf{d})$, for all $s, t \in {}_DQ_0 = I_D$.*

(c) *$C$ is left pure semisimple if and only if $\left[\begin{smallmatrix} D & {}_DU_D \\ 0 & D \end{smallmatrix}\right]$ is left pure semisimple, and if and only if the left separated valued quiver of $C$ is a disjoint union of Dynkin valued quivers.*

*Proof.* Apply Proposition 4.10, Theorem 5.11, and Proposition 6.2. ∎

Example 6.4. Let $\mathbb{N}$ be the set of positive integers and let

$$C = \bigoplus_{n\in\mathbb{N}} Ke_n \oplus \bigoplus_{m\in\mathbb{N}} K\eta_m$$

be a $K$-vector space with a countable basis $\{e_n, \eta_m\}_{n,m\in\mathbb{N}}$ equipped with the comultiplication $\Delta : C \to C \otimes C$ and the counit $\varepsilon : C \to K$, defined by the formulae:

- $\Delta(e_n) = e_n \otimes e_n$ and $\Delta(\eta_m) = e_m \otimes \eta_m + \eta_m \otimes e_{m+1}$,
- $\varepsilon(e_n) = 1$ and $\varepsilon(\eta_m) = 0$ for $n, m \in \mathbb{N}$.

It is straightforward to check that $C = (C, \Delta, \varepsilon)$ is a basic $K$-coalgebra, $C_0 = \operatorname{soc} C = \bigoplus_{n\in\mathbb{N}} S(n)$, where $S(n) = Ke_n$ is a simple subcoalgebra of $C$, and $C = C_1 = C_0 \wedge C_0$, that is, $C$ is coradical square complete.

It is easy to check that, for each $i \in \mathbb{N}$, we have $\operatorname{Ext}^1_C(S(i), S(i+1)) \cong K$ and $\operatorname{Ext}^1_C(S(i), S(j)) = 0$ for $j \neq i+1$. It follows that the separated valued quiver $({}^{s}_{C}Q, {}^{s}_{C}\mathbf{d})$ has the form

$$\begin{array}{ccccccccccccc} 1' & & 2' & & 3' & & 4' & & 5' & & 6' & & \cdots \\ & \searrow & & \searrow & & \searrow & & \searrow & & \searrow & & \searrow & \\ 1'' & & 2'' & & 3'' & & 4'' & & 5'' & & 6'' & & \cdots \end{array}$$

and, by Proposition 6.2, $C$ is left pure semisimple.

Note also that $C$ is isomorphic to the trivial extension coalgebra $D \ltimes {}_DU_D$, where $D = \operatorname{soc} C$ is a basic semisimple subcoalgebra of $C$ and ${}_DU_D = \bigoplus_{m\in\mathbb{N}} K\eta_m \subseteq C$ is viewed as a $D$-$D$-bicomodule in the obvious way.

It follows from Theorem 5.11 and Corollary 6.3 that the left Gabriel quiver of the bipartite coalgebra

$$H_C = \begin{bmatrix} D & {}_DU_D \\ 0 & D \end{bmatrix}$$

is the quiver presented above, whereas the left Gabriel quiver of $C \cong D \ltimes {}_DU_D$ is the infinite linear quiver

$$Q: 1 \xrightarrow{\beta_1} 2 \xrightarrow{\beta_2} \cdots \to s-1 \xrightarrow{\beta_{s-1}} s \xrightarrow{\beta_s} s+1 \xrightarrow{\beta_{s+1}} \cdots$$

obtained from the above by the identification $n \equiv n' \equiv n''$ for each $n \in \mathbb{N}$.

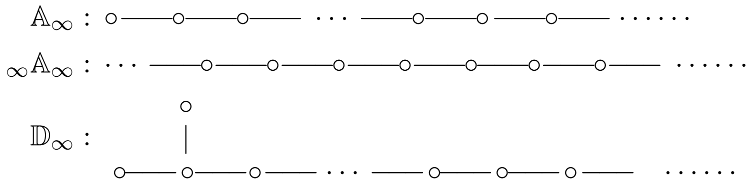
Let $K^{\square}Q$ be the path coalgebra of the quiver $Q$. One can show that there is a coalgebra isomorphism $C \cong (K^{\square}Q)_1 = KQ_0 \oplus KQ_1$ given by $e_n \mapsto \widehat{e}_n$ (the stationary path at the vertex $n \in Q_0$) and $\eta_n \mapsto \beta_n \in KQ_1$. Hence, by applying the results in [29], [31] and [33], one can show that $C$ is isomorphic to the path coalgebra $K^{\square}(Q,\Omega) = C(Q,\Omega)$ with the ideal $\Omega \subseteq KQ$ of relations generated by all compositions $\beta_n\beta_{n+1}$ with $n \in \mathbb{N}$. Consequently, the category $C$-comod $\cong K^{\square}(Q,\Omega)$-comod is equivalent to the category $\operatorname{rep}_K(Q,\Omega)$ of finite-dimensional representations of $Q$ satisfying the relation $\beta_n\beta_{n+1} = 0$ for each $n \in \mathbb{N}$. ■

We finish the paper by a discussion of tame and wild comodule type of any basic coalgebra $C$ by means of its separated valued quiver. For the definition of tame and wild comodule type the reader is referred to [29, Definition 6.6], [30], and [31]. In particular, the tame-wild dichotomy for coalgebras over an algebraically closed field is discussed in [31].

Proposition 6.5. *Assume that $K$ is an algebraically closed field. Let $C$ be a basic $K$-coalgebra, $C_1$ the first term of the coradical filtration of $C$, and $H = H_{C_1}$ the associated hereditary bipartite coalgebra.*

(a) *The quiver ${}_HQ$ coincides with the left separated quiver ${}^{s}_{C}Q$.*
(b) *If $H_{C_1}$ is of wild comodule type, then so is $C$.*
(c) *If $C$ is of tame comodule type, then so is $H = H_{C_1}$, and the underlying non-oriented graph of each of the connected components of ${}_HQ$ $(= {}^{s}_{C}Q)$ is of one of the types:*

- *the Dynkin diagrams* $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$,
- *the Euclidean diagrams* $\widetilde{\mathbb{A}}_n$, $\widetilde{\mathbb{D}}_n$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$,
- *the infinite locally Dynkin diagrams* (*see* [14], [29]–[31]),




*Proof.* We recall from Theorem 5.11 that $H_{C_1}$ is hereditary.

(a) By Proposition 3.5, the left Gabriel quiver $_{C_1}Q$ coincides with $_CQ$. Then (a) follows from Theorem 5.11(d).

(b) Assume that $H_{C_1}$ is of wild comodule type. Then there exists a $K$-linear representation embedding functor $T : \operatorname{mod} \Gamma_3(K) \to H_{C_1}$-comod, where $\Gamma_3(K) = \left[\begin{smallmatrix} K & K^3 \\ 0 & K \end{smallmatrix}\right]$ is the path $K$-algebra of the wild quiver $\circ \Rrightarrow \circ$. By [38, Corollary XVIII.4.2], there exists a full, faithful, exact $K$-linear endofunctor

$$F : \operatorname{mod} \Gamma_3(K) \to \operatorname{mod} \Gamma_3(K)$$

such that $\operatorname{Im} F$ is contained in the category $\operatorname{add} \mathcal{R}(\Gamma_3(K))$ of all regular $\Gamma_3(K)$-modules. It follows that the image of

$$T \circ F : \operatorname{mod} \Gamma_3(K) \to H_{C_1}\text{-comod}$$

does not contain simple comodules. Indeed, given a non-zero module $X$ in $\operatorname{mod} \Gamma_3(K)$, the module $F(X)$ is regular, and hence not simple. It follows that there exists a non-split exact sequence $0 \to Y' \to F(X) \to Y'' \to 0$ in $\operatorname{mod} \Gamma_3(K)$, where $Y'$ and $Y''$ are non-zero. Since $T$ is exact, we derive the exact sequence $0 \to T(Y') \to T(F(X)) \to T(Y'') \to 0$ in $H_{C_1}$-comod, where $T(Y')$ and $T(Y'')$ are non-zero. This shows that $\dim_K T(F(X)) \geq 2$, and consequently $T(F(X))$ lies in $H_{C_1}\text{-comod}^\bullet_{\mathrm{sp}}$.

It follows that $T \circ F : \operatorname{mod} \Gamma_3(K) \to H_{C_1}$-comod defines a representation embedding $(T \circ F)' : \operatorname{mod} \Gamma_3(K) \to H_{C_1}\text{-comod}^\bullet_{\mathrm{sp}}$. Since, by Corollary 5.13, $\mathbb{H}^\bullet_{C_1} : H_{C_1}\text{-comod}^\bullet_{\mathrm{sp}} \to C_1$-comod is a representation embedding, so is

$$\mathbb{H}^\bullet_{C_1} \circ (T \circ F)' : \operatorname{mod} \Gamma_3(K) \to C_1\text{-comod} \hookrightarrow C\text{-comod}.$$

This shows that $C$ is of wild comodule type.

(c) Assume that $C$ is of tame comodule type. By [29, Theorem 6.11(a)] and its proof, the subcoalgebra $C_1$ of $C$ is also of tame comodule type. Suppose that $H_{C_1}$ is not tame. Since, by [31, Theorem 5.12], the tame-wild dichotomy holds for hereditary basic coalgebras, $H_{C_1}$ is of wild comodule type. Hence, by (b), $C$ is of wild comodule type and, according to [31, Corollary 5.6] (a weak version of tame-wild dichotomy for coalgebras), we get a contradiction.

We recall that $H_{C_1}$ is hereditary. Since it is of tame comodule type, every indecomposable coalgebra direct summand $H'$ of $H_{C_1}$ is also of tame comodule type and, obviously, the left Gabriel quiver $Q'$ of $H'$ is a connected component of ${}_HQ$. Then, by [29, Theorem 9.4] and [31, Theorem 5.12], the underlying unoriented graph of $Q'$ is of one of the types listed in (c). ∎

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